SOLUTION MANUAL FOR

Problem 1.1

1.1 A number of common substances are

Tar	Sand
‘‘Silly Putty’’	Jello
Modeling clay	Toothpaste
Wax	Shaving cream

Some of these materials exhibit characteristics of both solid and fluid behavior under different conditions. Explain and give examples.

Given: Common Substances

Tar	Sand
“Silly Putty”	Jello
Modeling clay	Toothpaste
Wax	Shaving cream

Some of these substances exhibit characteristics of solids and fluids under different conditions.

Find: Explain and give examples.

Solution: Tar, Wax, and Jello behave as solids at room temperature or below at ordinary pressures. At high pressures or over long periods, they exhibit fluid characteristics. At higher temperatures, all three liquefy and become viscous fluids.

Modeling clay and silly putty show fluid behavior when sheared slowly. However, they fracture under suddenly applied stress, which is a characteristic of solids.

Toothpaste behaves as a solid when at rest in the tube. When the tube is squeezed hard, toothpaste “flows” out the spout, showing fluid behavior. Shaving cream behaves similarly.

Sand acts solid when in repose (a sand “pile”). However, it “flows” from a spout or down a steep incline.

Problem 1.2

[2]

1.2 Give a word statement of each of the five basic conservation laws stated in Section 1-4, as they apply to a system.

Given: Five basic conservation laws stated in Section 1-4.

Write: A word statement of each, as they apply to a system.

Solution: Assume that laws are to be written for a *system*.

a. Conservation of mass — The mass of a system is constant by definition.

b. Newton's second law of motion — The net force acting on a system is directly proportional to the product of the system mass times its acceleration.

c. First law of thermodynamics — The change in stored energy of a system equals the net energy added to the system as heat and work.

d. Second law of thermodynamics — The entropy of any isolated system cannot decrease during any process between equilibrium states.

e. Principle of angular momentum — The net torque acting on a system is equal to the rate of change of angular momentum of the system.

Problem 1.3

1.3 Discuss the physics of skipping a stone across the water surface of a lake. Compare these mechanisms with a stone as it bounces after being thrown along a roadway.

Open-Ended Problem Statement: Consider the physics of "skipping" a stone across the water surface of a lake. Compare these mechanisms with a stone as it bounces after being thrown along a roadway.

Discussion: Observation and experience suggest two behaviors when a stone is thrown along a water surface:

1. If the angle between the path of the stone and the water surface is steep the stone may penetrate the water surface. Some momentum of the stone will be converted to momentum of the water in the resulting splash. After penetrating the water surface, the high drag* of the water will slow the stone quickly. Then, because the stone is heavier than water it will sink.
2. If the angle between the path of the stone and the water surface is shallow the stone may not penetrate the water surface. The splash will be smaller than if the stone penetrated the water surface. This will transfer less momentum to the water, causing less reduction in speed of the stone. The only drag force on the stone will be from friction on the water surface. The drag will be momentary, causing the stone to lose only a portion of its kinetic energy. Instead of sinking, the stone may skip off the surface and become airborne again.

When the stone is thrown with speed and angle just right, it may skip several times across the water surface. With each skip the stone loses some forward speed. After several skips the stone loses enough forward speed to penetrate the surface and sink into the water.

Observation suggests that the shape of the stone significantly affects skipping. Essentially spherical stones may be made to skip with considerable effort and skill from the thrower. Flatter, more disc-shaped stones are more likely to skip, provided they are thrown with the flat surface(s) essentially parallel to the water surface; spin may be used to stabilize the stone in flight.

By contrast, no stone can ever penetrate the pavement of a roadway. Each collision between stone and roadway will be inelastic; friction between the road surface and stone will affect the motion of the stone only slightly. Regardless of the initial angle between the path of the stone and the surface of the roadway, the stone may bounce several times, then finally it will roll to a stop.

The shape of the stone is unlikely to affect trajectory of bouncing from a roadway significantly.

Problem 1.4

[3]

1.4 The barrel of a bicycle tire pump becomes quite warm during use. Explain the mechanisms responsible for the temperature increase.

Open-Ended Problem Statement: The barrel of a bicycle tire pump becomes quite warm during use. Explain the mechanisms responsible for the temperature increase.

Discussion: Two phenomena are responsible for the temperature increase: (1) friction between the pump piston and barrel and (2) temperature rise of the air as it is compressed in the pump barrel.

Friction between the pump piston and barrel converts mechanical energy (force on the piston moving through a distance) into thermal energy as a result of friction. Lubricating the piston helps to provide a good seal with the pump barrel and reduces friction (and therefore force) between the piston and barrel.

Temperature of the trapped air rises as it is compressed. The compression is not adiabatic because it occurs during a finite time interval. Heat is transferred from the warm compressed air in the pump barrel to the cooler surroundings. This raises the temperature of the barrel, making its outside surface warm (or even hot!) to the touch.

Problem 1.5

[1]

1.5 A spherical tank of inside diameter 500 cm contains compressed oxygen at 7 MPa and 25°C. What is the mass of oxygen?

Given: Data on oxygen tank.

Find: Mass of oxygen.

Solution: Compute tank volume, and then use oxygen density (Table A.6) to find the mass.

The given or available data is: $D = 500\cdot cm$ $\quad p = 7\cdot MPa$ $\quad T = (25 + 273)\cdot K$ $\quad T = 298\,K$

$$R_{O2} = 259.8\cdot\frac{J}{kg\cdot K} \qquad \text{(Table A.6)}$$

The governing equation is the ideal gas equation

$$p = \rho\cdot R_{O2}\cdot T \qquad \text{and} \quad \rho = \frac{M}{V}$$

where V is the tank volume $\quad V = \frac{\pi\cdot D^3}{6} \qquad V = \frac{\pi}{6}\times(5\cdot m)^3 \qquad V = 65.4\cdot m^3$

Hence $\quad M = V\cdot\rho = \frac{p\cdot V}{R_{O2}\cdot T} \qquad M = 7\times 10^6\cdot\frac{N}{m^2}\times 65.4\cdot m^3\times\frac{1}{259.8}\cdot\frac{kg\cdot K}{N\cdot m}\times\frac{1}{298}\cdot\frac{1}{K} \qquad M = 5913\,kg$

Problem 1.6

[1]

1.6 Make a guess at the order of magnitude of the mass (e.g., 0.01, 0.1, 1.0, 10, 100, or 1000 lbm or kg) of standard air that is in a room 10 ft by 10 ft by 8 ft, and then compute this mass in lbm and kg to see how close your estimate was.

Given: Dimensions of a room

Find: Mass of air

Solution:

Basic equation: $\rho = \frac{p}{R_{air} \cdot T}$

Given or available data $p = 14.7\text{psi}$ $\quad T = (59 + 460)\text{R}$ $\quad R_{air} = 53.33 \cdot \frac{\text{ft} \cdot \text{lbf}}{\text{lbm} \cdot \text{R}}$

$V = 10 \cdot \text{ft} \times 10 \cdot \text{ft} \times 8 \cdot \text{ft}$ $\quad V = 800\,\text{ft}^3$

Then $\rho = \frac{p}{R_{air} \cdot T}$ $\quad \rho = 0.076 \frac{\text{lbm}}{\text{ft}^3}$ $\quad \rho = 0.00238 \frac{\text{slug}}{\text{ft}^3}$ $\quad \rho = 1.23 \frac{\text{kg}}{\text{m}^3}$

$M = \rho \cdot V$ $\quad M = 61.2\,\text{lbm}$ $\quad M = 1.90\,\text{slug}$ $\quad M = 27.8\,\text{kg}$

Problem 1.7

1.7 A cylindrical tank for containing 10 lbm of compressed nitrogen at a pressure of 200 atm (gage) and 70°F must be designed. The design constraints are that the length must be twice the diameter and the wall thickness must be $\frac{1}{4}$ in. What are the external dimensions?

Given: Mass of nitrogen, and design constraints on tank dimensions.

Find: External dimensions.

Solution: Use given geometric data and nitrogen mass, with data from Table A.6.

The given or available data is: $M = 10 \cdot lbm$ $\quad p = (200 + 1) \cdot atm \quad p = 2.95 \times 10^3 \cdot psi$

$T = (70 + 460) \cdot K \quad T = 954 \cdot R \quad R_{N2} = 55.16 \cdot \frac{ft \cdot lbf}{lbm \cdot R}$ (Table A.6)

The governing equation is the ideal gas equation $\quad p = \rho \cdot R_{N2} \cdot T$ and $\rho = \frac{M}{V}$

where *V* is the tank volume $\quad V = \frac{\pi \cdot D^2}{4} \cdot L$ where $L = 2 \cdot D$

Combining these equations:

Hence

$$M = V \cdot \rho = \frac{p \cdot V}{R_{N2} \cdot T} = \frac{p}{R_{N2} \cdot T} \cdot \frac{\pi \cdot D^2}{4} \cdot L = \frac{p}{R_{N2} \cdot T} \cdot \frac{\pi \cdot D^2}{4} \cdot 2 \cdot D = \frac{p \cdot \pi \cdot D^3}{2 \cdot R_{N2} \cdot T}$$

Solving for D

$$D = \left(\frac{2 \cdot R_{N2} \cdot T \cdot M}{p \cdot \pi}\right)^{\frac{1}{3}} \qquad D = \left[\frac{2}{\pi} \times 55.16 \cdot \frac{ft \cdot lbf}{lbm \cdot R} \times 954 \cdot K \times 10 \cdot lbm \times \frac{1}{2950} \cdot \frac{in^2}{lbf} \times \left(\frac{ft}{12 \cdot in}\right)^2\right]^{\frac{1}{3}}$$

$D = 1.12 \cdot ft \quad D = 13.5 \cdot in \quad L = 2 \cdot D \quad L = 27 \cdot in$

These are internal dimensions; the external ones are 1/4 in. larger: $L = 27.25 \cdot in \quad D = 13.75 \cdot in$

Problem 1.8

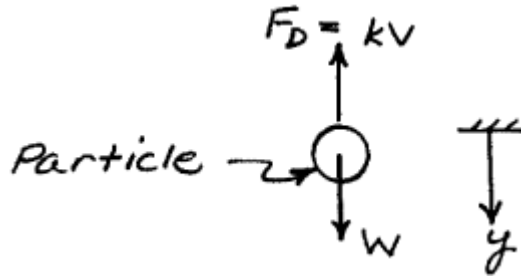
[3]

1.8 Very small particles moving in fluids are known to experience a drag force proportional to speed. Consider a particle of net weight *W* dropped in a fluid. The particle experiences a drag force, $F_D = kV$, where *V* is the particle speed. Determine the time required for the particle to accelerate from rest to 95 percent of its terminal speed, V_t, in terms of *k*, *W*, and *g*.

Given: Small particle accelerating from rest in a fluid. Net weight is W, resisting force F_D = kV, where V is speed.

Find: Time required to reach 95 percent of terminal speed, V_t.

Solution: Consider the particle to be a system. Apply Newton's second law.

Basic equation: $\sum F_y = ma_y$

Assumptions:

1. W is net weight
2. Resisting force acts opposite to V

Then
$$\sum F_y = W - kV = ma_y = m\frac{dV}{dt} = \frac{W}{g}\frac{dV}{dt}$$

or
$$\frac{dV}{dt} = g(1 - \frac{k}{W}V)$$

Separating variables,
$$\frac{dV}{1 - \frac{k}{W}V} = g\,dt$$

Integrating, noting that velocity is zero initially,
$$\int_0^V \frac{dV}{1 - \frac{k}{W}V} = -\frac{W}{k}\ln(1 - \frac{k}{W}V)\Big]_0^V = \int_0^t g dt = gt$$

or $$1-\frac{k}{W}V = e^{-\frac{kgt}{W}};\;\; V = \frac{W}{k}\left[1-e^{-\frac{kgt}{W}}\right]$$

But $V \to V_t$ as $t \to \infty$, so $V_t = \frac{W}{k}$. Therefore $$\frac{V}{V_t} = 1 - e^{-\frac{kgt}{W}}$$

When $\frac{V}{V_t} = 0.95$, then $e^{-\frac{kgt}{W}} = 0.05$ and $\frac{kgt}{W} = 3$. Thus t = 3 W/gk

Problem 1.9

[2]

1.9 Consider again the small particle of Problem 1.8. Express the distance required to reach 95 percent of its terminal speed in terms of *g*, *k*, and *W*.

Given: Small particle accelerating from rest in a fluid. Net weight is W, resisting force is $F_D = kV$, where V is speed.

Find: Distance required to reach 95 percent of terminal speed, V_t.

Solution: Consider the particle to be a system. Apply Newton's second law.

Basic equation: $\sum F_y = ma_y$

Assumptions:

1. W is net weight.
2. Resisting force acts opposite to V.

Then, $\sum F_y = W - kV = ma_y = m\frac{dV}{dt} = \frac{W}{g}V\frac{dV}{dy}$ or $1 - \frac{k}{W}V = \frac{V}{g}\frac{dV}{dy}$

At terminal speed, $a_y = 0$ and $V = V_t = \frac{W}{k}$. Then $1 - \frac{V}{V_g} = \frac{1}{g}V\frac{dV}{dy}$

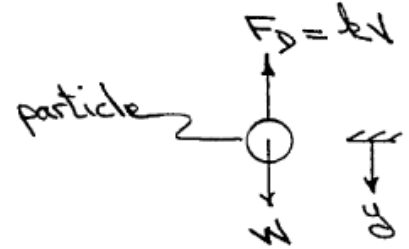
Separating variables $\frac{V\,dV}{1 - \frac{1}{V_t}V} = g\,dy$

Integrating, noting that velocity is zero initially

$$gy = \int_0^{0.95V_t} \frac{V\,dV}{1 - \frac{1}{V_t}V} = \left[-VV_t - V_t^2 \ln\left(1 - \frac{V}{V_t}\right)\right]_0^{0.95V_t}$$

$$gy = -0.95V_t^2 - V_t^2 \ln(1 - 0.95) - V_t^2 \ln(1)$$

$$gy = -V_t^2\left[0.95 + \ln 0.05\right] = 2.05\,V_t^2$$

$$\therefore y = \frac{2.05}{g}V_t^2 = 2.05\frac{W^2}{gt^2}$$

Problem 1.10

[3]

1.10 For a small particle of styrofoam (1 lbm/ft^3) (spherical, with diameter $d = 0.3$ mm) falling in standard air at speed V, the drag is given by $F_D = 3\pi\mu V d$, where μ is the air viscosity. Find the maximum speed starting from rest, and the time it takes to reach 95% of this speed. Plot the speed as a function of time.

Given: Data on sphere and formula for drag.

Find: Maximum speed, time to reach 95% of this speed, and plot speed as a function of time.

Solution: Use given data and data in Appendices, and integrate equation of motion by separating variables.

The data provided, or available in the Appendices, are:

$$\rho_{air} = 1.17 \cdot \frac{kg}{m^3} \qquad \mu = 1.8 \times 10^{-5} \cdot \frac{N \cdot s}{m^2} \qquad \rho_w = 999 \cdot \frac{kg}{m^3} \qquad SG_{Sty} = 0.016 \qquad d = 0.3 \cdot mm$$

Then the density of the sphere is $\rho_{Sty} = SG_{Sty} \cdot \rho_w$ $\qquad \rho_{Sty} = 16 \frac{kg}{m^3}$

The sphere mass is $M = \rho_{Sty} \cdot \frac{\pi \cdot d^3}{6} = 16 \cdot \frac{kg}{m^3} \times \pi \times \frac{(0.0003 \cdot m)^3}{6}$ $\qquad M = 2.26 \times 10^{-10} kg$

Newton's 2nd law for the steady state motion becomes (ignoring buoyancy effects) $\qquad M \cdot g = 3 \cdot \pi \cdot V \cdot d$

so

$$V_{max} = \frac{M \cdot g}{3 \cdot \pi \cdot \mu \cdot d} = \frac{1}{3 \cdot \pi} \times 2.26 \times 10^{-10} \cdot kg \times 9.81 \cdot \frac{m}{s^2} \times \frac{m^2}{1.8 \times 10^{-5} \cdot N \cdot s} \times \frac{1}{0.0003 \cdot m} \qquad V_{max} = 0.0435 \frac{m}{s}$$

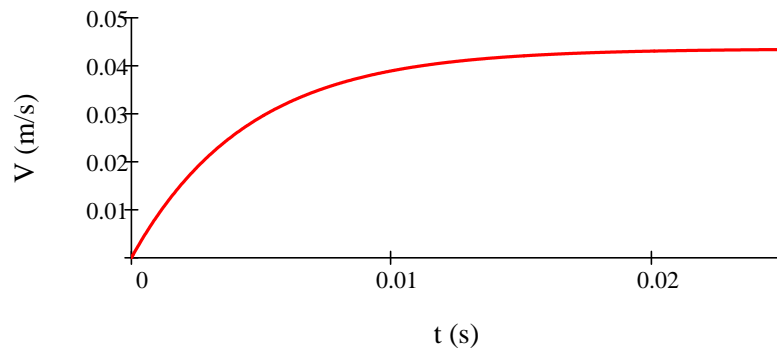
Newton's 2nd law for the general motion is (ignoring buoyancy effects) $\qquad M \cdot \frac{dV}{dt} = M \cdot g - 3 \cdot \pi \cdot \mu \cdot V \cdot d$

so

$$\frac{dV}{g - \frac{3 \cdot \pi \cdot \mu \cdot d}{M} \cdot V} = dt$$

Integrating and using limits

$$V(t) = \frac{M \cdot g}{3 \cdot \pi \cdot \mu \cdot d} \cdot \left(1 - e^{\frac{-3 \cdot \pi \cdot \mu \cdot d}{M} \cdot t} \right)$$

Using the given data

The time to reach 95% of maximum speed is obtained from $\frac{M \cdot g}{3 \cdot \pi \cdot \mu \cdot d} \cdot \left(1 - e^{\frac{-3 \cdot \pi \cdot \mu \cdot d}{M} \cdot t}\right) = 0.95 \cdot V_{max}$

so $t = -\frac{M}{3 \cdot \pi \cdot \mu \cdot d} \cdot \ln\left(1 - \frac{0.95 \cdot V_{max} \cdot 3 \cdot \pi \cdot \mu \cdot d}{M \cdot g}\right)$ Substituting values $t = 0.0133\,s$

The plot can also be done in *Excel*.

Problem 1.11

1.11 In a combustion process, gasoline particles are to be dropped in air. The particles must drop at least 25 cm in 1 s. Find the diameter d of droplets required for this. (The drag on these particles is given by $F_D = 3\pi\mu V d$, where V is the particle speed and μ is the air viscosity. To solve this problem use *Excel*'s *Goal Seek*.)

Given: Data on sphere and formula for drag.

Find: Diameter of gasoline droplets that take 1 second to fall 25 cm.

Solution: Use given data and data in Appendices; integrate equation of motion by separating variables.

The data provided, or available in the Appendices, are:

$$\mu = 1.8\times10^{-5}\cdot\frac{N\cdot s}{m^2} \qquad \rho_w = 999\cdot\frac{kg}{m^3} \qquad SG_{gas} = 0.72 \qquad \rho_{gas} = SG_{gas}\cdot\rho_w \qquad \rho_{gas} = 719\frac{kg}{m^3}$$

Newton's 2nd law for the sphere (mass M) is (ignoring buoyancy effects) $M\cdot\frac{dV}{dt} = M\cdot g - 3\cdot\pi\cdot\mu\cdot V\cdot d$

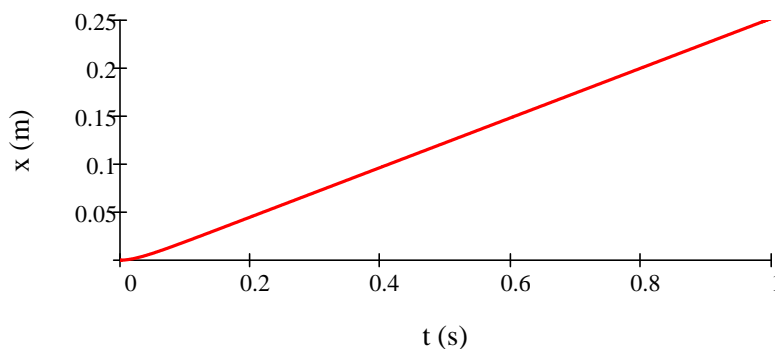
so $$\frac{dV}{g - \frac{3\cdot\pi\cdot\mu\cdot d}{M}\cdot V} = dt$$

Integrating and using limits $$V(t) = \frac{M\cdot g}{3\cdot\pi\cdot\mu\cdot d}\cdot\left(1 - e^{\frac{-3\cdot\pi\cdot\mu\cdot d}{M}\cdot t}\right)$$

Integrating again $$x(t) = \frac{M\cdot g}{3\cdot\pi\cdot\mu\cdot d}\cdot\left[t + \frac{M}{3\cdot\pi\cdot\mu\cdot d}\cdot\left(e^{\frac{-3\cdot\pi\cdot\mu\cdot d}{M}\cdot t} - 1\right)\right]$$

Replacing M with an expression involving diameter d $M = \rho_{gas}\cdot\frac{\pi\cdot d^3}{6}$ $$x(t) = \frac{\rho_{gas}\cdot d^2\cdot g}{18\cdot\mu}\cdot\left[t + \frac{\rho_{gas}\cdot d^2}{18\cdot\mu}\cdot\left(e^{\frac{-18\cdot\mu}{\rho_{gas}\cdot d^2}\cdot t} - 1\right)\right]$$

This equation must be solved for d so that $x(1\cdot s) = 1\cdot m$. The answer can be obtained from manual iteration, or by using *Excel's Goal Seek*. (See this in the corresponding Excel workbook.)

$$d = 0.109\cdot mm$$

Note That the particle quickly reaches terminal speed, so that a simpler approximate solution would be to solve $Mg = 3\pi\mu Vd$ for d, with $V = 0.25$ m/s (allowing for the fact that M is a function of d)!

Problem 1.12

1.12 A sky diver with a mass of 70 kg jumps from an aircraft. The aerodynamic drag force acting on the sky diver is known to be $F_D = kV^2$, where $k = 0.25\ \text{N} \cdot \text{s}^2/\text{m}^2$. Determine the maximum speed of free fall for the sky diver and the speed reached after 100 m of fall. Plot the speed of the sky diver as a function of time and as a function of distance fallen.

Given: Data on sky diver: $M = 70 \cdot \text{kg}$ $\quad k = 0.25 \cdot \frac{\text{N} \cdot \text{s}^2}{\text{m}^2}$

Find: Maximum speed; speed after 100 m; plot speed as function of time and distance.

Solution: Use given data; integrate equation of motion by separating variables.

Treat the sky diver as a system; apply Newton's 2nd law:

Newton's 2nd law for the sky diver (mass M) is (ignoring buoyancy effects): $M \cdot \frac{dV}{dt} = M \cdot g - k \cdot V^2$ (1)

(a) For terminal speed V_t, acceleration is zero, so $M \cdot g - k \cdot V^2 = 0$ so $V_t = \sqrt{\frac{M \cdot g}{k}}$

$$V_t = \left(75 \cdot \text{kg} \times 9.81 \cdot \frac{\text{m}}{\text{s}^2} \times \frac{\text{m}^2}{0.25 \cdot \text{N} \cdot \text{s}^2} \cdot \frac{\text{N} \cdot \text{s}^2}{\text{kg} \times \text{m}}\right)^{\frac{1}{2}} \qquad V_t = 54.2 \frac{\text{m}}{\text{s}}$$

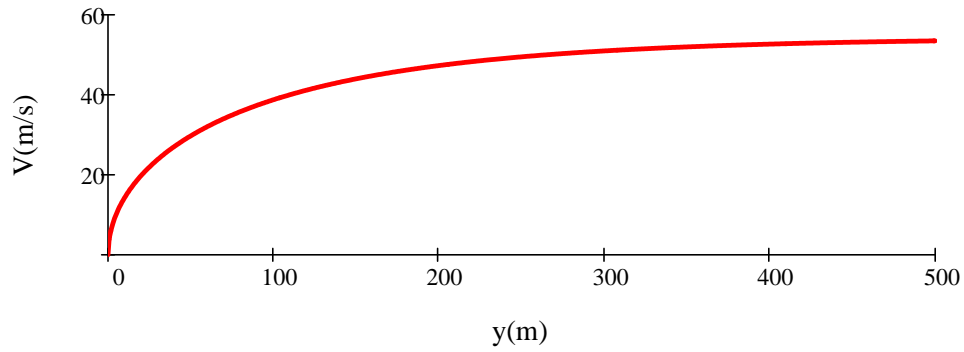
(b) For V at y = 100 m we need to find $V(y)$. From (1) $M \cdot \frac{dV}{dt} = M \cdot \frac{dV}{dy} \cdot \frac{dy}{dt} = M \cdot V \cdot \frac{dV}{dt} = M \cdot g - k \cdot V^2$

Separating variables and integrating:

$$\int_0^V \frac{V}{1 - \frac{k \cdot V^2}{M \cdot g}}\, dV = \int_0^y g\, dy$$

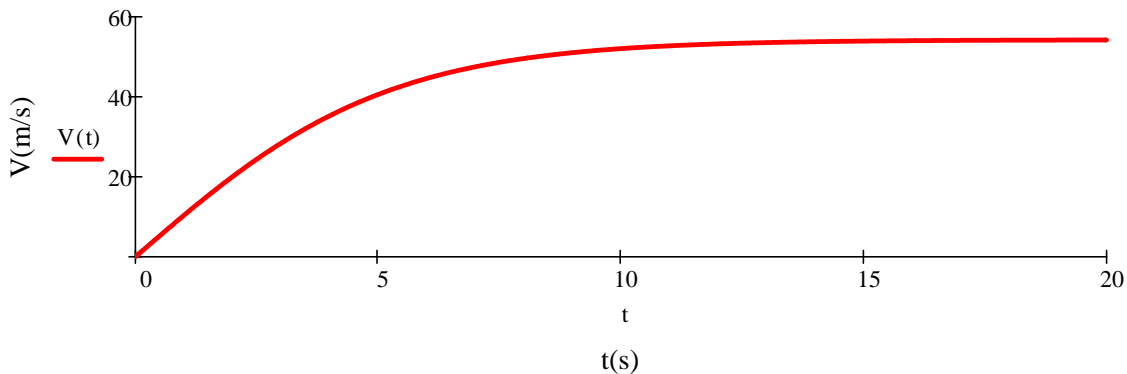
so $\ln\left(1 - \frac{k \cdot V^2}{M \cdot g}\right) = -\frac{2 \cdot k}{M} y$ or $V^2 = \frac{M \cdot g}{k} \cdot \left(1 - e^{-\frac{2 \cdot k \cdot y}{M}}\right)$

Hence $V(y) = V_t \cdot \left(1 - e^{-\frac{2 \cdot k \cdot y}{M}}\right)^{\frac{1}{2}}$

For y = 100 m: $V(100 \cdot \text{m}) = 54.2 \cdot \frac{\text{m}}{\text{s}} \cdot \left(1 - e^{-2 \times 0.25 \cdot \frac{\text{N} \cdot \text{s}^2}{\text{m}^2} \times 100 \cdot \text{m} \times \frac{1}{70 \cdot \text{kg}} \times \frac{\text{kg} \cdot \text{m}}{\text{s}^2 \cdot \text{N}}}\right)^{\frac{1}{2}}$ $\qquad V(100 \cdot \text{m}) = 38.8 \cdot \frac{\text{m}}{\text{s}}$

(c) For $V(t)$ we need to integrate (1) with respect to t: $M \cdot \frac{dV}{dt} = M \cdot g - k \cdot V^2$

Separating variables and integrating: $\int_0^V \frac{V}{\frac{M \cdot g}{k} - V^2} dV = \int_0^t 1\, dt$

so $$t = \frac{1}{2} \cdot \sqrt{\frac{M}{k \cdot g}} \cdot \ln\left(\left|\frac{\sqrt{\frac{M \cdot g}{k}} + V}{\sqrt{\frac{M \cdot g}{k}} - V}\right|\right) = \frac{1}{2} \cdot \sqrt{\frac{M}{k \cdot g}} \cdot \ln\left(\frac{|V_t + V|}{|V_t - V|}\right)$$

Rearranging $$V(t) = V_t \cdot \frac{\left(e^{2 \cdot \sqrt{\frac{k \cdot g}{M}} \cdot t} - 1\right)}{\left(e^{2 \cdot \sqrt{\frac{k \cdot g}{M}} \cdot t} + 1\right)}$$ or $$V(t) = V_t \cdot \tanh\left(V_t \cdot \frac{k}{M} \cdot t\right)$$

The two graphs can also be plotted in Excel.

Problem 1.13

1.13 For Problem 1.12, the initial horizontal speed of the skydiver is 70 m/s. As she falls, the k value for the vertical drag remains as before, but the value for horizontal motion is $k = 0.05$ N • s/m^2. Compute and plot the 2D trajectory of the skydiver.

Given: Data on sky diver: $M = 70\cdot kg$ $\quad k_{vert} = 0.25\cdot\frac{N\cdot s^2}{m^2}$ $\quad k_{horiz} = 0.05\cdot\frac{N\cdot s^2}{m^2}$ $\quad U_0 = 70\cdot\frac{m}{s}$

Find: Plot of trajectory.

Solution: Use given data; integrate equation of motion by separating variables.

Treat the sky diver as a system; apply Newton's 2nd law in horizontal and vertical directions:

Vertical: Newton's 2nd law for the sky diver (mass M) is (ignoring buoyancy effects): $M\cdot\frac{dV}{dt} = M\cdot g - k_{vert}\cdot V^2$ (1)

For $V(t)$ we need to integrate (1) with respect to t:

Separating variables and integrating:

$$\int_0^V \frac{V}{\frac{M\cdot g}{k_{vert}} - V^2}\,dV = \int_0^t 1\,dt$$

so

$$t = \frac{1}{2}\cdot\sqrt{\frac{M}{k_{vert}\cdot g}}\cdot\ln\left(\left|\frac{\sqrt{\frac{M\cdot g}{k_{vert}}} + V}{\sqrt{\frac{M\cdot g}{k_{vert}}} - V}\right|\right)$$

Rearranging

$$V(t) = \sqrt{\frac{M\cdot g}{k_{vert}}}\cdot\frac{\left(e^{2\cdot\sqrt{\frac{k_{vert}\cdot g}{M}}\cdot t} - 1\right)}{\left(e^{2\cdot\sqrt{\frac{k_{vert}\cdot g}{M}}\cdot t} + 1\right)}$$

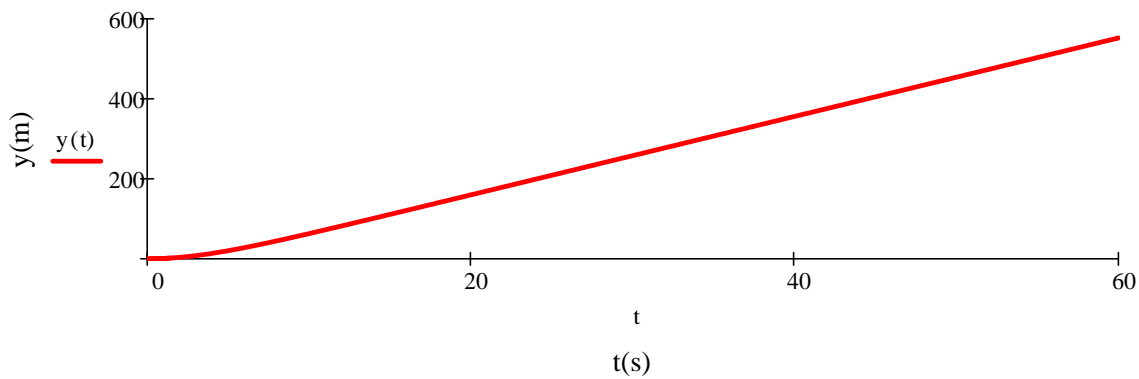
so

$$V(t) = \sqrt{\frac{M\cdot g}{k_{vert}}}\cdot\tanh\left(\sqrt{\frac{k_{vert}\cdot g}{M}}\cdot t\right)$$

For $y(t)$ we need to integrate again: $\frac{dy}{dt} = V$ or $y = \int V\,dt$

$$y(t) = \int_0^t V(t)\,dt = \int_0^t \sqrt{\frac{M\cdot g}{k_{vert}}}\cdot\tanh\left(\sqrt{\frac{k_{vert}\cdot g}{M}}\cdot t\right)dt = \sqrt{\frac{M\cdot g}{k_{vert}}}\cdot\ln\left(\cosh\left(\sqrt{\frac{k_{vert}\cdot g}{M}}\cdot t\right)\right)$$

$$y(t) = \sqrt{\frac{M\cdot g}{k_{vert}}}\cdot\ln\left(\cosh\left(\sqrt{\frac{k_{vert}\cdot g}{M}}\cdot t\right)\right)$$

t(s)

Horizontal: Newton's 2nd law for the sky diver (mass M) is:

$$M \cdot \frac{dU}{dt} = -k_{horiz} \cdot U^2 \qquad (2)$$

For *U*(*t*) we need to integrate (2) with respect to *t*:

Separating variables and integrating:

$$\int_{U_0}^{U} \frac{1}{U^2}\, dU = \int_0^t -\frac{k_{horiz}}{M}\, dt \qquad \text{so} \qquad -\frac{k_{horiz}}{M} \cdot t = -\frac{1}{U} + \frac{1}{U_0}$$

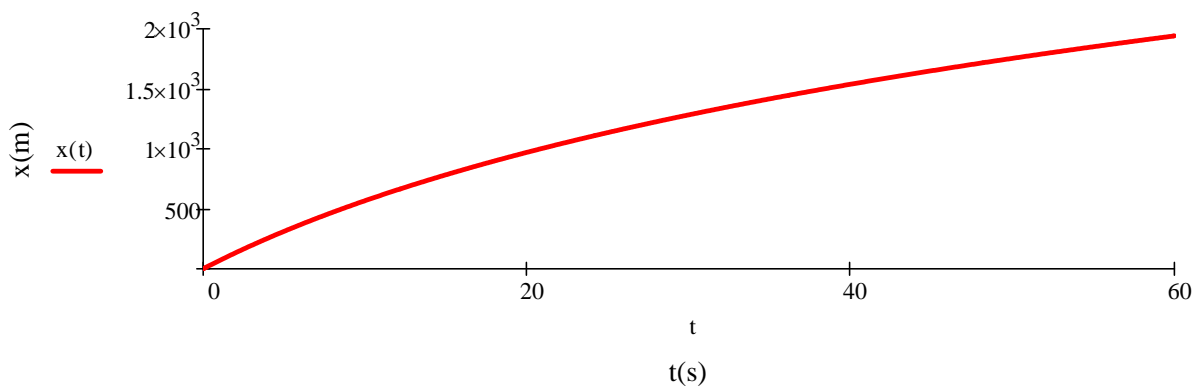
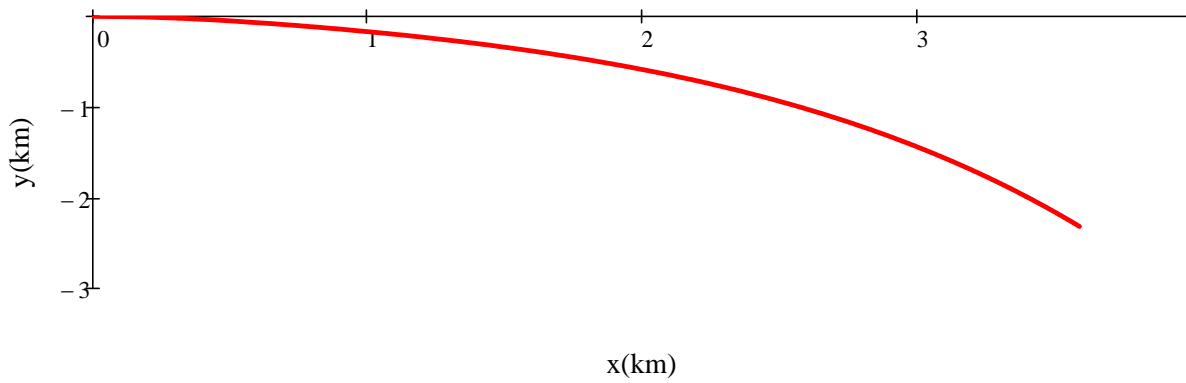
Rearranging or

$$U(t) = \frac{U_0}{1 + \frac{k_{horiz} \cdot U_0}{M} \cdot t}$$

For *x*(*t*) we need to integrate again:

$$\frac{dx}{dt} = U \qquad \text{or} \qquad x = \int U\, dt$$

$$x(t) = \int_0^t U(t)\, dt = \int_0^t \frac{U_0}{1 + \frac{k_{horiz} \cdot U_0}{M} \cdot t}\, dt = \frac{M}{k_{horiz}} \cdot \ln\left(\frac{k_{horiz} \cdot U_0}{M} \cdot t + 1\right)$$

$$x(t) = \frac{M}{k_{horiz}} \cdot \ln\left(\frac{k_{horiz} \cdot U_0}{M} \cdot t + 1\right)$$

Plotting the trajectory:

These plots can also be done in Excel.

Problem 1.14

[3]

1.14 In a pollution control experiment, minute solid particles (typical mass 5×10^{-11} kg) are dropped in the air. The terminal speed of the particles is measured to be 5 cm/s. The drag on these particles is given by $F_D = kV^2$, where V is the particle instantaneous speed. Find the value of constant k. Find the time required to reach 99 percent of terminal speed.

Given: Data on sphere and terminal speed.

Find: Drag constant *k*, and time to reach 99% of terminal speed.

Solution: Use given data; integrate equation of motion by separating variables.

The data provided are: $M = 5\cdot 10^{-11}\cdot\text{kg}$ $\quad V_t = 5\cdot\frac{\text{cm}}{\text{s}}$

Newton's 2nd law for the general motion is (ignoring buoyancy effects) $\quad M\cdot\frac{dV}{dt} = M\cdot g - k\cdot V \qquad (1)$

Newton's 2nd law for the steady state motion becomes (ignoring buoyancy effects) $\quad M\cdot g = k\cdot V_t \quad$ so $\quad k = \frac{M\cdot g}{V_t}$

$$k = \frac{M\cdot g}{V_t} = 5\times 10^{-11}\cdot\text{kg}\times 9.81\cdot\frac{\text{m}}{\text{s}^2}\times\frac{\text{s}}{0.05\cdot\text{m}} \qquad k = 9.81\times 10^{-9}\cdot\frac{\text{N}\cdot\text{s}}{\text{m}}$$

To find the time to reach 99% of V_t, we need $V(t)$. From 1, separating variables $\quad \frac{dV}{g - \frac{k}{M}\cdot V} = dt$

Integrating and using limits $\quad t = -\frac{M}{k}\cdot\ln\left(1 - \frac{k}{M\cdot g}\cdot V\right)$

We must evaluate this when $\quad V = 0.99\cdot V_t \quad V = 4.95\cdot\frac{\text{cm}}{\text{s}}$

$$t = 5\times 10^{-11}\cdot\text{kg}\times\frac{\text{m}}{9.81\times 10^{-9}\cdot\text{N}\cdot\text{s}}\times\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}\cdot\ln\left(1 - 9.81\cdot 10^{-9}\cdot\frac{\text{N}\cdot\text{s}}{\text{m}}\times\frac{1}{5\times 10^{-11}\cdot\text{kg}}\times\frac{\text{s}^2}{9.81\cdot\text{m}}\times\frac{0.0495\cdot\text{m}}{\text{s}}\times\frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right)$$

$t = 0.0235\,\text{s}$

Problem 1.15

[5]

1.15 For Problem 1.14, find the distance the particles travel before reaching 99 percent of terminal speed. Plot the distanced traveled as a function of time.

Given: Data on sphere and terminal speed from Problem 1.14.

Find: Distance traveled to reach 99% of terminal speed; plot of distance versus time.

Solution: Use given data; integrate equation of motion by separating variables.

The data provided are: $M = 5\cdot 10^{-11}\cdot kg$ $\quad V_t = 5\cdot\frac{cm}{s}$

Newton's 2nd law for the general motion is (ignoring buoyancy effects) $M\cdot\frac{dV}{dt} = M\cdot g - k\cdot V$ (1)

Newton's 2nd law for the steady state motion becomes (ignoring buoyancy effects) $M\cdot g = k\cdot V_t$ so $k = \frac{M\cdot g}{V_t}$

$$k = \frac{M\cdot g}{V_t} = 5\times 10^{-11}\cdot kg\times 9.81\cdot\frac{m}{s^2}\times\frac{s}{0.05\cdot m} \qquad k = 9.81\times 10^{-9}\cdot\frac{N\cdot s}{m}$$

To find the distance to reach 99% of V_t, we need $V(y)$. From 1: $M\cdot\frac{dV}{dt} = M\cdot\frac{dy}{dt}\cdot\frac{dV}{dy} = M\cdot V\cdot\frac{dV}{dy} = M\cdot g - k\cdot V$

Separating variables $\frac{V\cdot dV}{g-\frac{k}{M}\cdot V} = dy$

Integrating and using limits $y = -\frac{M^2\cdot g}{k^2}\cdot\ln\left(1-\frac{k}{M\cdot g}\cdot V\right) - \frac{M}{k}\cdot V$

We must evaluate this when $V = 0.99\cdot V_t$ $\quad V = 4.95\cdot\frac{cm}{s}$

$$y = \left(5\times 10^{-11}\cdot kg\right)^2\times\frac{9.81\cdot m}{s^2}\times\left(\frac{m}{9.81\times 10^{-9}\cdot N\cdot s}\right)^2\times\left(\frac{N\cdot s^2}{kg\cdot m}\right)^2\cdot\ln\left(1-9.81\cdot 10^{-9}\cdot\frac{N\cdot s}{m}\times\frac{1}{5\times 10^{-11}\cdot kg}\times\frac{s^2}{9.81\cdot m}\times\frac{0.0495\cdot m}{s}\times\frac{kg\cdot m}{N\cdot s^2}\right)\ldots$$
$$+ -5\times 10^{-11}\cdot kg\times\frac{m}{9.81\times 10^{-9}\cdot N\cdot s}\times\frac{0.0495\cdot m}{s}\times\frac{N\cdot s^2}{kg\cdot m}$$

$y = 0.922\cdot mm$

Alternatively we could use the approach of Problem 1.14 and first find the time to reach terminal speed, and use this time in y(t) to find the above value of y:

From 1, separating variables $\frac{dV}{g-\frac{k}{M}\cdot V} = dt$

Integrating and using limits $t = -\frac{M}{k}\cdot\ln\left(1-\frac{k}{M\cdot g}\cdot V\right)$ (2)

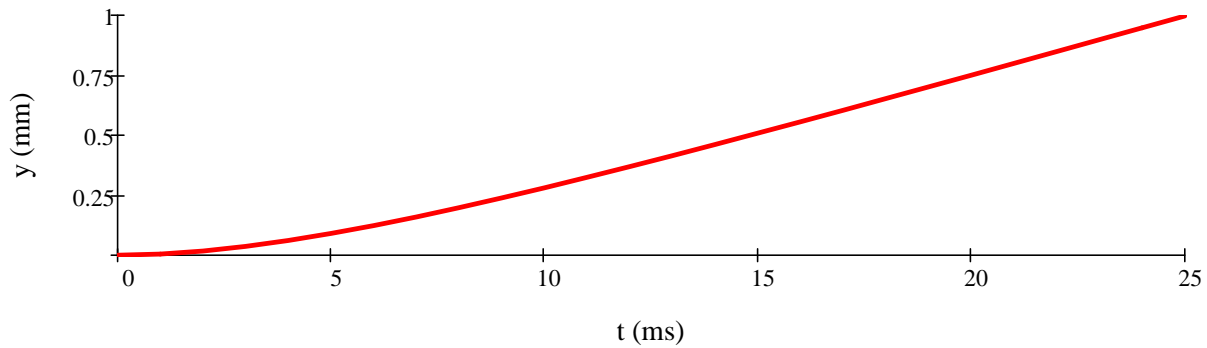
We must evaluate this when $V = 0.99 \cdot V_t$ $V = 4.95 \cdot \frac{cm}{s}$

$$t = 5\times 10^{-11} \cdot kg \times \frac{m}{9.81\times 10^{-9} \cdot N\cdot s} \times \frac{N\cdot s^2}{kg\cdot m} \cdot \ln\left(1 - 9.81\cdot 10^{-9} \cdot \frac{N\cdot s}{m} \times \frac{1}{5\times 10^{-11} \cdot kg} \times \frac{s^2}{9.81\cdot m} \times \frac{0.0495\cdot m}{s} \times \frac{kg\cdot m}{N\cdot s^2}\right) \qquad t = 0.0235\,s$$

From 2, after rearranging $V = \frac{dy}{dt} = \frac{M\cdot g}{k} \cdot \left(1 - e^{-\frac{k}{M}\cdot t}\right)$

Integrating and using limits $y = \frac{M\cdot g}{k} \cdot \left[t + \frac{M}{k} \cdot \left(e^{-\frac{k}{M}\cdot t} - 1\right)\right]$

$$y = 5\times 10^{-11} \cdot kg \times \frac{9.81\cdot m}{s^2} \times \frac{m}{9.81\times 10^{-9} \cdot N\cdot s} \times \frac{N\cdot s^2}{kg\cdot m} \cdot \left[0.0235\cdot s \ldots + 5\times 10^{-11} \cdot kg \times \frac{m}{9.81\times 10^{-9} \cdot N\cdot s} \times \frac{N\cdot s^2}{kg\cdot m} \cdot \left(e^{-\frac{9.81\cdot 10^{-9}}{5\cdot 10^{-11}} \cdot .0235} - 1\right)\right]$$

$y = 0.922 \cdot mm$

This plot can also be presented in Excel.

Problem 1.16

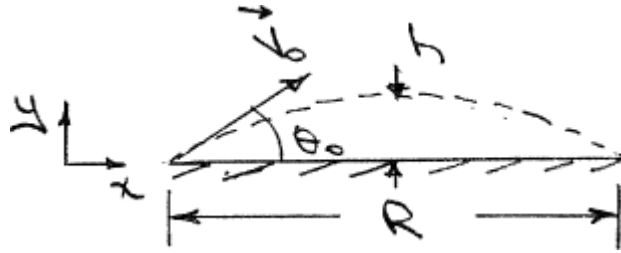
1.16 The English perfected the longbow as a weapon after the Medieval period. In the hands of a skilled archer, the longbow was reputed to be accurate at ranges to 100 meters or more. If the maximum altitude of an arrow is less than $h = 10$ m while traveling to a target 100 m away from the archer, and neglecting air resistance, estimate the speed and angle at which the arrow must leave the bow. Plot the required release speed and angle as a function of height *h*.

Given: Long bow at range, R = 100 m. Maximum height of arrow is h = 10 m. Neglect air resistance.

Find: Estimate of (a) speed, and (b) angle, of arrow leaving the bow.

Plot: (a) release speed, and (b) angle, as a function of h

Solution: Let $\overrightarrow{V_0} = u_0\hat{i} + v_0\hat{j} = V_0(\cos\theta_0\hat{i} + \sin\theta_0\hat{j})$

$\Sigma F_y = m\frac{dv}{dt} = -mg$, so $v = v_0 - gt$, and $t_f = 2t_{v=0} = 2v_0/g$

Also, $$mv\frac{dv}{dy} = -mg,\ v\,dv = -g\,dy,\ 0 - \frac{v_0^2}{2} = -gh$$

Thus $$h = v_0^2/2g \quad (1)$$

$$\Sigma F_x = m\frac{du}{dt} = 0,\ \text{so } u = u_0 = \text{const, and } R = u_0 t_f = \frac{2u_0v_0}{g} \quad (2)$$

From

1. $v_0^2 = 2gh$ (3)

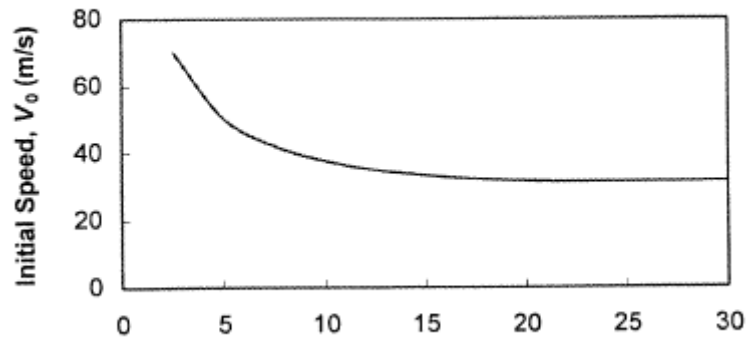
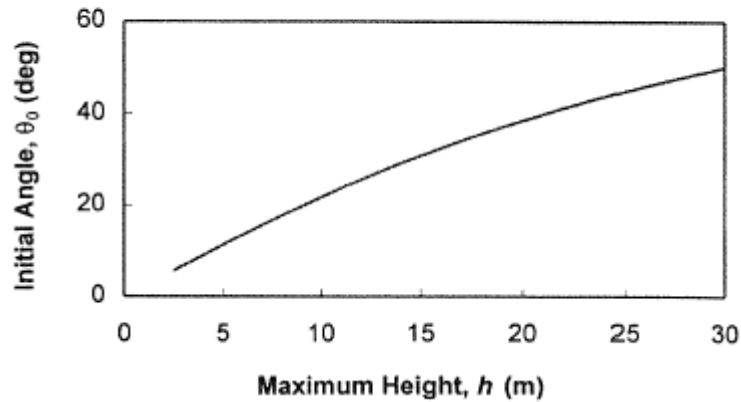
2. $u_0 = \frac{gR}{2v_0} = \frac{gR}{2\sqrt{2gh}} \quad \therefore\ u_0^2 = \frac{gR^2}{8h}$

Then $$V_0^2 = u_0^2 + v_0^2 = \frac{gR^2}{8h} + 2gh \quad \text{and} \quad V_0 = \left[2gh + \frac{gR^2}{8h}\right]^{\frac{1}{2}} \tag{4}$$

$$V_0 = \left[2 \times 9.81 \ \frac{m}{s^2} \times 10 \ m + \frac{9.81}{8} \frac{m}{s^2} \times (100)^2 \ m^2 \times \frac{1}{10 \ m}\right]^{\frac{1}{2}} = 37.7 \ m/s$$

From Eq. 3 $$v_0 = \sqrt{2gh} = V_0 \sin\theta, \theta = \sin^{-1}\frac{\sqrt{2gh}}{V_0} \tag{5}$$

$$\theta = \sin^{-1}\left[\left(2 \times 9.81 \frac{m}{s} \times 10 \ m\right)^{\frac{1}{2}} \frac{s}{37.7 \ m}\right] = 21.8°$$

Plots of $V_0 = V_0(h)$ {Eq. 4} and $\theta_0 = \theta_0(h)$ {Eq. 5} are presented below

Problem 1.17

[2]

1.17 For each quantity listed, indicate dimensions using force as a primary dimension, and give typical SI and English units:

a. Power
b. Pressure
c. Modulus of elasticity
d. Angular velocity
e. Energy
f. Momentum
g. Shear stress
h. Specific heat
i. Thermal expansion coefficient
j. Angular momentum

Given: Basic dimensions F, L, t and T.

Find: Dimensional representation of quantities below, and typical units in SI and English systems.

Solution:

(a) Power $\qquad \text{Power} = \frac{\text{Energy}}{\text{Time}} = \frac{\text{Force} \times \text{Distance}}{\text{Time}} = \frac{F \cdot L}{t} \qquad \frac{N \cdot m}{s} \qquad \frac{lbf \cdot ft}{s}$

(b) Pressure $\qquad \text{Pressure} = \frac{\text{Force}}{\text{Area}} = \frac{F}{L^2} \qquad \frac{N}{m^2} \qquad \frac{lbf}{ft^2}$

(c) Modulus of elasticity $\qquad \text{Pressure} = \frac{\text{Force}}{\text{Area}} = \frac{F}{L^2} \qquad \frac{N}{m^2} \qquad \frac{lbf}{ft^2}$

(d) Angular velocity $\qquad \text{AngularVelocity} = \frac{\text{Radians}}{\text{Time}} = \frac{1}{t} \qquad \frac{1}{s} \qquad \frac{1}{s}$

(e) Energy $\qquad \text{Energy} = \text{Force} \times \text{Distance} = F \cdot L \qquad N \cdot m \qquad lbf \cdot ft$

(f) Momentum $\qquad \text{Momentum} = \text{Mass} \times \text{Velocity} = M \cdot \frac{L}{t}$

From Newton's 2nd law $\quad \text{Force} = \text{Mass} \times \text{Acceleration}$ so $\quad F = M \cdot \frac{L}{t^2}$ or $\quad M = \frac{F \cdot t^2}{L}$

Hence $\quad \text{Momentum} = M \cdot \frac{L}{t} = \frac{F \cdot t^2 \cdot L}{L \cdot t} = F \cdot t \qquad N \cdot s \qquad lbf \cdot s$

(g) Shear stress $\qquad \text{ShearStress} = \frac{\text{Force}}{\text{Area}} = \frac{F}{L^2} \qquad \frac{N}{m^2} \qquad \frac{lbf}{ft^2}$

(h) Specific heat $\qquad \text{SpecificHeat} = \frac{\text{Energy}}{\text{Mass} \times \text{Temperature}} = \frac{F \cdot L}{M \cdot T} = \frac{F \cdot L}{\left(\frac{F \cdot t^2}{L}\right) \cdot T} = \frac{L^2}{t^2 \cdot T} \qquad \frac{m^2}{s^2 \cdot K} \qquad \frac{ft^2}{s^2 \cdot R}$

(i) Thermal expansion coefficient $\qquad \text{ThermalExpansionCoefficient} = \frac{\frac{\text{LengthChange}}{\text{Length}}}{\text{Temperature}} = \frac{1}{T} \qquad \frac{1}{K} \qquad \frac{1}{R}$

(j) Angular momentum $\qquad \text{AngularMomentum} = \text{Momentum} \times \text{Distance} = F \cdot t \cdot L \qquad N \cdot m \cdot s \qquad lbf \cdot ft \cdot s$

Problem 1.18

1.18 For each quantity listed, indicate dimensions using mass as a primary dimension, and give typical SI and English units:

a. Power
b. Pressure
c. Modulus of elasticity
d. Angular velocity
e. Energy
f. Moment of a force
g. Momentum
h. Shear stress
i. Strain
j. Angular momentum

Given: Basic dimensions M, L, t and T.

Find: Dimensional representation of quantities below, and typical units in SI and English systems.

Solution:

(a) Power

$$\text{Power} = \frac{\text{Energy}}{\text{Time}} = \frac{\text{Force} \times \text{Distance}}{\text{Time}} = \frac{F \cdot L}{t}$$

From Newton's 2nd law $\quad \text{Force} = \text{Mass} \times \text{Acceleration} \quad$ so $\quad F = \frac{M \cdot L}{t^2}$

Hence $\quad \text{Power} = \frac{F \cdot L}{t} = \frac{M \cdot L \cdot L}{t^2 \cdot t} = \frac{M \cdot L^2}{t^3} \qquad \frac{kg \cdot m^2}{s^3} \qquad \frac{slug\,ft^2}{s^3}$

(b) Pressure

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}} = \frac{F}{L^2} = \frac{M \cdot L}{t^2 \cdot L^2} = \frac{M}{L \cdot t^2} \qquad \frac{kg}{m \cdot s^2} \qquad \frac{slug}{ft \cdot s^2}$$

(c) Modulus of elasticity

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}} = \frac{F}{L^2} = \frac{M \cdot L}{t^2 \cdot L^2} = \frac{M}{L \cdot t^2} \qquad \frac{kg}{m \cdot s^2} \qquad \frac{slug}{ft \cdot s^2}$$

(d) Angular velocity

$$\text{AngularVelocity} = \frac{\text{Radians}}{\text{Time}} = \frac{1}{t} \qquad \frac{1}{s} \qquad \frac{1}{s}$$

(e) Energy

$$\text{Energy} = \text{Force} \times \text{Distance} = F \cdot L = \frac{M \cdot L \cdot L}{t^2} = \frac{M \cdot L^2}{t^2} \qquad \frac{kg \cdot m^2}{s^2} \qquad \frac{slug \cdot ft^2}{s^2}$$

(f) Moment of a force

$$\text{MomentOfForce} = \text{Force} \times \text{Length} = F \cdot L = \frac{M \cdot L \cdot L}{t^2} = \frac{M \cdot L^2}{t^2} \qquad \frac{kg \cdot m^2}{s^2} \qquad \frac{slug \cdot ft^2}{s^2}$$

(g) Momentum

$$\text{Momentum} = \text{Mass} \times \text{Velocity} = M \cdot \frac{L}{t} = \frac{M \cdot L}{t} \qquad \frac{kg \cdot m}{s} \qquad \frac{slug \cdot ft}{s}$$

(h) Shear stress

$$\text{ShearStress} = \frac{\text{Force}}{\text{Area}} = \frac{F}{L^2} = \frac{M \cdot L}{t^2 \cdot L^2} = \frac{M}{L \cdot t^2} \qquad \frac{kg}{m \cdot s^2} \qquad \frac{slug}{ft \cdot s^2}$$

(i) Strain

$$\text{Strain} = \frac{\text{LengthChange}}{\text{Length}} = \frac{L}{L}$$ Dimensionless

(j) Angular momentum

$$\text{AngularMomentum} = \text{Momentum} \times \text{Distance} = \frac{M \cdot L}{t} \cdot L = \frac{M \cdot L^2}{t} \qquad \frac{kg \cdot m^2}{s} \qquad \frac{slugs \cdot ft^2}{s}$$

Problem 1.19

1.19 Derive the following conversion factors:

a. Convert a pressure of 1 psi to kPa.

b. Convert a volume of 1 liter to gallons.

c. Convert a viscosity of 1 $\text{lbf} \cdot \text{s/ft}^2$ to $\text{N} \cdot \text{s/m}^2$.

Given: Pressure, volume and density data in certain units

Find: Convert to different units

Solution:

Using data from tables (e.g. Table G.2)

(a)
$$1 \cdot \text{psi} = 1 \cdot \text{psi} \times \frac{6895 \cdot \text{Pa}}{1 \cdot \text{psi}} \times \frac{1 \cdot \text{kPa}}{1000 \cdot \text{Pa}} = 6.89 \cdot \text{kPa}$$

(b)
$$1 \cdot \text{liter} = 1 \cdot \text{liter} \times \frac{1 \cdot \text{quart}}{0.946 \cdot \text{liter}} \times \frac{1 \cdot \text{gal}}{4 \cdot \text{quart}} = 0.264 \cdot \text{gal}$$

(c)
$$1 \cdot \frac{\text{lbf} \cdot \text{s}}{\text{ft}^2} = 1 \cdot \frac{\text{lbf} \cdot \text{s}}{\text{ft}^2} \times \frac{4.448 \cdot \text{N}}{1 \cdot \text{lbf}} \times \left(\frac{\frac{1}{12} \cdot \text{ft}}{0.0254 \cdot \text{m}} \right)^2 = 47.9 \cdot \frac{\text{N} \cdot \text{s}}{\text{m}^2}$$

Problem 1.20

[1]

1.20 Derive the following conversion factors:

a. Convert a viscosity of 1 m^2/s to ft^2/s.

b. Convert a power of 100 W to horsepower.

c. Convert a specific energy of 1 kJ/kg to Btu/lbm.

Given:
Viscosity, power, and specific energy data in certain units

Find:
Convert to different units

Solution:

Using data from tables (e.g. Table G.2)

(a) $$1\cdot\frac{m^2}{s} = 1\cdot\frac{m^2}{s}\times\left(\frac{\frac{1}{12}\cdot ft}{0.0254\cdot m}\right)^2 = 10.76\cdot\frac{ft^2}{s}$$

(b) $$100\cdot W = 100\cdot W\times\frac{1\cdot hp}{746\cdot W} = 0.134\cdot hp$$

(c) $$1\cdot\frac{kJ}{kg} = 1\cdot\frac{kJ}{kg}\times\frac{1000\cdot J}{1\cdot kJ}\times\frac{1\cdot Btu}{1055\cdot J}\times\frac{0.454\cdot kg}{1\cdot lbm} = 0.43\cdot\frac{Btu}{lbm}$$

Problem 1.21

[1]

1.21 Express the following in SI units:

a. 100 cfm (ft^3/min)
b. 5 gal
c. 65 mph
d. 5.4 acres

Given: Quantities in English Engineering (or customary) units.

Find: Quantities in SI units.

Solution: Use Table G.2 and other sources (e.g., Google)

(a)
$$100 \cdot \frac{ft^3}{m} = 100 \cdot \frac{ft^3}{min} \times \left(\frac{0.0254 \cdot m}{1 \cdot in} \times \frac{12 \cdot in}{1 \cdot ft}\right)^3 \times \frac{1 \cdot min}{60 \cdot s} = 0.0472 \cdot \frac{m^3}{s}$$

(b)
$$5 \cdot gal = 5 \cdot gal \times \frac{231 \cdot in^3}{1 \cdot gal} \times \left(\frac{0.0254 \cdot m}{1 \cdot in}\right)^3 = 0.0189 \cdot m^3$$

(c)
$$65 \cdot mph = 65 \cdot \frac{mile}{hr} \times \frac{1852 \cdot m}{1 \cdot mile} \times \frac{1 \cdot hr}{3600 \cdot s} = 29.1 \cdot \frac{m}{s}$$

(d)
$$5.4 \cdot acres = 5.4 \cdot acre \times \frac{4047 \cdot m^3}{1 \cdot acre} = 2.19 \times 10^4 \cdot m^2$$

Problem 1.22

[1]

1.22 Express the following in BG units:

a. 50 m^2
b. 250 cc
c. 100 kW
d. 5 lbf • s/ft^2

Given: Quantities in SI (or other) units.

Find: Quantities in BG units.

Solution: Use Table G.2.

(a)
$$50\cdot m^2 = 50\cdot m^2 \times \left(\frac{1\cdot in}{0.0254\cdot m}\times\frac{1\cdot ft}{12\cdot in}\right)^2 = 538\cdot ft^2$$

(b)
$$250\cdot cc = 250\cdot cm^3 \times \left(\frac{1\cdot m}{100\cdot cm}\times\frac{1\cdot in}{0.0254\cdot m}\times\frac{1\cdot ft}{12\cdot in}\right)^3 = 8.83\times 10^{-3}\cdot ft^3$$

(c)
$$100\cdot kW = 100\cdot kW\times\frac{1000\cdot W}{1\cdot kW}\times\frac{1\cdot hp}{746\cdot W} = 134\cdot hp$$

(d)
$$5\cdot\frac{lbf\cdot s}{ft^2}$$ is already in BG units

Problem 1.23

[1]

1.23 A farmer needs $1\frac{1}{2}$ in. of rain per week on his farm, with 25 acres of crops. If there is a drought, how much water (gpm) will have to be pumped in to maintain his crops?

Given: Acreage of land, and water needs.

Find: Water flow rate (gpm) to water crops.

Solution: Use Table G.2 and other sources (e.g., Google) as needed.

The volume flow rate needed is $Q = \frac{1.5 \cdot in}{week} \times 25 \cdot acres$

Performing unit conversions $Q = \frac{1.5 \cdot in \times 25 \cdot acre}{week} = \frac{1.5 \cdot in \times 25 \cdot acre}{week} \times \frac{4.36 \times 10^4 \cdot ft^2}{1 \cdot acre} \times \left(\frac{12 \cdot in}{1 \cdot ft}\right)^2 \times \frac{1 \cdot week}{7 \cdot day} \times \frac{1 \cdot day}{24 \cdot hr} \times \frac{1 \cdot hr}{60 \cdot min}$

$Q = 101 \cdot gpm$

Problem 1.24

[2]

1.24 While you're waiting for the ribs to cook, you muse about the propane tank of your barbecue. You're curious about the volume of propane versus the actual tank size. Find the liquid propane volume when full (the weight of the propane is specified on the tank). Compare this to the tank volume (take some measurements, and approximate the tank shape as a cylinder with a hemisphere on each end). Explain the discrepancy.

Given: Geometry of tank, and weight of propane.

Find: Volume of propane, and tank volume; explain the discrepancy.

Solution: Use Table G.2 and other sources (e.g., Google) as needed.

The author's tank is approximately 12 in in diameter, and the cylindrical part is about 8 in. The weight of propane specified is 17 lb.

The tank diameter is $D = 12 \cdot in$

The tank cylindrical height is $L = 8 \cdot in$

The mass of propane is $m_{prop} = 17 \cdot lbm$

The specific gravity of propane is $SG_{prop} = 0.495$

The density of water is $\rho = 998 \cdot \frac{kg}{m^3}$

The volume of propane is given by

$$V_{prop} = \frac{m_{prop}}{\rho_{prop}} = \frac{m_{prop}}{SG_{prop} \cdot \rho}$$

$$V_{prop} = 17 \cdot lbm \times \frac{1}{0.495} \times \frac{m^3}{998 \cdot kg} \times \frac{0.454 \cdot kg}{1 \cdot lbm} \times \left(\frac{1 \cdot in}{0.0254 \cdot m}\right)^3$$

$$V_{prop} = 953 \cdot in^3$$

The volume of the tank is given by a cylinder diameter D length L, $\pi D^2 L/4$ and a sphere (two halves) given by $\pi D^3/6$

$$V_{tank} = \frac{\pi \cdot D^2}{4} \cdot L + \frac{\pi \cdot D^3}{6}$$

$$V_{tank} = \frac{\pi \cdot (12 \cdot in)^2}{4} \cdot 8 \cdot in + \pi \cdot \frac{(12 \cdot in)^3}{6}$$

$$V_{tank} = 1810 \cdot in^3$$

The ratio of propane to tank volumes is $\frac{V_{prop}}{V_{tank}} = 53 \cdot \%$

This seems low, and can be explained by a) tanks are not filled completely, b) the geometry of the tank gave an overestimate of the volume (th ends are not really hemispheres, and we have not allowed for tank wall thickness).

Problem 1.25

[1]

1.25 The density of mercury is given as 26.3 $slug/ft^3$. Calculate the specific gravity and the specific volume in m^3/kg of the mercury. Calculate the specific weight in lbf/ft^3 on Earth and on the moon. Acceleration of gravity on the moon is 5.47 ft/s^2.

Given: Density of mercury is $\rho = 26.3\ slug/ft^3$.

Acceleration of gravity on moon is $g_m = 5.47\ ft/s^2$.

Find:

a. Specific gravity of mercury.

b. Specific volume of mercury, in m^3/kg.

c. Specific weight on Earth.

d. Specific weight on moon.

Solution: Apply definitions: $\gamma \equiv \rho g,\ v \equiv 1/\rho,\ SG \equiv \rho/\rho_{H_2O}$

Thus

$$SG = 26.3\frac{slug}{ft^3} \times \frac{ft^3}{1.94\ slug} = 13.6$$

$$v = \frac{ft^3}{26.3\ slug} \times (0.3048)^3 \frac{m^3}{ft^3} \times \frac{slug}{32.2\ lbm} \times \frac{lbm}{0.4536\ kg} = 7.37 \times 10^{-5}\ m^3/kg$$

On Earth,

$$\gamma_E = 26.3\frac{slug}{ft^3} \times 32.2\frac{ft}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} = 847\ lbf/ft^3$$

On the moon,

$$\gamma_m = 26.3\frac{slug}{ft^3} \times 5.47\frac{ft}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} = 144\ lbf/ft^3$$

{Note that the mass based quantities (SG and ν) are independent of gravity.}

Problem 1.26

[1]

1.26 Derive the following conversion factors:

a. Convert a volume flow rate in $\text{in.}^3/\text{min}$ to mm^3/s.

b. Convert a volume flow rate in cubic meters per second to gpm (gallons per minute).

c. Convert a volume flow rate in liters per minute to gpm (gallons per minute).

d. Convert a volume flow rate of air in standard cubic feet per minute (SCFM) to cubic meters per hour. A standard cubic foot of gas occupies one cubic foot at standard temperature and pressure ($T = 15°\text{C}$ and $p = 101.3$ kPa absolute).

Given: Data in given units

Find: Convert to different units

Solution:

(a)
$$1\cdot\frac{\text{in}^3}{\text{min}} = 1\cdot\frac{\text{in}^3}{\text{min}} \times \left(\frac{0.0254\cdot\text{m}}{1\cdot\text{in}} \times \frac{1000\cdot\text{mm}}{1\cdot\text{m}}\right)^3 \times \frac{1\cdot\text{min}}{60\cdot\text{s}} = 273\cdot\frac{\text{mm}^3}{\text{s}}$$

(b)
$$1\cdot\frac{\text{m}^3}{\text{s}} = 1\cdot\frac{\text{m}^3}{\text{s}} \times \frac{1\cdot\text{gal}}{4\times 0.000946\cdot\text{m}^3} \times \frac{60\cdot\text{s}}{1\cdot\text{min}} = 15850\cdot\text{gpm}$$

(c)
$$1\cdot\frac{\text{liter}}{\text{min}} = 1\cdot\frac{\text{liter}}{\text{min}} \times \frac{1\cdot\text{gal}}{4\times 0.946\cdot\text{liter}} \times \frac{60\cdot\text{s}}{1\cdot\text{min}} = 0.264\cdot\text{gpm}$$

(d)
$$1\cdot\text{SCFM} = 1\cdot\frac{\text{ft}^3}{\text{min}} \times \left(\frac{0.0254\cdot\text{m}}{\frac{1}{12}\cdot\text{ft}}\right)^3 \times \frac{60\cdot\text{min}}{1\cdot\text{hr}} = 1.70\cdot\frac{\text{m}^3}{\text{hr}}$$

Problem 1.27

[1]

1.27 The kilogram force is commonly used in Europe as a unit of force. (As in the U.S. customary system, where 1 lbf is the force exerted by a mass of 1 lbm in standard gravity, 1 kgf is the force exerted by a mass of 1 kg in standard gravity.) Moderate pressures, such as those for auto or truck tires, are conveniently expressed in units of kgf/cm^2. Convert 32 psig to these units.

Given: In European usage, 1 kgf is the force exerted on 1 kg mass in standard gravity.

Find: Convert 32 psi to units of kgf/cm^2.

Solution: Apply Newton's second law.

Basic equation: F = ma

The force exerted on 1 kg in standard gravity is

$$F = 1\ \text{kg} \times 9.81 \frac{\text{m}}{\text{s}^2} \times \frac{\text{N} \cdot \text{s}^2}{\text{kg} \cdot \text{m}} = 9.81\ \text{N} = 1\ \text{kgf}$$

Setting up a conversion from psi to kgf/cm^2,

$$1 \frac{\text{lbf}}{\text{in.}^2} = 1 \frac{\text{lbf}}{\text{in.}^2} \times 4.448 \frac{\text{N}}{\text{lbf}} \times \frac{\text{in.}^2}{(2.54)^2\ \text{cm}^2} \times \frac{\text{kgf}}{9.81\ \text{N}} = 0.0703 \frac{\text{kgf}}{\text{cm}^2}$$

or

$$1 \equiv \frac{0.0703\ \text{kgf}/\text{cm}^2}{\text{psi}}$$

Thus

$$32\ \text{psi} = 32\ \text{psi} \times \frac{0.0703\ \text{kgf}/\text{cm}^2}{\text{psi}}$$

$$32\ \text{psi} = 2.25\ \text{kgf}/\text{cm}^2$$

Problem 1.28

1.28 In Section 1-6 we learned that the Manning equation computes the flow speed V (m/s) in a canal made from unfinished concrete, given the hydraulic radius R_h (m), the channel slope S_0, and a Manning resistance coefficient constant value $n \approx 0.014$. For a canal with $R_h = 7.5$ m and a slope of 1/10, find the flow speed. Compare this result with that obtained using the same n value, but with R_h first converted to ft, with the answer assumed to be in ft/s. Finally, find the value of n if we wish to *correctly* use the equation for BG units (and compute V to check!)

Given: Information on canal geometry.

Find: Flow speed using the Manning equation, correctly and incorrectly!

Solution: Use Table G.2 and other sources (e.g., Google) as needed.

The Manning equation is $V = \dfrac{R_h^{\frac{2}{3}} \cdot S_0^{\frac{1}{2}}}{n}$ which assumes R_h in meters and V in m/s.

The given data is $R_h = 7.5 \cdot m$ $\quad S_0 = \dfrac{1}{10}$ $\quad n = 0.014$

Hence $V = \dfrac{7.5^{\frac{2}{3}} \cdot \left(\frac{1}{10}\right)^{\frac{1}{2}}}{0.014}$ $\quad V = 86.5 \cdot \dfrac{m}{s}$ (Note that we don't cancel units; we just write m/s next to the answer! Note also this is a very high speed due to the extreme slope S_0.)

Using the equation incorrectly: $R_h = 7.5 \cdot m \times \dfrac{1 \cdot in}{0.0254 \cdot m} \times \dfrac{1 \cdot ft}{12 \cdot in}$ $\quad R_h = 24.6 \cdot ft$

Hence $V = \dfrac{24.6^{\frac{2}{3}} \cdot \left(\frac{1}{10}\right)^{\frac{1}{2}}}{0.014}$ $\quad V = 191 \cdot \dfrac{ft}{s}$ (Note that we again don't cancel units; we just write ft/s next to the answer!)

This incorrect use does not provide the correct answer $V = 191 \cdot \dfrac{ft}{s} \times \dfrac{12 \cdot in}{1 \cdot ft} \times \dfrac{0.0254 \cdot m}{1 \cdot in}$ $\quad V = 58.2 \dfrac{m}{s}$ which is wrong!

This demonstrates that for this "engineering" equation we must be careful in its use!

To generate a Manning equation valid for R_h in ft and V in ft/s, we need to do the following:

$$V\left(\frac{ft}{s}\right) = V\left(\frac{m}{s}\right) \times \frac{1 \cdot in}{0.0254 \cdot m} \times \frac{1 \cdot ft}{12 \cdot in} = \frac{R_h(m)^{\frac{2}{3}} \cdot S_0^{\frac{1}{2}}}{n} \times \left(\frac{1 \cdot in}{0.0254 \cdot m} \times \frac{1 \cdot ft}{12 \cdot in}\right)$$

$$V\left(\frac{ft}{s}\right) = \frac{R_h(ft)^{\frac{2}{3}} \cdot S_0^{\frac{1}{2}}}{n} \times \left(\frac{1 \cdot in}{0.0254 \cdot m} \times \frac{1 \cdot ft}{12 \cdot in}\right)^{-\frac{2}{3}} \times \left(\frac{1 \cdot in}{0.0254 \cdot m} \times \frac{1 \cdot ft}{12 \cdot in}\right) = \frac{R_h(ft)^{\frac{2}{3}} \cdot S_0^{\frac{1}{2}}}{n} \times \left(\frac{1 \cdot in}{0.0254 \cdot m} \times \frac{1 \cdot ft}{12 \cdot in}\right)^{\frac{1}{3}}$$

In using this equation, we ignore the units and just evaluate the conversion factor $\left(\frac{1}{.0254} \cdot \frac{1}{12}\right)^{\frac{1}{3}} = 1.49$

Hence $\quad V\left(\frac{ft}{s}\right) = \frac{1.49 \cdot R_h(ft)^{\frac{2}{3}} \cdot S_0^{\frac{1}{2}}}{n}$

Handbooks sometimes provide this form of the Manning equation for direct use with BG units. In our case we are asked to instead define a new value for n:

$$n_{BG} = \frac{n}{1.49} \qquad n_{BG} = 0.0094 \qquad \text{where} \qquad V\left(\frac{ft}{s}\right) = \frac{R_h(ft)^{\frac{2}{3}} \cdot S_0^{\frac{1}{2}}}{n_{BG}}$$

Using this equation with Rh = 24.6 ft: $\quad V = \frac{24.6^{\frac{2}{3}} \cdot \left(\frac{1}{10}\right)^{\frac{1}{2}}}{0.0094} \qquad V = 284\frac{ft}{s}$

Converting to m/s $\quad V = 284 \cdot \frac{ft}{s} \times \frac{12 \cdot in}{1 \cdot ft} \times \frac{0.0254 \cdot m}{1 \cdot in} \qquad V = 86.6\frac{m}{s}$ $\qquad$ which is the correct answer!

Problem 1.29

1.29 The maximum theoretical flow rate (kg/s) through a supersonic nozzle is

$$\dot{m}_{max} = 0.04\frac{A_t p_0}{\sqrt{T_0}}$$

where A_t (m^2) is the nozzle throat area, p_0 (Pa) is the tank pressure, and T_0 (K) is the tank temperature. Is this equation dimensionally correct? If not, find the units of the 0.04 term. Write the equivalent equation in BG units.

Given: Equation for maximum flow rate.

Find: Whether it is dimensionally correct. If not, find units of 0.04 term. Write a BG version of the equation

Solution: Rearrange equation to check units of 0.04 term. Then use conversions from Table G.2 or other sources (e.g., Google)

"Solving" the equation for the constant 0.04: $0.04 = \frac{m_{max}\cdot\sqrt{T_0}}{A_t\cdot p_0}$

Substituting the units of the terms on the right, the units of the constant are

$$\frac{kg}{s} \times K^{\frac{1}{2}} \times \frac{1}{m^2} \times \frac{1}{Pa} = \frac{kg}{s} \times K^{\frac{1}{2}} \times \frac{1}{m^2} \times \frac{m^2}{N} \times \frac{N\cdot s^2}{kg\cdot m} = \frac{K^{\frac{1}{2}}\cdot s}{m}$$

Hence the constant is actually $c = 0.04\cdot\frac{K^{\frac{1}{2}}\cdot s}{m}$

For BG units we could start with the equation and convert each term (e.g., A_t), and combine the result into a new constant, or simply convert c directly:

$$c = 0.04\cdot\frac{K^{\frac{1}{2}}\cdot s}{m} = 0.04 \times \left(\frac{1.8\cdot R}{K}\right)^{\frac{1}{2}} \times \frac{0.0254\cdot m}{1\cdot in} \times \frac{12\cdot in}{1\cdot ft}$$

$$c = 0.0164\cdot\frac{R^{\frac{1}{2}}\cdot s}{ft} \qquad \text{so} \qquad m_{max} = 0.0164\cdot\frac{A_t\cdot p_0}{\sqrt{T_0}}$$

with A_t in ft^2, p_0 in lbf/ft^2, and T_0 in R.

This value of c assumes p is in lbf/ft^2. For p in psi we need an additional conversion:

$$c = 0.0164\cdot\frac{R^{\frac{1}{2}}\cdot s}{ft} \times \left(\frac{12\cdot in}{1\cdot ft}\right)^2 \qquad c = 2.36\cdot\frac{R^{\frac{1}{2}}\cdot in^2\cdot s}{ft^3} \qquad \text{so} \qquad m_{max} = 2.36\cdot\frac{A_t\cdot p_0}{\sqrt{T_0}}$$

with A_t in ft^2, p_0 in psi, and T_0 in R.

Problem 1.30

1.30 From thermodynamics, we know that the coefficient of performance of an ideal air conditioner is given by

$$COP_{Ideal} = \frac{T_L}{T_H - T_L}$$

where T_L and T_H are the room and outside temperatures (absolute). If an AC is to keep a room at 68°F when it is 95°F outside, find the COP_{Ideal}. Convert to an EER value, and compare this to a typical Energy Star compliant EER value.

Given: Equation for COP and temperature data.

Find: COP_{Ideal}, EER, and compare to a typical Energy Star compliant EER value.

Solution: Use the COP equation. Then use conversions from Table G.2 or other sources (e.g., Google) to find the EER.

The given data is $T_L = (68 + 460) \cdot R$ $\quad T_L = 528 \cdot R$ $\quad T_H = (95 + 460) \cdot R$ $\quad T_H = 555 \cdot R$

The COP_{Ideal} is

$$COP_{Ideal} = \frac{T_L}{T_H - T_L} = \frac{525}{555 - 528} = 19.4$$

The EER is a similar measure to COP except the cooling rate (numerator) is in BTU/hr and the electrical input (denominator) is in W:

$$EER_{Ideal} = COP_{Ideal} \times \frac{\frac{BTU}{hr}}{W} = 19.4 \times \frac{2545 \cdot \frac{BTU}{hr}}{746 \cdot W} = 66.2 \cdot \frac{\frac{BTU}{hr}}{W}$$

This compares to Energy Star compliant values of about 15 BTU/hr/W! We have some way to go! We can define the isentropic efficiency as

$$\eta_{isen} = \frac{EER_{Actual}}{EER_{Ideal}}$$

Hence the isentropic efficiency of a very good AC is about 22.5%.

Problem 1.31

[1]

1.31 In Chapter 9 we will study aerodynamics and learn that the drag force F_D on a body is given by

$$F_D = \frac{1}{2}\rho V^2 A C_D$$

Hence the drag depends on speed V, fluid density ρ, and body size (indicated by frontal area A) and shape (indicated by drag coefficient C_D). What are the dimensions of C_D?

Given: Equation for drag on a body.

Find: Dimensions of C_D.

Solution: Use the drag equation. Then "solve" for CD and use dimensions.

The drag equation is $$F_D = \frac{1}{2}\cdot\rho\cdot V^2\cdot A\cdot C_D$$

"Solving" for C_D, and using dimensions $$C_D = \frac{2\cdot F_D}{\rho\cdot V^2\cdot A}$$

$$C_D = \frac{F}{\frac{M}{L^3}\times\left(\frac{L}{t}\right)^2\times L^2}$$

But, From Newton's 2nd law $\quad \text{Force} = \text{Mass}\cdot\text{Acceleration} \quad$ or $\quad F = M\cdot\frac{L}{t^2}$

Hence $$C_D = \frac{F}{\frac{M}{L^3}\times\left(\frac{L}{t}\right)^2\times L^2} = \frac{M\cdot L}{t^2}\times\frac{L^3}{M}\times\frac{t^2}{L^2}\times\frac{1}{L^2} = 0$$

The drag coefficient is dimensionless.

Problem 1.32

1.32 The mean free path λ of a molecule of gas is the average distance it travels before collision with another molecule. It is given by

$$\lambda = C\frac{m}{\rho d^2}$$

where m and d are the molecule's mass and diameter, respectively, and ρ is the gas density. What are the dimensions of constant C for a dimensionally consistent equation?

Given: Equation for mean free path of a molecule.

Find: Dimensions of C for a diemsionally consistent equation.

Solution: Use the mean free path equation. Then "solve" for C and use dimensions.

The mean free path equation is $\lambda = C \cdot \frac{m}{\rho \cdot d^2}$

"Solving" for C, and using dimensions $C = \frac{\lambda \cdot \rho \cdot d^2}{m}$

$$C = \frac{L \times \frac{M}{L^3} \times L^2}{M} = 0$$

The drag constant C is dimensionless.

Problem 1.33

1.33 An important equation in the theory of vibrations is

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t)$$

where m (kg) is the mass and x (m) is the position at time t (s). For a dimensionally consistent equation, what are the dimensions of c, k, and f? What would be suitable units for c, k, and f in the SI and BG systems?

Given: Equation for vibrations.

Find: Dimensions of c, k and f for a dimensionally consistent equation. Also, suitable units in SI and BG systems.

Solution: Use the vibration equation to find the diemsions of each quantity

The first term of the equation is $m\cdot\frac{d^2x}{dt^2}$

The dimensions of this are $M\times\frac{L}{t^2}$

Each of the other terms must also have these dimensions.

Hence $c\cdot\frac{dx}{dt} = \frac{M\cdot L}{t^2}$ so $c\times\frac{L}{t} = \frac{M\cdot L}{t^2}$ and $c = \frac{M}{t}$

$k\cdot x = \frac{M\cdot L}{t^2}$ so $k\times L = \frac{M\cdot L}{t^2}$ and $k = \frac{M}{t^2}$

$f = \frac{M\cdot L}{t^2}$

Suitable units for c, k, and f are c: $\frac{kg}{s}$ $\frac{slug}{s}$ k: $\frac{kg}{s^2}$ $\frac{slug}{s^2}$ f: $\frac{kg\cdot m}{s^2}$ $\frac{slug\cdot ft}{s^2}$

Note that c is a damping (viscous) friction term, k is a spring constant, and f is a forcing function. These are more typically expressed using F rather than M (mass). From Newton's 2nd law:

$F = M\cdot\frac{L}{t^2}$ or $M = \frac{F\cdot t^2}{L}$

Using this in the dimensions and units for c, k, and f we fin $c = \frac{F\cdot t^2}{L\cdot t} = \frac{F\cdot t}{L}$ $k = \frac{F\cdot t^2}{L\cdot t^2} = \frac{F}{L}$ $f = F$

c: $\frac{N\cdot s}{m}$ $\frac{lbf\cdot s}{ft}$ k: $\frac{N}{m}$ $\frac{lbf}{ft}$ f: N lbf

Problem 1.34

[1]

1.34 A parameter that is often used in describing pump performance is the specific speed, $N_{S_{cu}}$, given by

$$N_{S_{cu}} = \frac{N(\text{rpm})[Q(\text{gpm})]^{1/2}}{[H(\text{ft})]^{3/4}}$$

What are the units of specific speed? A particular pump has a specific speed of 2000. What will be the specific speed in SI units (angular velocity in rad/s)?

Given: Specific speed in customary units

Find: Units; Specific speed in SI units

Solution:

The units are $\dfrac{\text{rpm}\cdot\text{gpm}^{\frac{1}{2}}}{\text{ft}^{\frac{3}{4}}}$ or $\dfrac{\text{ft}^{\frac{3}{4}}}{\text{s}^{\frac{3}{2}}}$

Using data from tables (e.g. Table G.2)

$$N_{Scu} = 2000\cdot\frac{\text{rpm}\cdot\text{gpm}^{\frac{1}{2}}}{\text{ft}^{\frac{3}{4}}}$$

$$N_{Scu} = 2000\times\frac{\text{rpm}\cdot\text{gpm}^{\frac{1}{2}}}{\text{ft}^{\frac{3}{4}}}\times\frac{2\cdot\pi\cdot\text{rad}}{1\cdot\text{rev}}\times\frac{1\cdot\text{min}}{60\cdot\text{s}}\times\left(\frac{4\times 0.000946\cdot\text{m}^3}{1\cdot\text{gal}}\cdot\frac{1\cdot\text{min}}{60\cdot\text{s}}\right)^{\frac{1}{2}}\times\left(\frac{\frac{1}{12}\cdot\text{ft}}{0.0254\cdot\text{m}}\right)^{\frac{3}{4}}$$

$$N_{Scu} = 4.06\cdot\frac{\frac{\text{rad}}{\text{s}}\cdot\left(\frac{\text{m}^3}{\text{s}}\right)^{\frac{1}{2}}}{\text{m}^{\frac{3}{4}}}$$

Problem 1.35

[1]

1.35 A particular pump has an "engineering" equation form of the performance characteristic equation given by H (ft) $= 1.5 - 4.5 \times 10^{-5}$ $[Q \text{ (gpm)}]^2$, relating the head H and flow rate Q. What are the units of the coefficients 1.5 and 4.5×10^{-5}? Derive an SI version of this equation.

Given: "Engineering" equation for a pump

Find: SI version

Solution:

The dimensions of "1.5" are ft.

The dimensions of "4.5×10^{-5}" are ft/gpm^2.

Using data from tables (e.g. Table G.2), the SI versions of these coefficients can be obtained

$$1.5\cdot\text{ft} = 1.5\cdot\text{ft}\times\frac{0.0254\cdot\text{m}}{\frac{1}{12}\cdot\text{ft}} = 0.457\cdot\text{m}$$

$$4.5\times10^{-5}\cdot\frac{\text{ft}}{\text{gpm}^2} = 4.5\cdot10^{-5}\cdot\frac{\text{ft}}{\text{gpm}^2}\times\frac{0.0254\cdot\text{m}}{\frac{1}{12}\cdot\text{ft}}\times\left(\frac{1\cdot\text{gal}}{4\cdot\text{quart}}\cdot\frac{1\text{quart}}{0.000946\cdot\text{m}^3}\cdot\frac{60\cdot\text{s}}{1\text{min}}\right)^2$$

$$4.5\cdot10^{-5}\cdot\frac{\text{ft}}{\text{gpm}^2} = 3450\cdot\frac{\text{m}}{\left(\frac{\text{m}^3}{\text{s}}\right)^2}$$

The equation is

$$H(\text{m}) = 0.457 - 3450\cdot\left(Q\left(\frac{\text{m}^3}{\text{s}}\right)\right)^2$$

Problem 1.36

[2]

1.36 A container weighs 3.5 lbf when empty. When filled with water at 90°F, the mass of the container and its contents is 2.5 slug. Find the weight of water in the container, and its volume in cubic feet, using data from Appendix A.

Given: Empty container weighing 3.5 lbf when empty, has a mass of 2.5 slug when filled with water at 90°F.

Find:

a. Weight of water in the container

b. Container volume in ft^3

Solution: Basic equation: $F = ma$

Weight is the force of gravity on a body, W = mg

Then

$$W_t = W_{H_2O} + W_c$$
$$W_{H_2O} = W_t - W_c = mg - W_c$$
$$W_{H_2O} = 2.5\,\text{slug} \times 32.2\,\frac{\text{ft}}{\text{s}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} - 3.5\,\text{lbf} = 77.0\,\text{lbf}$$

The volume is given by

$$\forall = \frac{M_{H_2O}}{\rho} = \frac{M_{H_2O}g}{\rho g} = \frac{W_{H_2O}}{\rho g}$$

From Table A.7, $\rho = 1.93$ slug/ft^3 at T = 90°F

$$\therefore \forall = 77.0\,\text{lbf} \times \frac{\text{ft}^3}{1.93\,\text{slug}} \times \frac{\text{s}^2}{32.2\,\text{ft}} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2} = 1.24\,\text{ft}^3$$

Problem 1.37

[2]

1.37 Calculate the density of standard air in a laboratory from the ideal gas equation of state. Estimate the experimental uncertainty in the air density calculated for standard conditions (29.9 in. of mercury and 59°F) if the uncertainty in measuring the barometer height is ±0.1 in. of mercury and the uncertainty in measuring temperature is ±0.5°F. (Note that 29.9 in. of mercury corresponds to 14.7 psia.)

Given:

Air at standard conditions – p = 29.9 in Hg, T = 59°F

Uncertainty: in p is ± 0.1 in Hg, in T is ± 0.5°F

Note that 29.9 in Hg corresponds to 14.7 psia

Find:

a. air density using ideal gas equation of state.

b. estimate of uncertainty in calculated value.

Solution:

$$\rho = \frac{p}{RT} = 14.7\frac{\text{lbf}}{\text{in}^2} \times \frac{\text{lb}\cdot{}^\circ\text{R}}{53.3\text{ ft}\cdot\text{lbf}} \times \frac{1}{519^\circ\text{R}} \times 144\frac{\text{in}^2}{\text{ft}^2}$$

$$\rho = 0.0765\ \text{lbm}/\text{ft}^3$$

The uncertainty in density is given by

$$u_\rho = \left[\left(\frac{p}{\rho}\frac{\partial\rho}{\partial p}u_p\right)^2 + \left(\frac{T}{\rho}\frac{\partial\rho}{\partial T}u_T\right)^2\right]^{1/2}$$

$$\frac{p}{\rho}\frac{\partial\rho}{\partial p} = RT\frac{1}{RT} = \frac{RT}{RT} = 1; \qquad u_p = \frac{\pm 0.1}{29.9} = \pm 0.334\%$$

$$\frac{T}{\rho}\frac{\partial\rho}{\partial T} = \frac{T}{\rho}\left(-\frac{p}{RT^2}\right) = -\frac{p}{\rho RT} = -1; \qquad u_T = \frac{\pm 0.5}{460+59} = \pm 0.0963\%$$

Then

$$u_\rho = \left[(u_p)^2 + (-u_T)^2\right]^{1/2} = \pm\left[(0.334)^2 + (-0.0963)^2\right]$$

$$u_\rho = \pm 0.348\%\ \left(\pm 2.66\times 10^{-4}\ \text{lbm}/\text{ft}^3\right)$$

Problem 1.38

[2]

1.38 Repeat the calculation of uncertainty described in Problem 1.37 for air in a freezer. Assume the measured barometer height is 759 ± 1 mm of mercury and the temperature is –20 ± 0.5 C. [Note that 759 mm of mercury corresponds to 101 kPa (abs).]

Given:

Air at pressure, p = 759 ± 1 mm Hg and temperature, T = –20 ± 0.5°C.

Note that 759 mm Hg corresponds to 101 kPa.

Find:

a. Air density using ideal gas equation of state

b. Estimate of uncertainty in calculated value

Solution:

$$\rho = \frac{p}{RT} = 101 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287\ N \cdot m} \times \frac{1}{253\ K} = 1.39\ kg/m^3$$

The uncertainty in density is given by

$$u_\rho = \left[\left(\frac{p}{\rho}\frac{\partial \rho}{\partial p} u_p\right)^2 + \left(\frac{T}{\rho}\frac{\partial \rho}{\partial T} u_T\right)^2\right]^{1/2}$$

$$\frac{p}{\rho}\frac{\partial \rho}{\partial p} = RT\frac{1}{RT} = 1; \qquad u_p = \frac{\pm 1}{759} = \pm 0.132\%$$

$$\frac{T}{\rho}\frac{\partial \rho}{\partial T} = \frac{T}{\rho}\left(-\frac{p}{RT^2}\right) = -\frac{p}{\rho RT} = -1; \qquad u_T = \frac{\pm 0.5}{273 - 20} = \pm 0.198\%$$

Then

$$u_\rho = \left[\left(u_p\right)^2 + \left(-u_T\right)^2\right]^{1/2} = \pm\left[(0.132)^2 + (-0.198)^2\right]^{1/2}$$

$$u_\rho = \pm 0.238\% \quad \left(\pm 3.31 \times 10^{-3}\ kg/m^3\right)$$

Problem 1.39 [2]

1.39 The mass of the standard American golf ball is 1.62 ± 0.01 oz and its mean diameter is 1.68 ± 0.01 in. Determine the density and specific gravity of the American golf ball. Estimate the uncertainties in the calculated values.

Given: Standard American golf ball: $m = 1.62 \pm 0.01\ \text{oz} \quad (20\text{ to }1)$

$D = 1.68 \pm 0.01\ \text{in.} \quad (20\text{ to }1)$

Find:

a. Density and specific gravity.

b. Estimate uncertainties in calculated values.

Solution: Density is mass per unit volume, so

$$\rho = \frac{m}{\forall} = \frac{m}{\frac{4}{3}\pi R^3} = \frac{3}{4\pi}\frac{m}{(D/2)^3} = \frac{6}{\pi}\frac{m}{D^3}$$

$$\rho = \frac{6}{\pi} \times 1.62\ \text{oz} \times \frac{1}{(1.68)^3\ \text{in.}^3} \times \frac{0.4536\ \text{kg}}{16\ \text{oz}} \times \frac{\text{in.}^3}{(0.0254)^3\ \text{m}^3} = 1130\ \text{kg/m}^3$$

and

$$SG = \frac{\rho}{\rho_{H_2O}} = 1130\ \frac{\text{kg}}{\text{m}^3} \times \frac{\text{m}^3}{1000\ \text{kg}} = 1.13$$

The uncertainty in density is given by

$$u_\rho = \pm\left[\left(\frac{m}{\rho}\frac{\partial\rho}{\partial m}u_m\right)^2 + \left(\frac{D}{\rho}\frac{\partial\rho}{\partial D}u_D\right)^2\right]^{1/2}$$

$$\frac{m}{\rho}\frac{\partial\rho}{\partial m} = \frac{m}{\rho}\frac{1}{\forall} = \frac{\forall}{\forall} = 1; \quad u_m = \pm\frac{0.01}{1.62} = \pm 0.617\ \text{percent}$$

$$\frac{D}{\rho}\frac{\partial\rho}{\partial D} = \frac{D}{\rho}\left(-3\frac{6}{\pi}\frac{m}{D^4}\right) = \frac{\pi D^4}{6\ m}\left(-3\frac{6}{\pi}\frac{m}{D^4}\right) = -3; \quad u_D = \pm 0.595\ \text{percent}$$

Thus

$$u_\rho = \pm\left[(u_m)^2 + (-3u_D)^2\right]^{1/2}$$

$$= \pm\left\{(0.617)^2 + \left[-3(0.595)^2\right]\right\}^{\frac{1}{2}}$$

$$u_\rho = \pm 1.89\ \text{percent}\ \left(\pm 21.4\ \text{kg/m}^3\right)$$

$$u_{SG} = u_\rho = \pm 1.89\ \text{percent}\ (\pm 0.0214)$$

Finally,

$$\rho = 1130 \pm 21.4\ \text{kg/m}^3\ (20\text{ to }1)$$

$$SG = 1.13 \pm 0.0214\ (20\text{ to }1)$$

Problem 1.40

1.40 The mass flow rate in a water flow system determined by collecting the discharge over a timed interval is 0.2 kg/s. The scales used can be read to the nearest 0.05 kg and the stopwatch is accurate to 0.2 s. Estimate the precision with which the flow rate can be calculated for time intervals of (a) 10 s and (b) 1 min.

Given: Mass flow rate of water determined by collecting discharge over a timed interval is 0.2 kg/s.

Scales can be read to nearest 0.05 kg.

Stopwatch can be read to nearest 0.2 s.

Find: Estimate precision of flow rate calculation for time intervals of (a) 10 s, and (b) 1 min.

Solution: Apply methodology of uncertainty analysis, Appendix F:

Computing equations:

$$\dot{m} = \frac{\Delta m}{\Delta t}$$

$$u_{\dot{m}} = \pm\left[\left(\frac{\Delta m}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta m}u_{\Delta m}\right)^2 + \left(\frac{\Delta t}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta t}u_{\Delta t}\right)^2\right]^{\frac{1}{2}}$$

Thus

$$\frac{\Delta m}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta m} = \Delta t\left(\frac{1}{\Delta t}\right) = 1 \quad \text{and} \quad \frac{\Delta t}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta t} = \frac{\Delta t^2}{\Delta m}\left[(-1)\frac{\Delta m}{\Delta t^2}\right] = -1$$

The uncertainties are expected to be ± half the least counts of the measuring instruments.

Tabulating results:

Time Interval, Δt(s)	Error in Δt(s)	Uncertainty in Δt (percent)	Water Collected, Δm(kg)	Error in Δm(kg)	Uncertainty in Δm (percent)	Uncertainty in $\dot{m}$ (percent)
10	± 0.10	± 1.0	2.0	± 0.025	± 1.25	± 1.60
60	± 0.10	± 0.167	12.0	± 0.025	± 0.208	± 0.267

A time interval of about 15 seconds should be chosen to reduce the uncertainty in results to ± 1 percent.

Problem 1.41

1.41 A can of pet food has the following internal dimensions: 102 mm height and 73 mm diameter (each ±1 mm at odds of 20 to 1). The label lists the mass of the contents as 397 g. Evaluate the magnitude and estimated uncertainty of the density of the pet food if the mass value is accurate to ±1 g at the same odds.

Given: Pet food can

$$H = 102 \pm 1 \text{ mm (20 to 1)}$$
$$D = 73 \pm 1 \text{ mm (20 to 1)}$$
$$m = 397 \pm 1 \text{ g (20 to 1)}$$

Find: Magnitude and estimated uncertainty of pet food density.

Solution: Density is

$$\rho = \frac{m}{\forall} = \frac{m}{\pi R^2 H} = \frac{4}{\pi}\frac{m}{D^2 H} \quad \text{or} \quad \rho = \rho(m, D, H)$$

From uncertainty analysis

$$u_\rho = \pm\left[\left(\frac{m}{\rho}\frac{\partial\rho}{\partial m}u_m\right)^2 + \left(\frac{D}{\rho}\frac{\partial\rho}{\partial D}u_D\right)^2 + \left(\frac{H}{\rho}\frac{\partial\rho}{\partial H}u_H\right)^2\right]^{\frac{1}{2}}$$

Evaluating,

$$\frac{m}{\rho}\frac{\partial\rho}{\partial m} = \frac{m}{\rho}\frac{4}{\pi}\frac{1}{D^2 H} = \frac{1}{\rho}\frac{4m}{\pi D^2 H} = 1; \qquad u_m = \frac{\pm 1}{397} = \pm 0.252\%$$

$$\frac{D}{\rho}\frac{\partial\rho}{\partial D} = \frac{D}{\rho}(-2)\frac{4m}{\pi D^3 H} = (-2)\frac{1}{\rho}\frac{4m}{\pi D^2 H} = -2; \qquad u_D = \frac{\pm 1}{73} = \pm 1.37\%$$

$$\frac{H}{\rho}\frac{\partial\rho}{\partial H} = \frac{H}{\rho}(-1)\frac{4m}{\pi D^2 H^2} = (-1)\frac{1}{\rho}\frac{4m}{\pi D^2 H} = -1; \qquad u_H = \frac{\pm 1}{102} = \pm 0.980\%$$

Substituting

$$u_\rho = \pm\left\{[(1)(0.252)]^2 + [(-2)(1.37)]^2 + [(-1)(0.980)]^2\right\}^{\frac{1}{2}}$$
$$u_\rho = \pm 2.92 \text{ percent}$$

$$\forall = \frac{\pi}{4}D^2 H = \frac{\pi}{4}\times(73)^2\ \text{mm}^2 \times 102\ \text{mm} \times \frac{\text{m}^3}{10^9\ \text{mm}^3} = 4.27\times 10^{-4}\,\text{m}^3$$

$$\rho = \frac{m}{\forall} = \frac{397\ \text{g}}{4.27\times 10^{-4}\,\text{m}^3}\times\frac{\text{kg}}{1000\ \text{g}} = 930\ \text{kg/m}^3$$

Thus

$$\rho = 930 \pm 27.2\ \text{kg/m}^3 \text{ (20 to 1)}$$

Problem 1.42 [2]

1.42 The mass of the standard British golf ball is 45.9 ± 0.3 g and its mean diameter is 41.1 ± 0.3 mm. Determine the density and specific gravity of the British golf ball. Estimate the uncertainties in the calculated values.

Given: Standard British golf ball:

$$m = 45.9 \pm 0.3\ \text{g} \quad (20\ \text{to}\ 1)$$
$$D = 41.1 \pm 0.3\ \text{mm} \quad (20\ \text{to}\ 1)$$

Find:

a. Density and specific gravity

b. Estimate of uncertainties in calculated values.

Solution: Density is mass per unit volume, so

$$\rho = \frac{m}{\forall} = \frac{m}{\frac{4}{3}\pi R^3} = \frac{3}{4\pi}\frac{m}{(D/2)^3} = \frac{6}{\pi}\frac{m}{D^3}$$

$$\rho = \frac{6}{\pi} \times 0.0459\ \text{kg} \times \frac{1}{(0.0411)^3}\ \text{m}^3 = 1260\ \text{kg/m}^3$$

and

$$SG = \frac{\rho}{\rho H_2O} = 1260\ \frac{\text{kg}}{\text{m}^3} \times \frac{\text{m}^3}{1000\ \text{kg}} = 1.26$$

The uncertainty in density is given by

$$u_\rho = \pm\left[\left(\frac{m}{\rho}\frac{\partial\rho}{\partial m}u_m\right)^2 + \left(\frac{D}{\rho}\frac{\partial\rho}{\partial D}u_D\right)^2\right]^{1/2}$$

$$\frac{m}{\rho}\frac{\partial\rho}{\partial m} = \frac{m}{\rho}\frac{1}{\forall} = \frac{\forall}{\forall} = 1;\quad u_m = \pm\frac{0.3}{45.9} = \pm 0.654\%$$

$$\frac{D}{\rho}\frac{\partial\rho}{\partial D} = \frac{D}{\rho}\left(-3\frac{6}{\pi}\frac{m}{D^4}\right) = -3\left(\frac{6m}{\pi D^3\rho}\right) = -3$$

$$u_D = \pm\frac{0.3}{41.1} = 0.730\%$$

Thus

$$u_\rho = \pm[(u_m)^2 + (-3u_D)^2]^{1/2} = \pm\left\{(0.654)^2 + [-3(0.730)]^2\right\}^{1/2}$$

$$u_\rho = \pm 2.29\%\ (\pm 28.9\ \text{kg/m}^3)$$

$$u_{SG} = u_\rho = \pm 2.29\%\ (\pm 0.0289)$$

Summarizing

$$\rho = 1260 \pm 28.9\ \text{kg/m}^3\ (20\ \text{to}\ 1)$$

$$SG = 1.26 \pm 0.0289\ (20\ \text{to}\ 1)$$

Problem 1.43

[3]

1.43 The mass flow rate of water in a tube is measured using a beaker to catch water during a timed interval. The nominal mass flow rate is 100 g/s. Assume that mass is measured using a balance with a least count of 1 g and a maximum capacity of 1 kg, and that the timer has a least count of 0.1 s. Estimate the time intervals and uncertainties in measured mass flow rate that would result from using 100, 500, and 1000 mL beakers. Would there be any advantage in using the largest beaker? Assume the tare mass of the empty 1000 mL beaker is 500 g.

Given: Nominal mass flow rate of water determined by collecting discharge (in a beaker) over a timed interval is $\dot{m} = 100\ \text{g/s}$

- Scales have capacity of 1 kg, with least count of 1 g.
- Timer has least count of 0.1 s.
- Beakers with volume of 100, 500, 1000 mL are available – tare mass of 1000 mL beaker is 500 g.

Find: Estimate (a) time intervals, and (b) uncertainties, in measuring mass flow rate from using each of the three beakers.

Solution: To estimate time intervals assume beaker is filled to maximum volume in case of 100 and 500 mL beakers and to maximum allowable mass of water (500 g) in case of 1000 mL beaker.

Then $\dot{m} = \dfrac{\Delta m}{\Delta t}$ and $\Delta t = \dfrac{\Delta m}{\dot{m}} = \dfrac{\rho \Delta \forall}{\dot{m}}$

Tabulating results

$$\begin{array}{llll} \Delta\forall = & 100\ \text{mL} & 500\ \text{mL} & 1000\ \text{mL} \\ \Delta t = & 1\ \text{s} & 5\ \text{s} & 5\ \text{s} \end{array}$$

Apply the methodology of uncertainty analysis, Appendix E Computing equation:

$$u_{\dot{m}} = \pm\left[\left(\frac{\Delta m}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta m}u_{\Delta m}\right)^2 + \left(\frac{\Delta t}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta t}u_{\Delta t}\right)^2\right]^{1/2}$$

The uncertainties are expected to be ± half the least counts of the measuring instruments

$$\delta\Delta m = \pm 0.5\ \text{g} \qquad \delta\Delta t = 0.05\ \text{s}$$

$$\frac{\Delta m}{\dot{m}} = \frac{\partial \dot{m}}{\partial \Delta m} = \Delta t\left(\frac{1}{\Delta t}\right) = 1 \qquad \text{and} \qquad \frac{\Delta t}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta t} = \frac{(\Delta t)^2}{\Delta m}\left[-\frac{\Delta m}{(\Delta t)^2}\right] = -1$$

$$\therefore u_{\dot{m}} = \pm\left[\left(u_{\Delta m}\right)^2 + \left(-u_{\Delta t}\right)^2\right]^{1/2}$$

Tabulating results:

Beaker Volume $\Delta \forall$ (mL)	Water Collected Δm(g)	Error in Δm(g)	Uncertainty in Δm (percent)	Time Interval Δt(s)	Error in Δt(s)	Uncertainty in Δt (percent)	in $\dot{m}$ (percent)
100	100	± 0.50	± 0.50	1.0	± 0.05	± 5.0	± 5.03
500	500	± 0.50	± 0.10	5.0	± 0.05	± 1.0	± 1.0
1000	500	± 0.50	± 0.10	5.0	± 0.05	± 1.0	± 1.0

Since the scales have a capacity of 1 kg and the tare mass of the 1000 mL beaker is 500 g, there is no advantage in using the larger beaker. The uncertainty in $\dot{m}$ could be reduced to ± 0.50 percent by using the large beaker if a scale with greater capacity the same least count were available

Problem 1.44

[3]

1.44 The estimated dimensions of a soda can are $D = 66.0 \pm 0.5$ mm and $H = 110 \pm 0.5$ mm. Measure the mass of a full can and an empty can using a kitchen scale or postal scale. Estimate the volume of soda contained in the can. From your measurements estimate the depth to which the can is filled and the uncertainty in the estimate. Assume the value of SG = 1.055, as supplied by the bottler.

Given: Soda can with estimated dimensions D = 66.0 ± 0.5 mm, H = 110 ± 0.5 mm. Soda has SG = 1.055

Find:

a. volume of soda in the can (based on measured mass of full and empty can).

b. estimate average depth to which the can is filled and the uncertainty in the estimate.

Solution: Measurements on a can of coke give

$$m_f = 386.5 \pm 0.50 \text{ g}, \quad m_e = 17.5 \pm 0.50 \text{ g} \therefore m = m_f - m_e = 369 \pm u_m \text{ g}$$

$$u_m = \pm\left[\left(\frac{m_f}{m}\frac{\partial m}{\partial m_f}u_{m_f}\right)^2 + \left(\frac{m_e}{m}\frac{\partial m}{\partial m_e}u_{m_e}\right)^2\right]^{1/2}$$

$$u_{m_f} = \pm\frac{0.5 \text{ g}}{386.5 \text{ g}} = \pm 0.00129, \quad u_{m_e} = \pm\frac{0.50}{17.5} = 0.0286$$

$$\therefore u_m = \pm\left\{\left[\frac{386.5}{369}(1)(0.00129)\right]^2 + \left[\frac{17.5}{369}(-1)(0.0286)\right]^2\right\}^{1/2} = 0.0019$$

Density is mass per unit volume and SG = $\rho/\rho H_2O$ so

$$\forall = \frac{m}{\rho} = \frac{m}{\rho H_2O \text{ SG}} = 369 \text{ g} \times \frac{\text{m}^3}{1000 \text{ kg}} \times \frac{1}{1.055} \times \frac{\text{kg}}{1000 \text{ g}} = 350 \times 10^{-6} \text{ m}^3$$

The reference value ρH_2O is assumed to be precise. Since SG is specified to three places beyond the decimal point, assume $u_{SG} = \pm 0.001$. Then

$$u_v = \pm\left[\left(\frac{m}{v}\frac{\partial v}{\partial m}u_m\right)^2 + \left(\frac{m}{SG}\frac{\partial v}{\partial SG}\right)^2\right]^{1/2} = \pm\left\{[(1)\ u_m]^2 + [(-1)\ u_{SG}]^2\right\}^{1/2}$$

$$u_v = \pm\left\{[(1)(0.0019)]^2 + [(-1)(0.001)]^2\right\}^{1/2} = 0.0021 \text{ or } 0.21\%$$

$$\forall = \frac{\pi D^2}{4}L \text{ or } L = \frac{4\forall}{\pi D^2} = \frac{4}{\pi} \times \frac{350 \times 10^{-6}\ m^3}{(0.066)^2\ m^2} \times \frac{10^3\ mm}{m} = 102\ mm$$

$$u_L = \pm\left[\left(\frac{\forall}{L}\frac{\partial L}{\partial \forall}u_\forall\right)^2\right] + \left[\left(\frac{D}{L}\frac{\partial L}{\partial D}u_D\right)^2\right]^{1/2}$$

$$\frac{\forall}{L}\frac{\partial L}{\partial \forall} = \frac{\pi D^2}{4} \times \frac{4}{\pi D^2} = 1\ u_D = \pm\frac{0.5\ mm}{66\ mm} = 0.0076$$

$$\frac{D}{L}\frac{\partial L}{\partial D} = D\frac{\pi D^2}{4\forall} \times \frac{4\forall}{\pi}\left(-\frac{2}{D^3}\right) = -2$$

$$u_L = \pm\left\{[(1)(0.0021)]^2 + [(-2)(0.0076)]^2\right\}^{1/2} = 0.0153 \text{ or } 1.53\%$$

Note:

1. Printing on the can states the content as 355 ml. This suggests that the implied accuracy of the SG value may be over stated.
2. Results suggest that over seven percent of the can height is void of soda.

Problem 1.45

[3]

1.45 From Appendix A, the viscosity μ(N • s/m^2) of water at temperature **T** (K) can be computed from $\mu = A10^{B/(T-C)}$, where $A = 2.414 \times 10^{-5}$ N • s/m^2, $B = 247.8$ K, and $C = 140$ K. Determine the viscosity of water at 20°C, and estimate its uncertainty if the uncertainty in temperature measurement is ±0.25°C.

Given: Data on water

Find: Viscosity; Uncertainty in viscosity

Solution:

The data is: $A = 2.414 \times 10^{-5} \cdot \frac{N \cdot s}{m^2}$ $\quad B = 247.8 \cdot K$ $\quad C = 140 \cdot K$ $\quad T = 293 \cdot K$

The uncertainty in temperature is $\quad u_T = \frac{0.25 \cdot K}{293 \cdot K}$ $\quad u_T = 0.085 \cdot \%$

Also $\quad \mu(T) = A \cdot 10^{\frac{B}{(T-C)}}$ $\quad$ Evaluating $\quad \mu(T) = 1.01 \times 10^{-3} \cdot \frac{N \cdot s}{m^2}$

For the uncertainty

$$\frac{d}{dT}\mu(T) = -\frac{A \cdot B \cdot \ln(10)}{10^{\frac{B}{C-T}} \cdot (C - T)^2}$$

Hence

$$u_\mu(T) = \left| \frac{T}{\mu(T)} \cdot \frac{d}{dT}\mu(T) \cdot u_T \right| = \frac{\ln(10) \cdot |B \cdot T \cdot u_T|}{(|C - T|)^2}$$

Evaluating $\quad u_\mu(T) = 0.609 \cdot \%$

Problem 1.46

[3]

1.46 An enthusiast magazine publishes data from its road tests on the lateral acceleration capability of cars. The measurements are made using a 150-ft-diameter skid pad. Assume the vehicle path deviates from the circle by ±2 ft and that the vehicle speed is read from a fifth-wheel speed-measuring system to ±0.5 mph. Estimate the experimental uncertainty in a reported lateral acceleration of 0.7 *g*. How would you improve the experimental procedure to reduce the uncertainty?

Given: Lateral acceleration, a = 0.70 g, measured on 150-ft diameter skid pad.

$$\left.\begin{array}{l}\text{Path deviation: } \pm 2 \text{ ft} \\ \text{Vehicle speed: } \pm 0.5 \text{ mph}\end{array}\right\} \text{measurement uncertainty}$$

Find:

a. Estimate uncertainty in lateral acceleration.

b. How could experimental procedure be improved?

Solution: Lateral acceleration is given by $a = V^2/R$.

From Appendix F, $u_a = \pm[(2u_v)^2 + (u_R)^2]^{1/2}$

From the given data, $V^2 = aR; \quad V = \sqrt{aR} = \left[0.70 \times \frac{32.2 \text{ ft}}{s^2} \times 75 \text{ ft}\right]^{1/2} = 41.1 \text{ ft/s}$

Then $u_v = \pm\frac{\delta V}{V} = \pm 0.5 \frac{\text{mi}}{\text{hr}} \times \frac{\text{s}}{41.1 \text{ ft}} \times 5280 \frac{\text{ft}}{\text{mi}} \times \frac{\text{hr}}{3600 \text{ s}} = \pm 0.0178$

and $u_R = \pm\frac{\delta R}{R} = \pm 2 \text{ ft} \times \frac{1}{75 \text{ ft}} = \pm 0.0267$

so
$$u_a = \pm\left[(2 \times 0.0178)^2 + (0.0267)^2\right]^{1/2} = \pm 0.0445$$
$$u_a = \pm 4.45 \text{ percent}$$

Experimental procedure could be improved by using a larger circle, assuming the absolute errors in measurement are constant.

For

$$D = 400 \text{ ft}, \quad R = 200 \text{ ft}$$

$$V = \sqrt{aR} = \left[0.70 \times \frac{32.2 \text{ ft}}{s^2} \times 200 \text{ ft}\right]^{1/2} = 67.1 \text{ ft / s} = 45.8 \text{ mph}$$

$$u_v = \pm\frac{0.5 \text{ mph}}{45.8 \text{ mph}} = \pm 0.0109; \quad u_R = \pm\frac{2 \text{ ft}}{200 \text{ ft}} = \pm 0.0100$$

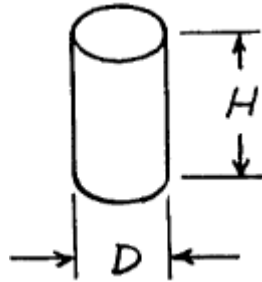
$$u_a = \pm\left[(2 \times 0.0109)^2 + (0.0100)^2\right]^{1/2} = \pm 0.0240 \text{ or } \pm 2.4 \text{ percent}$$

Problem 1.47

[4]

1.47 Using the nominal dimensions of the soda can given in Problem 1.44, determine the precision with which the diameter and height must be measured to estimate the volume of the can within an uncertainty of ±0.5 percent.

Given: Dimensions of soda can:

$$D = 66 \text{ mm}$$
$$H = 110 \text{ mm}$$

Find: Measurement precision needed to allow volume to be estimated with an uncertainty of ± 0.5 percent or less.

Solution: Use the methods of Appendix F:

Computing equations:

$$\forall = \frac{\pi D^2 H}{4}$$

$$u_\forall = \pm\left[\left(\frac{H}{\forall}\frac{\partial\forall}{\partial H}u_H\right)^2 + \left(\frac{D}{\forall}\frac{\partial\forall}{\partial D}u_D\right)^2\right]^{\frac{1}{2}}$$

Since $\forall = \frac{\pi D^2 H}{4}$, then $\frac{\partial\forall}{\partial H} = \frac{\pi D^2}{4}$ and $\frac{\partial\forall}{\partial D} = \frac{\pi DH}{2}$

Let $u_D = \pm\frac{\delta x}{D}$ and $u_H = \pm\frac{\delta x}{H}$, substituting,

$$u_\forall = \pm\left[\left(\frac{4H}{\pi D^2 H}\frac{\pi D^2}{4}\frac{\delta x}{H}\right)^2 + \left(\frac{4D}{\pi D^2 H}\frac{\pi DH}{2}\frac{\delta x}{D}\right)^2\right]^{\frac{1}{2}} = \pm\left[\left(\frac{\delta x}{H}\right)^2 + \left(\frac{2\delta x}{D}\right)^2\right]^{\frac{1}{2}}$$

Solving,

$$u_\forall{}^2 = \left(\frac{\delta x}{H}\right)^2 + \left(\frac{2\delta x}{D}\right)^2 = (\delta x)^2\left[\left(\frac{1}{H}\right)^2 + \left(\frac{2}{D}\right)^2\right]$$

$$\delta x = \pm \frac{u_\forall}{\left[\left(\frac{1}{H}\right)^2 + \left(\frac{2}{D}\right)^2 \right]^{\frac{1}{2}}} = \pm \frac{0.005}{\left[\left(\frac{1}{110\text{ mm}}\right)^2 + \left(\frac{2}{66\text{ mm}}\right)^2 \right]^{\frac{1}{2}}} = \pm 0.158\text{ mm}$$

Check:

$$u_H = \pm \frac{\delta x}{H} = \pm \frac{0.158\text{ mm}}{110\text{ mm}} = \pm 1.44 \times 10^{-3}$$

$$u_D = \pm \frac{\delta x}{D} = \pm \frac{0.158\text{ mm}}{66\text{ mm}} = \pm 2.39 \times 10^{-3}$$

$$u_\forall = \pm[(u_H)^2 + (2u_D)^2]^{\frac{1}{2}} = \pm[(0.00144)^2 + (0.00478)^2]^{\frac{1}{2}} = \pm 0.00499$$

If δx represents half the least count, a minimum resolution of about $2\,\delta x \approx 0.32$ mm is needed.

Problem 1.48

Given: American golf ball, $m = 1.62 \pm 0.01$ oz, $D = 1.68$ in.

Find: Precision to which D must be measured to estimate density within uncertainty of ± 1 percent.

Solution: Apply uncertainty concepts

Definition: Density, $\rho \equiv \frac{m}{\forall}$ $\forall = \frac{4}{3}\pi R^3 = \frac{\pi D^3}{6}$

Computing equation: $u_R = \pm\left[\left(\frac{x_1}{R}\frac{\partial R}{\partial x_1} u_{x_1}\right)^2 + \cdots\right]^{1/2}$

From the definition, $\rho = \frac{m}{\pi D^3/6} = \frac{6m}{\pi D^3} = \rho(m, D)$

Thus $\frac{m}{\rho}\frac{\partial \rho}{\partial m} = 1$ and $\frac{D}{\rho}\frac{\partial \rho}{\partial D} = 3$, so

$$u_\rho = \pm\left[(1\, u_m)^2 + (3 u_D)^2\right]^{1/2}$$

$$u_\rho^2 = u_m^2 + 9 u_D^2$$

Solving, $u_D = \pm \frac{1}{3}\left[u_\rho^2 - u_m^2\right]^{\frac{1}{2}}$

From the data given, $u_\rho = \pm 0.0100$

$$u_m = \frac{\pm 0.01 \text{ oz}}{1.62 \text{ oz}} = \pm 0.00617$$

$$u_D = \pm \frac{1}{3}\left[(0.0100)^2 - (0.00617)^2\right]^{\frac{1}{2}} = \pm 0.00262 \text{ or } \pm 0.262\,\%$$

Since $u_D = \pm \frac{\delta D}{D}$, then

$$\delta D = \pm D u_D = \pm 1.68 \text{ in.} \times 0.00262 = \pm 0.00441 \text{ in.}$$

δ

The ball diameter must be measured to a precision of ± 0.00441 in. (± 0.112 mm) or better to estimate density within ± 1 percent. A micrometer or caliper could be used.

Problem 1.49

1.49 The height of a building may be estimated by measuring the horizontal distance to a point on the ground and the angle from this point to the top of the building. Assuming these measurements are $L = 100 \pm 0.5$ ft and $\theta = 30 \pm 0.2$ degrees, estimate the height H of the building and the uncertainty in the estimate. For the same building height and measurement uncertainties, use *Excel's Solver* to determine the angle (and the corresponding distance from the building) at which measurements should be made to minimize the uncertainty in estimated height. Evaluate and plot the optimum measurement angle as a function of building height for $50 \leq H \leq 1000$ ft.

Given: Data on length and angle measurements

Find: Height; Angle for minimum uncertainty in height; Plot

Solution:

The data is: $L = 100\cdot ft$ $\quad \delta L = 0.5\cdot ft$ $\quad \theta = 30\cdot deg$ $\quad \delta\theta = 0.2\cdot deg$

Uncertainties: $u_L = \frac{\delta L}{L}$ $\quad u_L = 0.5\%$ $\quad u_\theta = \frac{\delta\theta}{\theta}$ $\quad u_\theta = 0.667\%$

The height is: $H = L\cdot\tan(\theta)$ $\quad H = 57.7 ft$ $\quad$ with uncertainty $\quad u_H = \sqrt{\left(\frac{L}{H}\cdot\frac{\partial}{\partial L}H\cdot u_L\right)^2 + \left(\frac{\theta}{H}\cdot\frac{\partial}{\partial \theta}H\cdot u_\theta\right)^2}$

Hence with $\frac{\partial}{\partial L}H = \tan(\theta)$ $\quad \frac{\partial}{\partial \theta}H = L\cdot\left(1 + \tan(\theta)^2\right)$ $\quad u_H = \sqrt{\left(\frac{L}{H}\cdot\tan(\theta)\cdot u_L\right)^2 + \left[\frac{L\cdot\theta}{H}\cdot\left(1+\tan(\theta)^2\right)\cdot u_\theta\right]^2}$

Evaluating $u_H = 0.949\%$ $\quad$ and $\quad \delta H = u_H\cdot H$ $\quad \delta H = 0.548 ft$

The height is then $H = 57.7 ft$ +/- $\delta H = 0.548 ft$

To plot u_H versus θ for a given H we need to replace L, u_L and u_θ with functions of θ. Doing this and simplifying

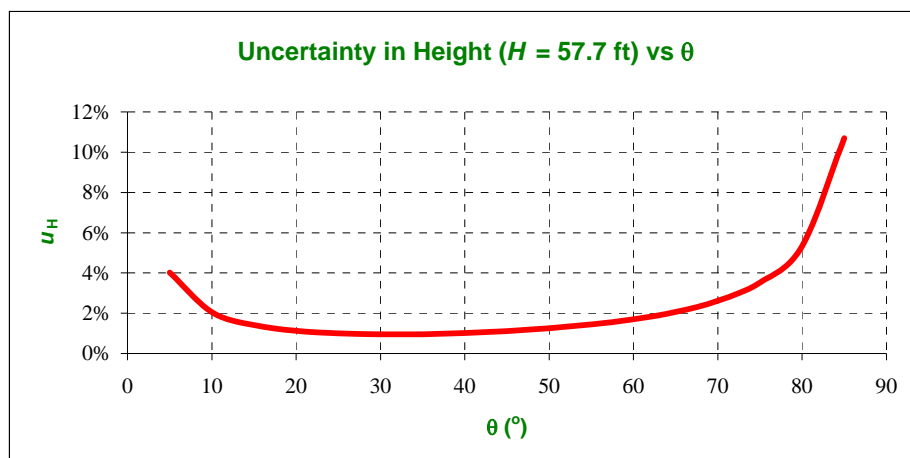
$$u_H(\theta) = \sqrt{\left(\tan(\theta)\cdot\frac{\delta L}{H}\right)^2 + \left[\frac{\delta\theta}{\tan(\theta)}\cdot\left(1+\tan(\theta)^2\right)\right]^2}$$

Given data:

H =	57.7	ft
δL =	0.5	ft
$\delta\theta$ =	0.2	deg

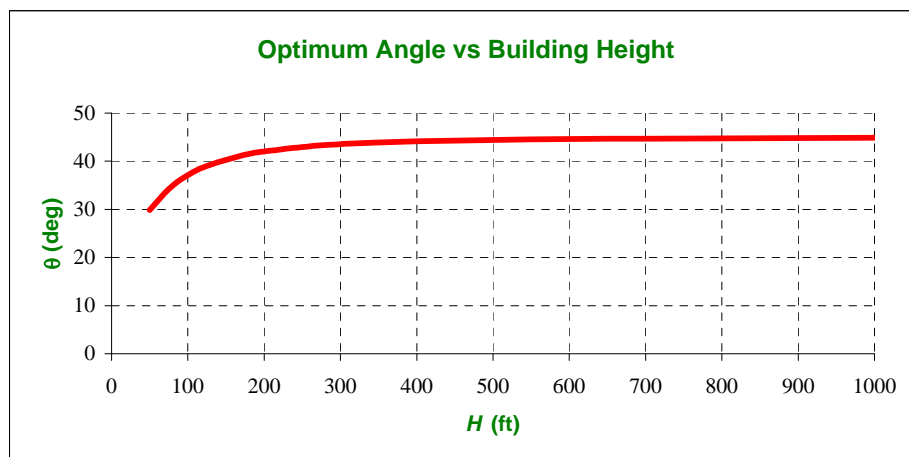
For this building height, we are to vary θ (and therefore L) to minimize the uncertainty u_H.

Plotting u_H vs θ

θ (deg)	u_H
5	4.02%
10	2.05%
15	1.42%
20	1.13%
25	1.00%
30	0.95%
35	0.96%
40	1.02%
45	1.11%
50	1.25%
55	1.44%
60	1.70%
65	2.07%
70	2.62%
75	3.52%
80	5.32%
85	10.69%

Optimizing using *Solver*

θ (deg)	u_H
31.4	0.947%

To find the optimum θ as a function of building height H we need a more complex *Solver*

H (ft)	θ (deg)	u_H
50	29.9	0.992%
75	34.3	0.877%
100	37.1	0.818%
125	39.0	0.784%
175	41.3	0.747%
200	42.0	0.737%
250	43.0	0.724%
300	43.5	0.717%
400	44.1	0.709%
500	44.4	0.705%
600	44.6	0.703%
700	44.7	0.702%
800	44.8	0.701%
900	44.8	0.700%
1000	44.9	0.700%

Use *Solver* to vary ALL θ's to minimize the total u_H!

Total u_H's: 11.3%

Problem 1.50

[5]

1.50 In the design of a medical instrument it is desired to dispense 1 cubic millimeter of liquid using a piston-cylinder syringe made from molded plastic. The molding operation produces plastic parts with estimated dimensional uncertainties of ±0.002 in. Estimate the uncertainty in dispensed volume that results from the uncertainties in the dimensions of the device. Plot on the same graph the uncertainty in length, diameter, and volume dispensed as a function of cylinder diameter *D* from *D* = 0.5 to 2 mm. Determine the ratio of stroke length to bore diameter that gives a design with minimum uncertainty in volume dispensed. Is the result influenced by the magnitude of the dimensional uncertainty?

Given: Piston-cylinder device to have $\forall = 1\,\text{mm}^3$.

Molded plastic parts with dimensional uncertainties, $\delta = \pm\, 0.002$ in.

Find:

a. Estimate of uncertainty in dispensed volume that results from the dimensional uncertainties.

b. Determine the ratio of stroke length to bore diameter that minimizes $u_\forall$; plot of the results.

c. Is this result influenced by the magnitude of δ?

Solution: Apply uncertainty concepts from Appendix F:

Computing equation: $$\forall = \frac{\pi D^2 L}{4};\quad u_\forall = \pm\left[\left(\frac{L}{\forall}\frac{\partial\forall}{\partial L}u_L\right)^2 + \left(\frac{D}{\forall}\frac{\partial\forall}{\partial D}u_D\right)^2\right]^{\frac{1}{2}}$$

From $\forall$, $\frac{L}{\forall}\frac{\partial\forall}{\partial L} = 1$, and $\frac{D}{\forall}\frac{\partial\forall}{\partial D} = 2$, so $u_\forall = \pm[u_L^2 + (2u_D)^2]^{\frac{1}{2}}$

The dimensional uncertainty is $\delta = \pm 0.002\ \text{in.} \times 25.4\ \frac{\text{mm}}{\text{in.}} = \pm 0.0508\ \text{mm}$

Assume D = 1 mm. Then $L = \frac{4\forall}{\pi D^2} = \frac{4}{\pi} \times 1\,\text{mm}^3 \times \frac{1}{(1)^2\ \text{mm}^2} = 1.27\ \text{mm}$

$$\left.\begin{aligned} u_D &= \pm\frac{\delta}{D} = \pm\frac{0.0508}{1} = \pm 5.08\ \text{percent} \\ u_L &= \pm\frac{\delta}{L} = \pm\frac{0.0508}{1.27} = \pm 4.00\ \text{percent}\end{aligned}\right\} u_\forall = \pm[(4.00)^2 + (2(5.08))^2]^{\frac{1}{2}}$$

$$u_\forall = \pm 10.9\ \text{percent}$$

To minimize $u_\forall$, substitute in terms of D:

$$u_\forall = \pm[(u_L)^2 + (2u_D)^2] = \pm\left[\left(\frac{\delta}{L}\right)^2 + \left(2\frac{\delta}{D}\right)^2\right]^{\frac{1}{2}} = \pm\left[\left(\frac{\pi D^2}{4\forall}\delta\right)^2 + \left(2\frac{\delta}{D}\right)^2\right]^{\frac{1}{2}}$$

This will be minimum when D is such that ∂[]/∂D = 0, or

$$\frac{\partial[]}{\partial D} = \left(\frac{\pi\delta}{4\forall}\right)^2 4D^3 + (2\delta)^2\left(-2\frac{1}{D^3}\right) = 0;\ \ D^6 = 2\left(\frac{4\forall}{\pi}\right)^2;\ \ D = 2^{\frac{1}{6}}\left(\frac{4\forall}{\pi}\right)^{\frac{1}{3}}$$

Thus $$D_{opt} = 2^{\frac{1}{6}}\left(\frac{4}{\pi}\times 1\text{ mm}^3\right)^{\frac{1}{3}} = 1.22\text{ mm}$$

The corresponding L is $$L_{opt} = \frac{4\forall}{\pi D^2} = \frac{4}{\pi}\times 1\text{ mm}^3 \times \frac{1}{(1.22)^2\text{ mm}^2} = 0.855\text{ mm}$$

The optimum stroke-to-bore ratio is $L/D)_{opt} = \dfrac{0.855\text{ mm}}{1.22\text{ mm}} = 0.701$ (see table and plot on next page)

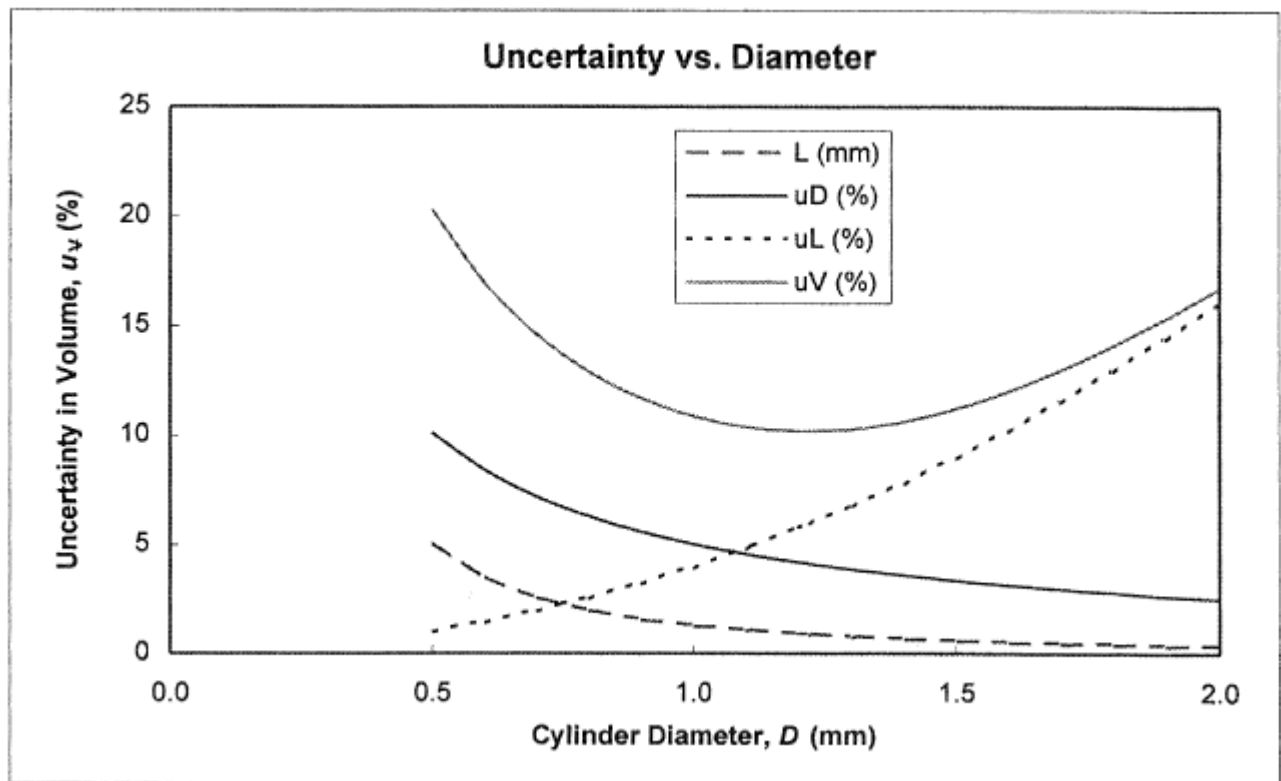
Note that δ drops out of the optimization equation. This optimum L/D is independent of the magnitude of δ

However, the magnitude of the optimum $u_\forall$ increases as δ increases.

Uncertainty in volume of cylinder: $\delta = 0.002$ in. 0.0508 mm

$\forall =$ 1 mm^3

***D* (mm)**	***L* (mm)**	***L/D* (---)**	**u_D(%)**	**u_L(%)**	**$u_\forall$ (%)**
0.5	5.09	10.2	10.2	1.00	20.3
0.6	3.54	5.89	8.47	1.44	17.0
0.7	2.60	3.71	7.26	1.96	14.6
0.8	1.99	2.49	6.35	2.55	13.0
0.9	1.57	1.75	5.64	3.23	11.7
1.0	1.27	1.27	5.08	3.99	10.9
1.1	1.05	0.957	4.62	4.83	10.4
1.2	0.884	0.737	4.23	5.75	10.2
1.22	0.855	0.701	4.16	5.94	10.2
1.3	0.753	0.580	3.91	6.74	10.3

1.4	0.650	0.464	3.63	7.82	10.7
1.5	0.566	0.377	3.39	8.98	11.2
1.6	0.497	0.311	3.18	10.2	12.0
1.7	0.441	0.259	2.99	11.5	13.0
1.8	0.393	0.218	2.82	12.9	14.1
1.9	0.353	0.186	2.67	14.4	15.4
2.0	0.318	0.159	2.54	16.0	16.7
2.1	0.289	0.137	2.42	17.6	18.2
2.2	0.263	0.120	2.31	19.3	19.9
2.3	0.241	0.105	2.21	21.1	21.6
2.4	0.221	0.092	2.12	23.0	23.4
2.5	0.204	0.081	2.03	24.9	25.3

Problem 2.1

[1]

2.1 For the velocity fields given below, determine:

a. whether the flow field is one-, two-, or three-dimensional, and why.

b. whether the flow is steady or unsteady, and why.

c. (The quantities a and b are constants.)

(1) $\vec{V} = [ay^2 e^{-bt}]\hat{i}$ (2) $\vec{V} = ax^2\hat{i} + bx\hat{j} + c\hat{k}$

(3) $\vec{V} = axy\hat{i} - byt\hat{j}$ (4) $\vec{V} = ax\hat{i} - by\hat{j} + ct\hat{k}$

(5) $\vec{V} = [ae^{-bx}]\hat{i} + bt^2\hat{j}$ (6) $\vec{V} = a(x^2 + y^2)^{1/2}(1/z^3)\hat{k}$

(7) $\vec{V} = (ax + t)\hat{i} - by^2\hat{j}$ (8) $\vec{V} = ax^2\hat{i} + bxz\hat{j} + cy\hat{k}$

Given: Velocity fields

Find: Whether flows are 1, 2 or 3D, steady or unsteady.

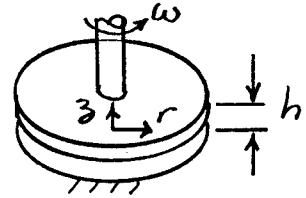
Solution:

(1)	$\vec{V} = \vec{V}(y)$	1D	$\vec{V} = \vec{V}(t)$	Unsteady
(2)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} \neq \vec{V}(t)$	Steady
(3)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} = \vec{V}(t)$	Unsteady
(4)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} = \vec{V}(t)$	Unsteady
(5)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} = \vec{V}(t)$	Unsteady
(6)	$\vec{V} = \vec{V}(x,y,z)$	3D	$\vec{V} \neq \vec{V}(t)$	Steady
(7)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} = \vec{V}(t)$	Unsteady
(8)	$\vec{V} = \vec{V}(x,y,z)$	3D	$\vec{V} \neq \vec{V}(t)$	Steady

Problem 2.2

Given: Viscous liquid sheared between parallel disks.

Upper disk rotates, lower fixed.

Velocity field is $\vec{V} = \hat{e}_\theta r\omega z/h$.

Find: (a) Dimensions of velocity field.
(b) Satisfy physical boundary conditions.

Solution: To find dimensions, compare to $\vec{V} = \vec{V}(x, y, z)$ form.

The given field is $\vec{V} = \vec{V}(r, z)$. Two space coordinates are included, so field is 2-D.

2-D

Flow must satisfy the no-slip condition:

(1) At lower disk, $\vec{V} = 0$, since stationary.

$z = 0$, so $\vec{V} = \hat{e}_\theta r\omega(0)/h = 0$ $\therefore$ satisfied

$z = $

(2) At upper disk, $\vec{V} = \hat{e}_\theta r\omega$, since it rotates as a solid body.

$z = h$, so $\vec{V} = \hat{e}_\theta r\omega(h)/h = \hat{e}_\theta r\omega$ $\therefore$ satisfied

$z = $

Problem 2.3

2.3 For the velocity field $\vec{V} = Ax^2\hat{j} + Bxy\hat{j}$, where $A = 1$ $m^{-1}s^{-1}$, $B = -½$ $m^{-1}s^{-1}$, and the coordinates are measured in meters, obtain an equation for the flow streamlines. Plot several streamlines for positive y.

Given: Velocity field

Find: Equation for streamlines

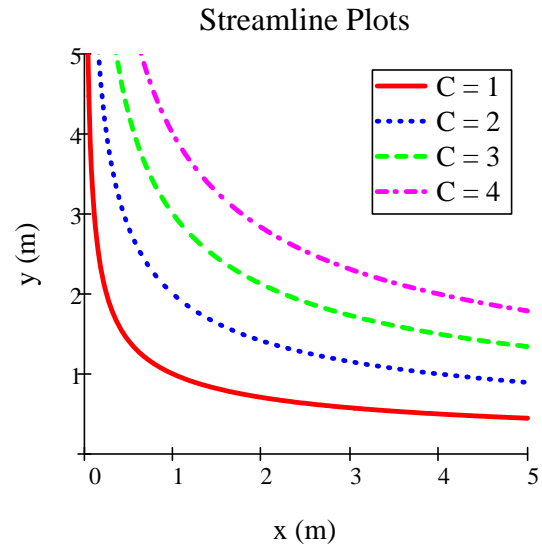
Solution:

For streamlines
$$\frac{v}{u} = \frac{dy}{dx} = \frac{B \cdot x \cdot y}{A \cdot x^2} = \frac{B \cdot y}{A \cdot x}$$

So, separating variables
$$\frac{dy}{y} = \frac{B}{A} \cdot \frac{dx}{x}$$

Integrating
$$\ln(y) = \frac{B}{A} \cdot \ln(x) + c = -\frac{1}{2} \cdot \ln(x) + c$$

The solution is
$$y = \frac{C}{\sqrt{x}}$$

The plot can be easily done in *Excel*.

Problem 2.4

Given: Velocity field, $\vec{V} = ax\hat{i} - by\hat{j}$ $\quad (a = b = 1\ \text{sec}^{-1})$

Find: Equation for the flow streamlines, and

Plot: Representative streamlines for $x \geq 0$ and $y \geq 0$

Solution:

The slope of the streamlines in the x-y plane is given by

$$\frac{dy}{dx} = \frac{v}{u}$$

For $\vec{V} = ax\hat{i} - by\hat{j}$, then $u = ax$, $v = -by$. Hence

$$\frac{dy}{dx} = \frac{v}{u} = -\frac{b}{a}\frac{y}{x}$$

To solve the differential equation, separate variables and integrate

$$\int \frac{dy}{y} = -\int \frac{b}{a}\frac{dx}{x}$$

$$\ln y = -\frac{b}{a}\ln x + \text{constant}$$

$$\ln y = \ln x^{-\frac{b}{a}} + \ln c \qquad \text{where constant} = \ln c$$

Then

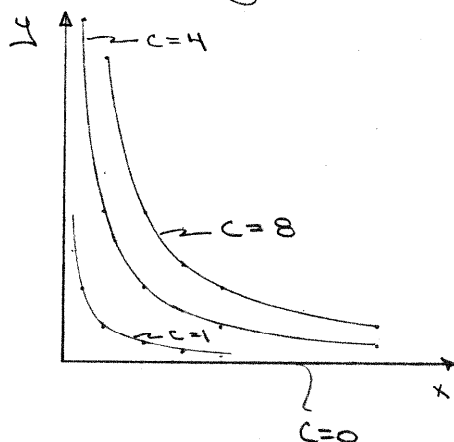
$$y = c\,x^{-\frac{b}{a}} \qquad y(x)$$

or alternately $\quad x = \left(\frac{y}{c}\right)^{-\frac{a}{b}} = \left(\frac{c}{y}\right)^{\frac{a}{b}}$

For a given velocity field, the constants a and b are fixed. Different streamlines are obtained by assigning different values to the constant of integration, c.

Since $a = b = 1\ \text{sec}^{-1}$, then $a/b = 1$, and the streamlines are given by the equation

$$y = cx^{-1} = \frac{c}{x} \qquad \text{or} \qquad x = \frac{c}{y}$$

For $c = 0$ $\quad y = 0$ for all x and $x = 0$ for all y.

The equation $y = \frac{c}{x}$ is the equation of a hyperbola.

Curves are shown for different values of c

Problem 2.5

2.5 A velocity field is given by $\vec{V} = ax\hat{i} - bty\hat{j}$, where $a = 1\ \text{s}^{-1}$ and $b = 1\ \text{s}^{-2}$. Find the equation of the streamlines at any time t. Plot several streamlines in the first quadrant at $t = 0$ s, $t = 1$ s, and $t = 20$ s.

Given: Velocity field

Find: Equation for streamlines; Plot streamlines

Solution:

For streamlines $$\frac{v}{u} = \frac{dy}{dx} = \frac{-b \cdot t \cdot y}{a \cdot x}$$

So, separating variables $$\frac{dy}{y} = \frac{-b \cdot t}{a} \cdot \frac{dx}{x}$$

Integrating $$\ln(y) = \frac{-b \cdot t}{a} \cdot \ln(x)$$

The solution is $$y = c \cdot x^{\frac{-b}{a} \cdot t}$$

For $t = 0$ s $\quad y = c \qquad$ For $t = 1$ s $\quad y = \frac{c}{x} \qquad$ For $t = 20$ s $\quad y = c \cdot x^{-20}$

t = 0

	c = 1	c = 2	c = 3
x	**y**	**y**	**y**
0.05	1.00	2.00	3.00
0.10	1.00	2.00	3.00
0.20	1.00	2.00	3.00
0.30	1.00	2.00	3.00
0.40	1.00	2.00	3.00
0.50	1.00	2.00	3.00
0.60	1.00	2.00	3.00
0.70	1.00	2.00	3.00
0.80	1.00	2.00	3.00
0.90	1.00	2.00	3.00
1.00	1.00	2.00	3.00
1.10	1.00	2.00	3.00
1.20	1.00	2.00	3.00
1.30	1.00	2.00	3.00
1.40	1.00	2.00	3.00
1.50	1.00	2.00	3.00
1.60	1.00	2.00	3.00
1.70	1.00	2.00	3.00
1.80	1.00	2.00	3.00
1.90	1.00	2.00	3.00
2.00	1.00	2.00	3.00

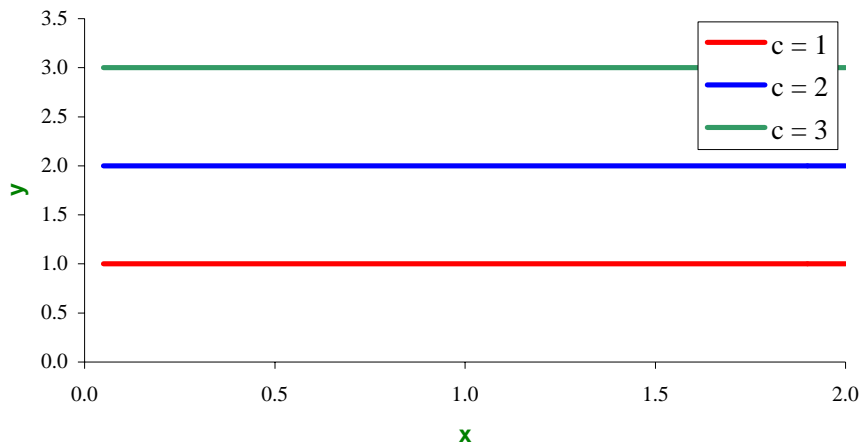
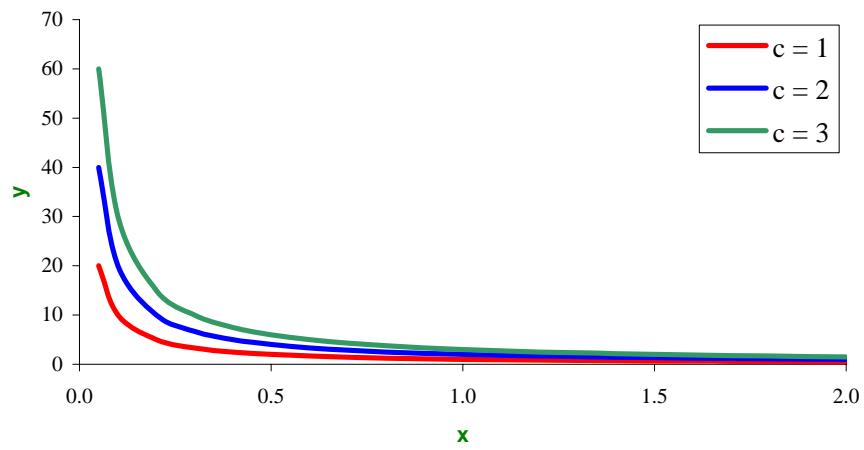
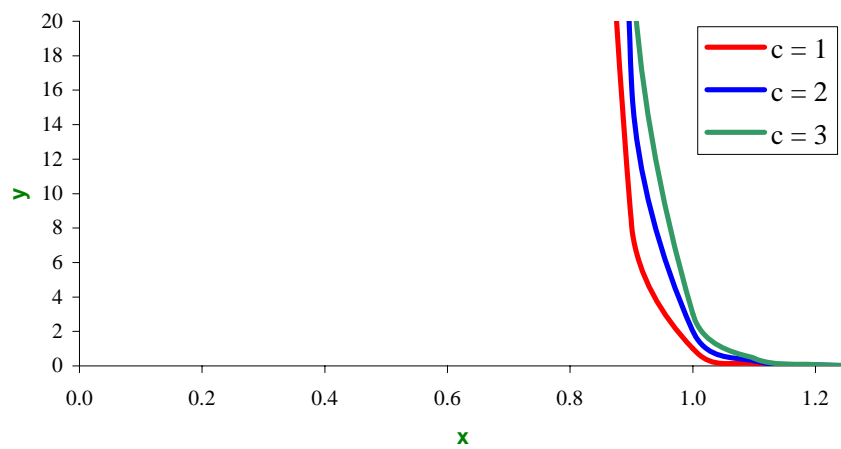
t =1 s

(### means too large to view)

	c = 1	c = 2	c = 3
x	**y**	**y**	**y**
0.05	20.00	40.00	60.00
0.10	10.00	20.00	30.00
0.20	5.00	10.00	15.00
0.30	3.33	6.67	10.00
0.40	2.50	5.00	7.50
0.50	2.00	4.00	6.00
0.60	1.67	3.33	5.00
0.70	1.43	2.86	4.29
0.80	1.25	2.50	3.75
0.90	1.11	2.22	3.33
1.00	1.00	2.00	3.00
1.10	0.91	1.82	2.73
1.20	0.83	1.67	2.50
1.30	0.77	1.54	2.31
1.40	0.71	1.43	2.14
1.50	0.67	1.33	2.00
1.60	0.63	1.25	1.88
1.70	0.59	1.18	1.76
1.80	0.56	1.11	1.67
1.90	0.53	1.05	1.58
2.00	0.50	1.00	1.50

t = 20 s

	c = 1	c = 2	c = 3
x	**y**	**y**	**y**
0.05	######	######	######
0.10	######	######	######
0.20	######	######	######
0.30	######	######	######
0.40	######	######	######
0.50	######	######	######
0.60	######	######	######
0.70	######	######	######
0.80	86.74	173.47	260.21
0.90	8.23	16.45	24.68
1.00	1.00	2.00	3.00
1.10	0.15	0.30	0.45
1.20	0.03	0.05	0.08
1.30	0.01	0.01	0.02
1.40	0.00	0.00	0.00
1.50	0.00	0.00	0.00
1.60	0.00	0.00	0.00
1.70	0.00	0.00	0.00
1.80	0.00	0.00	0.00
1.90	0.00	0.00	0.00
2.00	0.00	0.00	0.00

Streamline Plot (t = 0)
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
y
0.0
0.5
1.0
1.5
2.0
x
c = 1
c = 2
c = 3
Streamline Plot (t = 1 s)
70
60
50
40
30
20
10
0
y
0.0
0.5
1.0
1.5
2.0
x
c = 1
c = 2
c = 3
Streamline Plot (t = 20 s)
20
18
16
14
12
10
8
6
4
2
0
y
0.0
0.2
0.4
0.6
0.8
1.0
1.2
x
c = 1
c = 2
c = 3

Problem 2.6

2.6 A velocity field is specified as $\vec{V} = axy\hat{i} + by^2\hat{j}$, where $a = 2$ $\text{m}^{-1}\text{s}^{-1}$, $b = -6$ $\text{m}^{-1}\text{s}^{-1}$, and the coordinates are measured in meters. Is the flow field one-, two-, or three-dimensional? Why? Calculate the velocity components at the point (2, ½). Develop an equation for the streamline passing through this point. Plot several streamlines in the first quadrant including the one that passes through the point (2, ½).

Given: Velocity field

Find: Whether field is 1D, 2D or 3D; Velocity components at (2,1/2); Equation for streamlines; Plot

Solution:

The velocity field is a function of *x* and *y*. It is therefore 2D.

At point (2,1/2), the velocity components are $u = a \cdot x \cdot y = 2 \cdot \frac{1}{m \cdot s} \times 2 \cdot m \times \frac{1}{2} \cdot m$ $\qquad u = 2 \cdot \frac{m}{s}$

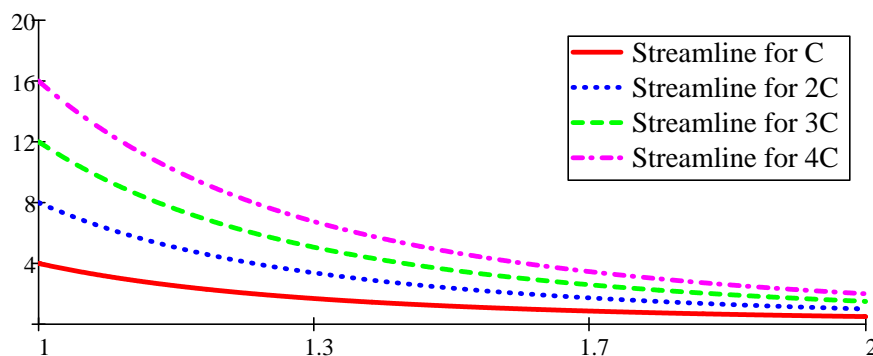
$$v = b \cdot y^2 = -6 \cdot \frac{1}{m \cdot s} \times \left(\frac{1}{2} \cdot m\right)^2 \qquad v = -\frac{3}{2} \cdot \frac{m}{s}$$

For streamlines $\quad \frac{v}{u} = \frac{dy}{dx} = \frac{b \cdot y^2}{a \cdot x \cdot y} = \frac{b \cdot y}{a \cdot x}$

So, separating variables $\quad \frac{dy}{y} = \frac{b}{a} \cdot \frac{dx}{x}$

Integrating $\quad \ln(y) = \frac{b}{a} \cdot \ln(x) + c \qquad y = C \cdot x^{\frac{b}{a}}$

The solution is $\quad y = C \cdot x^{-3}$

The streamline passing through point (2,1/2) is given by $\quad \frac{1}{2} = C \cdot 2^{-3} \qquad C = \frac{1}{2} \cdot 2^3 \qquad C = 4 \qquad y = \frac{4}{x^3}$

This can be plotted in *Excel*.

Problem 2.7

2.7 A velocity field is given by $\vec{V} = ax^3\hat{i} + bxy^3\hat{j}$, where $a = 1$ $m^{-2}\,s^{-1}$ and $b = 1$ $m^{-3}\,s^{-1}$. Find the equation of the streamlines. Plot several streamlines in the first quadrant.

Given: Velocity field

Find: Equation for streamlines; Plot streamlines

Solution:

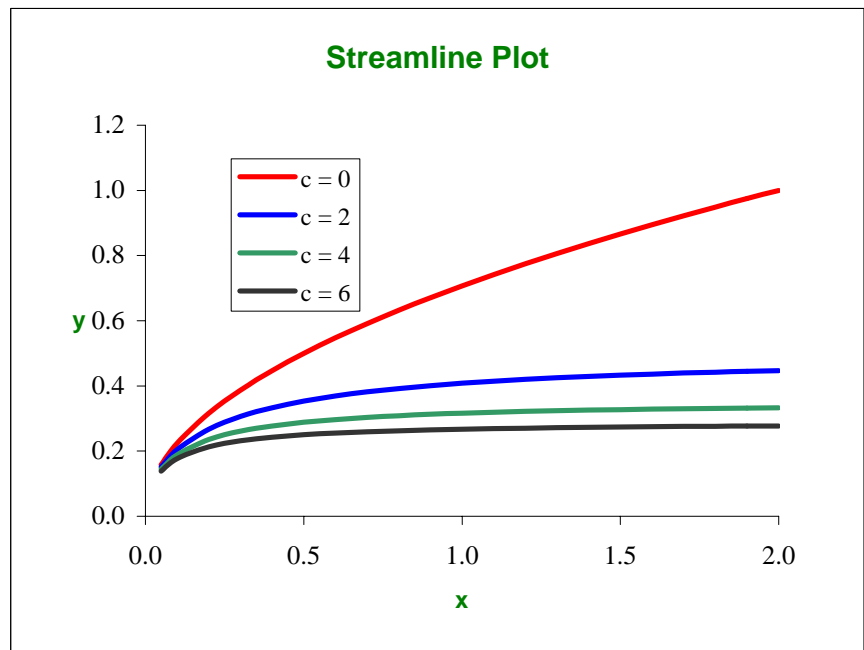
Streamlines are given by $\frac{v}{u} = \frac{dy}{dx} = \frac{b \cdot x \cdot y^3}{a \cdot x^3}$

So, separating variables $\frac{dy}{y^3} = \frac{b \cdot dx}{a \cdot x^2}$

Integrating $-\frac{1}{2 \cdot y^2} = \frac{b}{a} \cdot \left(-\frac{1}{x}\right) + C$

The solution is $y = \frac{1}{\sqrt{2 \cdot \left(\frac{b}{a \cdot x} + C\right)}}$ Note: For convenience the sign of C is changed.

a = **1**
b = **1**

C =	**0**	**2**	**4**	**6**
x	**y**	**y**	**y**	**y**
0.05	0.16	0.15	0.14	0.14
0.10	0.22	0.20	0.19	0.18
0.20	0.32	0.27	0.24	0.21
0.30	0.39	0.31	0.26	0.23
0.40	0.45	0.33	0.28	0.24
0.50	0.50	0.35	0.29	0.25
0.60	0.55	0.37	0.30	0.26
0.70	0.59	0.38	0.30	0.26
0.80	0.63	0.39	0.31	0.26
0.90	0.67	0.40	0.31	0.27
1.00	0.71	0.41	0.32	0.27
1.10	0.74	0.41	0.32	0.27
1.20	0.77	0.42	0.32	0.27
1.30	0.81	0.42	0.32	0.27
1.40	0.84	0.43	0.33	0.27
1.50	0.87	0.43	0.33	0.27
1.60	0.89	0.44	0.33	0.27
1.70	0.92	0.44	0.33	0.28
1.80	0.95	0.44	0.33	0.28
1.90	0.97	0.44	0.33	0.28
2.00	1.00	0.45	0.33	0.28

Problem 2.8

[2]

2.8 A flow is described by the velocity field $\vec{V} = (Ax + B)\hat{i} + (-Ay)\hat{j}$, where $A = 10$ ft/s/ft and $B = 20$ ft/ s. Plot a few streamlines in the xy plane, including the one that passes through the point $(x, y) = (1, 2)$.

Given: Velocity field

Find: Plot streamlines

Solution:

Streamlines are given by $\frac{v}{u} = \frac{dy}{dx} = \frac{-A \cdot y}{A \cdot x + B}$

So, separating variables $\frac{dy}{-A \cdot y} = \frac{dx}{A \cdot x + B}$

Integrating $-\frac{1}{A} \ln(y) = \frac{1}{A} \cdot \ln\left(x + \frac{B}{A}\right)$

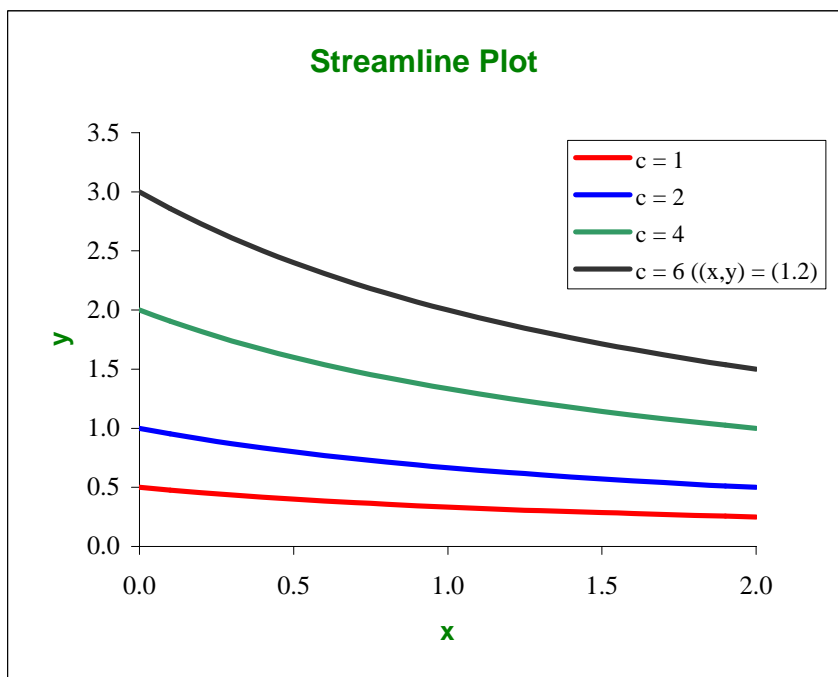
The solution is

$$y = \frac{C}{x + \frac{B}{A}}$$

For the streamline that passes through point $(x,y) = (1,2)$ $\quad C = y \cdot \left(x + \frac{B}{A}\right) = 2 \cdot \left(1 + \frac{20}{10}\right) = 6 \qquad y = \frac{6}{x + \frac{20}{10}}$

$$y = \frac{6}{x + 2}$$

A = **10**
B = **20**
C =

	1	2	4	6
x	**y**	**y**	**y**	**y**
0.00	0.50	1.00	2.00	3.00
0.10	0.48	0.95	1.90	2.86
0.20	0.45	0.91	1.82	2.73
0.30	0.43	0.87	1.74	2.61
0.40	0.42	0.83	1.67	2.50
0.50	0.40	0.80	1.60	2.40
0.60	0.38	0.77	1.54	2.31
0.70	0.37	0.74	1.48	2.22
0.80	0.36	0.71	1.43	2.14
0.90	0.34	0.69	1.38	2.07
1.00	0.33	0.67	1.33	2.00
1.10	0.32	0.65	1.29	1.94
1.20	0.31	0.63	1.25	1.88
1.30	0.30	0.61	1.21	1.82
1.40	0.29	0.59	1.18	1.76
1.50	0.29	0.57	1.14	1.71
1.60	0.28	0.56	1.11	1.67
1.70	0.27	0.54	1.08	1.62
1.80	0.26	0.53	1.05	1.58
1.90	0.26	0.51	1.03	1.54
2.00	0.25	0.50	1.00	1.50

Problem 2.9

[2]

2.9 The velocity for a steady, incompressible flow in the xy plane is given by $\vec{V} = \hat{i}A/x + \hat{j}Ay/x^2$, where $A = 2\ \text{m}^2/\text{s}$, and the coordinates are measured in meters. Obtain an equation for the streamline that passes through the point $(x, y) = (1, 3)$. Calculate the time required for a fluid particle to move from $x = 1$ m to $x = 2$ m in this flow field.

Given: Velocity field

Find: Equation for streamline through (1,3)

Solution:

For streamlines
$$\frac{v}{u} = \frac{dy}{dx} = \frac{A\cdot\frac{y}{x^2}}{\frac{A}{x}} = \frac{y}{x}$$

So, separating variables
$$\frac{dy}{y} = \frac{dx}{x}$$

Integrating
$$\ln(y) = \ln(x) + c$$

The solution is $y = C\cdot x$ which is the equation of a straight line.

For the streamline through point (1,3) $\quad 3 = C\cdot 1 \quad C = 3$ and $y = 3\cdot x$

For a particle
$$u_p = \frac{dx}{dt} = \frac{A}{x} \quad \text{or} \quad x\cdot dx = A\cdot dt \qquad x = \sqrt{2\cdot A\cdot t + c} \qquad t = \frac{x^2}{2\cdot A} - \frac{c}{2\cdot A}$$

Hence the time for a particle to go from x = 1 to x = 2 m is

$$\Delta t = t(x = 2) - t(x = 1) \qquad \Delta t = \frac{(2\cdot m)^2 - c}{2\cdot A} - \frac{(1\cdot m)^2 - c}{2\cdot A} = \frac{4\cdot m^2 - 1\cdot m^2}{2\times 2\cdot\frac{m^2}{s}} \qquad \Delta t = 0.75\cdot s$$

Problem 2.10

2.10 The flow field for an atmospheric flow is given by

$$\vec{V} = -\frac{Ky}{2\pi(x^2+y^2)}\hat{i} + \frac{Kx}{2\pi(x^2+y^2)}\hat{j}$$

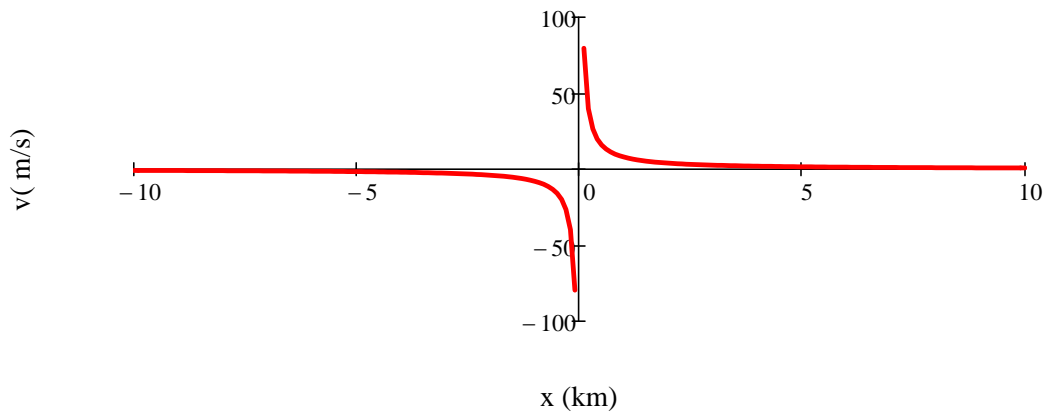
where $K = 5 \times 10^4\ \text{m}^2/\text{s}$ and the x and y coordinates are parallel to the local latitude and longitude. Plot the velocity magnitude along the x axis, along the y axis, and along the line $y = x$. For each plot use the range $-10\ \text{km} \le x$ or $y \le 10\ \text{km}$, excluding $|x|$ or $|y| \le 100\ \text{m}$. Find the equation for the streamlines and sketch several of them. What does this flow field model?

Given: Flow field

Find: Plot of velocity magnitude along axes, and y = x; Equation of streamlines

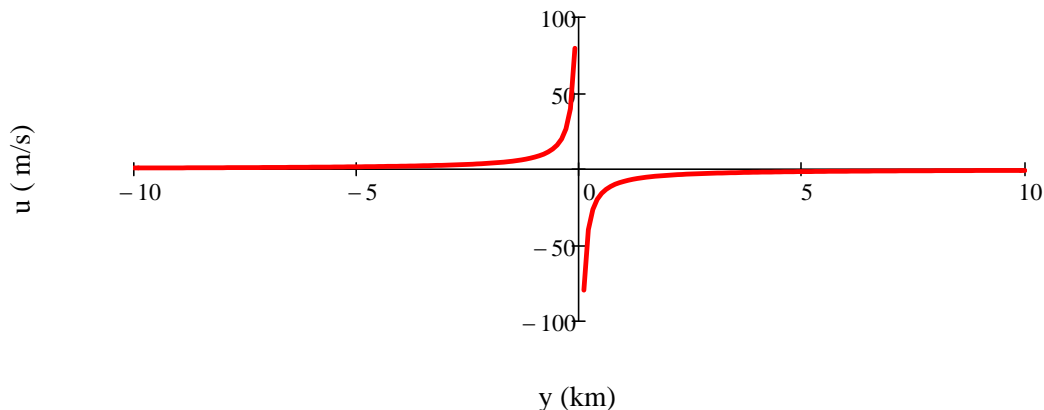
Solution:

On the x axis, y = 0, so $\quad u = -\frac{K \cdot y}{2 \cdot \pi \cdot (x^2 + y^2)} = 0 \qquad v = \frac{K \cdot x}{2 \cdot \pi \cdot (x^2 + y^2)} = \frac{K}{2 \cdot \pi \cdot x}$

Plotting

The velocity is perpendicular to the axis, is very high close to the origin, and falls off to zero.

This can also be plotted in Excel.

On the y axis, x = 0, so $\quad u = -\frac{K \cdot y}{2 \cdot \pi \cdot (x^2 + y^2)} = -\frac{K}{2 \cdot \pi \cdot y} \qquad v = \frac{K \cdot x}{2 \cdot \pi \cdot (x^2 + y^2)} = 0$

Plotting

The velocity is perpendicular to the axis, is very high close to the origin, and falls off to zero.

This can also be plotted in Excel.

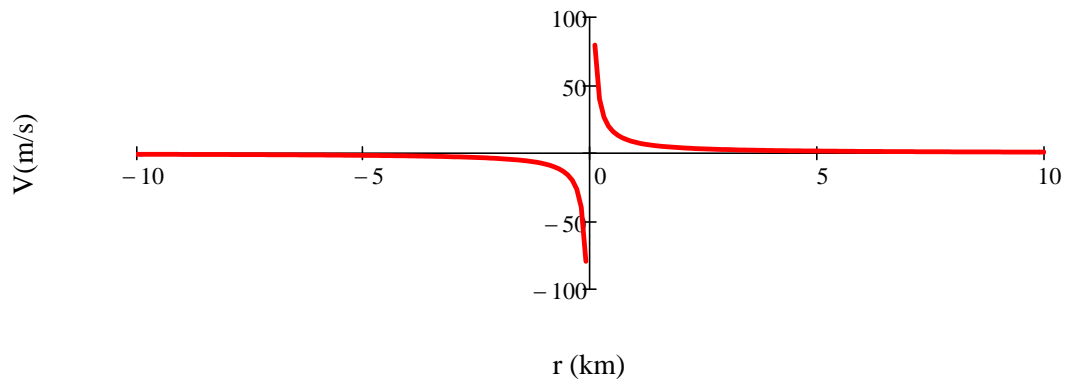
On the y = x axis
$$u = -\frac{K\cdot x}{2\cdot\pi\cdot\left(x^2+x^2\right)} = -\frac{K}{4\cdot\pi\cdot x} \qquad v = \frac{K\cdot x}{2\cdot\pi\cdot\left(x^2+x^2\right)} = \frac{K}{4\cdot\pi\cdot x}$$

The flow is perpendicular to line y = x: Slope of line y = x: 1

Slope of trajectory of motion: $\frac{u}{v} = -1$

If we define the radial position: $r = \sqrt{x^2+y^2}$ then along y = x $r = \sqrt{x^2+x^2} = \sqrt{2}\cdot x$

Then the magnitude of the velocity along y = x is $V = \sqrt{u^2+v^2} = \frac{K}{4\cdot\pi}\cdot\sqrt{\frac{1}{x^2}+\frac{1}{x^2}} = \frac{K}{2\cdot\pi\cdot\sqrt{2}\cdot x} = \frac{K}{2\cdot\pi\cdot r}$

Plotting

This can also be plotted in Excel.

For streamlines
$$\frac{v}{u} = \frac{dy}{dx} = \frac{\dfrac{K\cdot x}{2\cdot\pi\cdot\left(x^2+y^2\right)}}{-\dfrac{K\cdot y}{2\cdot\pi\cdot\left(x^2+y^2\right)}} = -\frac{x}{y}$$

So, separating variables $y\cdot dy = -x\cdot dx$

Integrating $\frac{y^2}{2} = -\frac{x^2}{2} + c$

The solution is $x^2 + y^2 = C$ which is the equation of a circle.

Streamlines form a set of concentric circles.

This flow models a vortex flow. See Example 5.6 for streamline plots. Streamlines are circular, and the velocity approaches infinity as we approach the center. In Problem 2.11, we see that the streamlines are also circular. In a real tornado, at large distances from the center, the velocities behave as in this problem; close to the center, they behave as in Problem 2.11.

Problem 2.11

2.11 The flow field for an atmospheric flow is given by

$$\vec{V} = -\frac{My}{2\pi}\hat{i} + \frac{Mx}{2\pi}\hat{j}$$

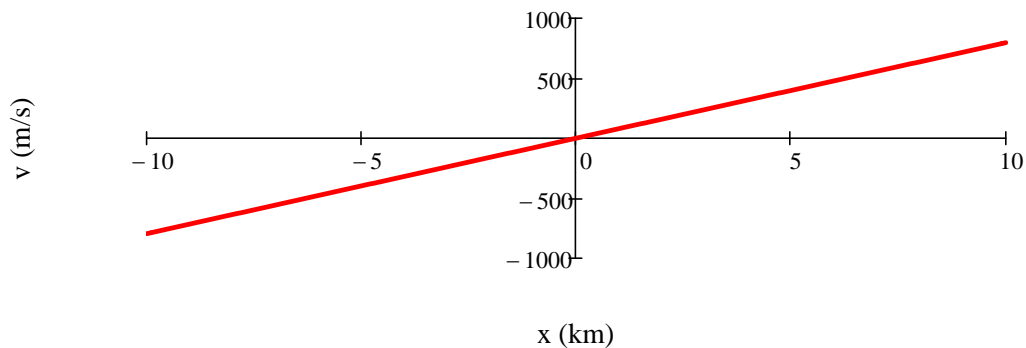
where $M = 0.5\ s^{-1}$ and the x and y coordinates are parallel to the local latitude and longitude. Plot the velocity magnitude along the x axis, along the y axis, and along the line $y = x$. For each plot use the range $-10\ km \le x$ or $y \le 10\ km$, excluding $|x|$ or $|y| \le 100\ m$. Find the equation for the streamlines and sketch several of them. What does this flow field model?

Given: Flow field

Find: Plot of velocity magnitude along axes, and y = x; Equation for streamlines

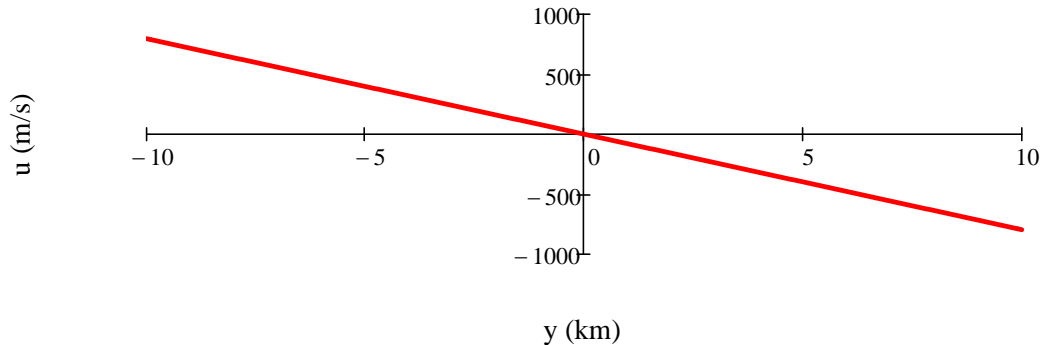
Solution:

On the x axis, y = 0, so $u = -\frac{M\cdot y}{2\cdot\pi} = 0$ $\quad v = \frac{M\cdot x}{2\cdot\pi}$

Plotting

The velocity is perpendicular to the axis and increases linearly with distance x.

This can also be plotted in Excel.

On the y axis, x = 0, so $u = -\frac{M\cdot y}{2\cdot\pi}$ $\quad v = \frac{M\cdot x}{2\cdot\pi} = 0$

Plotting

The velocity is perpendicular to the axis and increases linearly with distance y.

This can also be plotted in Excel.

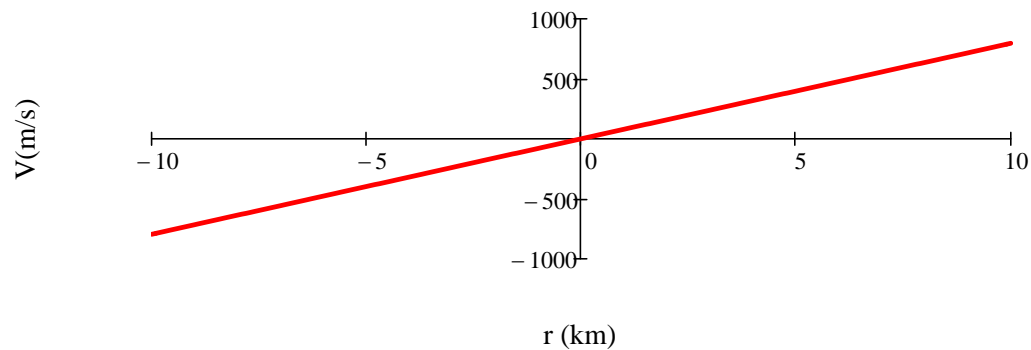
On the y = x axis $\qquad u = -\frac{M\cdot y}{2\cdot\pi} = -\frac{M\cdot x}{2\cdot\pi} \qquad v = \frac{M\cdot x}{2\cdot\pi}$

The flow is perpendicular to line y = x: $\qquad$ Slope of line y = x: $\quad 1$

Slope of trajectory of motion: $\frac{u}{v} = -1$

If we define the radial position: $\quad r = \sqrt{x^2 + y^2} \quad$ then along y = x $\quad r = \sqrt{x^2 + x^2} = \sqrt{2}\cdot x$

Then the magnitude of the velocity along y = x is $V = \sqrt{u^2 + v^2} = \frac{M}{2\cdot\pi}\cdot\sqrt{x^2 + x^2} = \frac{M\cdot\sqrt{2}\cdot x}{2\cdot\pi} = \frac{M\cdot r}{2\cdot\pi}$

Plotting

This can also be plotted in Excel.

For streamlines $\qquad \frac{v}{u} = \frac{dy}{dx} = \frac{\frac{M\cdot x}{2\cdot\pi}}{-\frac{M\cdot y}{2\cdot\pi}} = -\frac{x}{y}$

So, separating variables $\qquad y\cdot dy = -x\cdot dx$

Integrating $\qquad \frac{y^2}{2} = -\frac{x^2}{2} + c$

The solution is $\qquad x^2 + y^2 = C \qquad$ which is the equation of a circle.

The streamlines form a set of concentric circles.

This flow models a rigid body vortex flow. See Example 5.6 for streamline plots. Streamlines are circular, and the velocity approaches zer as we approach the center. In Problem 2.10, we see that the streamlines are also circular. In a real tornado, at large distances from the center, the velocities behave as in Problem 2.10; close to the center, they behave as in this problem.

Problem 2.12

2.12 A flow field flow is given by

$$\vec{V} = -\frac{qx}{2\pi(x^2+y^2)}\hat{i} - \frac{qy}{2\pi(x^2+y^2)}\hat{j}$$

where $q = 2 \times 10^4\ \text{m}^2/\text{s}$. Plot the velocity magnitude along the x axis, along the y axis, and along the line $y = x$. For each plot use the range $-10\ \text{km} \leq x$ or $y \leq 10\ \text{km}$, excluding $|x|$ or $|y| \leq 100\ \text{m}$. Find the equation for the streamlines and sketch several of them. What does this flow field model?

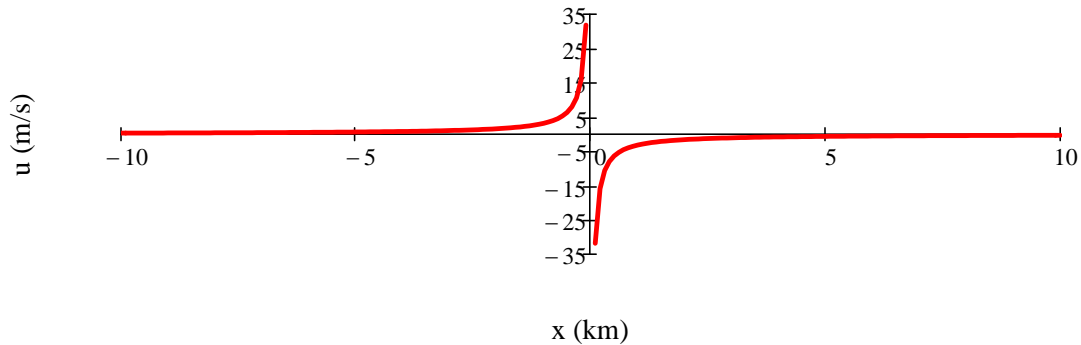
Given: Flow field

Find: Plot of velocity magnitude along axes, and y = x; Equations of streamlines

Solution:

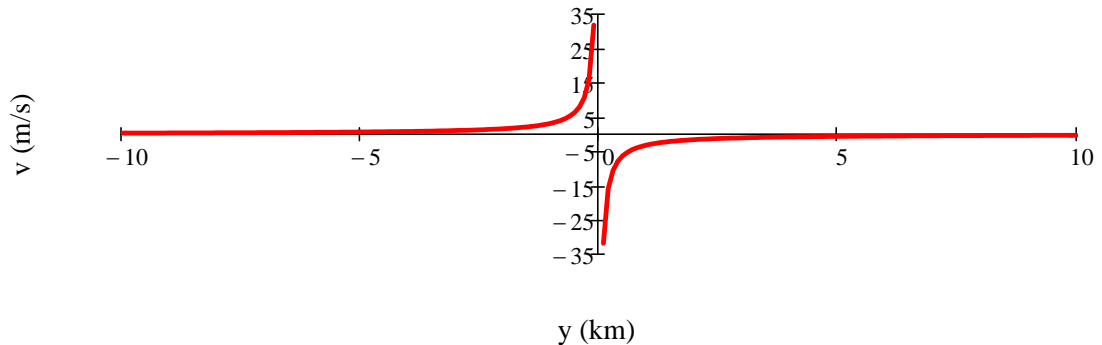
On the x axis, y = 0, so

$$u = -\frac{q \cdot x}{2\cdot\pi\cdot\left(x^2+y^2\right)} = -\frac{q}{2\cdot\pi\cdot x} \qquad v = -\frac{q\cdot y}{2\cdot\pi\cdot\left(x^2+y^2\right)} = 0$$

Plotting

The velocity is very high close to the origin, and falls off to zero. It is also along the axis. This can be plotted in *Excel*.

On the y axis, x = 0, so

$$u = -\frac{q \cdot x}{2\cdot\pi\cdot\left(x^2+y^2\right)} = 0 \qquad v = -\frac{q\cdot y}{2\cdot\pi\cdot\left(x^2+y^2\right)} = -\frac{q}{2\cdot\pi\cdot y}$$

Plotting

The velocity is again very high close to the origin, and falls off to zero. It is also along the axis.

This can also be plotted in *Excel*.

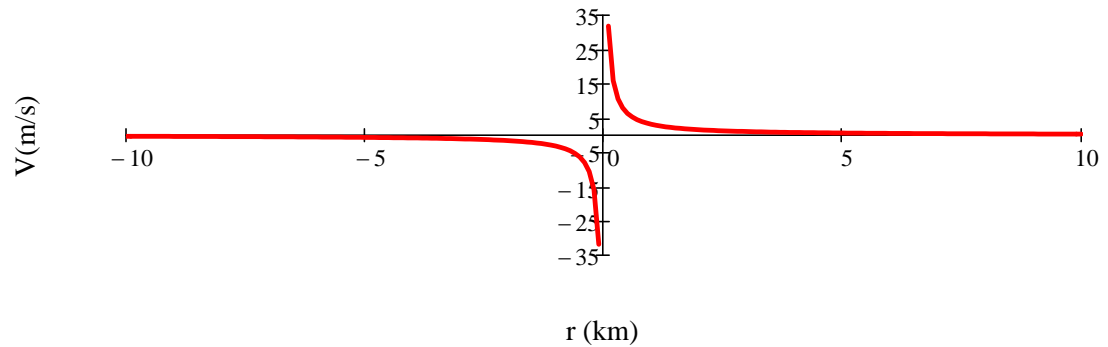
On the y = x axis $u = -\frac{q \cdot x}{2\cdot\pi\cdot(x^2 + x^2)} = -\frac{q}{4\cdot\pi\cdot x}$ $v = -\frac{q \cdot x}{2\cdot\pi\cdot(x^2 + x^2)} = -\frac{q}{4\cdot\pi\cdot x}$

The flow is parallel to line y = x: Slope of line y = x: 1

Slope of trajectory of motion: $\frac{v}{u} = 1$

If we define the radial position: $r = \sqrt{x^2 + y^2}$ then along y = x $r = \sqrt{x^2 + x^2} = \sqrt{2}\cdot x$

Then the magnitude of the velocity along y = x is $V = \sqrt{u^2 + v^2} = \frac{q}{4\cdot\pi}\cdot\sqrt{\frac{1}{x^2} + \frac{1}{x^2}} = \frac{q}{2\cdot\pi\cdot\sqrt{2}\cdot x} = \frac{q}{2\cdot\pi\cdot r}$

Plotting

This can also be plotted in Excel.

For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{-\frac{q\cdot y}{2\cdot\pi\cdot(x^2 + y^2)}}{-\frac{q\cdot x}{2\cdot\pi\cdot(x^2 + y^2)}} = \frac{y}{x}$

So, separating variables $\frac{dy}{y} = \frac{dx}{x}$

Integrating $\ln(y) = \ln(x) + c$

The solution is $y = C\cdot x$ which is the equation of a straight line.

This flow field corresponds to a sink (discussed in Chapter 6).

Problem 2.13

Given: Velocity field $\vec{V} = ax\hat{i} - by\hat{j}$, where $a = b = 1\ s^{-1}$.

Find: (a) Show that particle motion is described by the parametric equations $x_p = c_1 e^{at}$ and $y_p = c_2 e^{-bt}$
(b) Obtain equation of pathline for particle located at (1,2) at $t=0$.
(c) Compare pathline with streamline through same point

Solution

(a) A particle moving in the velocity field $\vec{V} = ax\hat{i} - by\hat{j}$ will have velocity components $u = ax$, $v = -by$

Thus $u_p = \frac{dx}{dt} = ax$ or $\frac{dx}{x} = a\,dt$ and $\int \frac{dx}{x} = \int a\,dt$ ...(1)

$v_p = \frac{dy}{dt} = -by$ or $\frac{dy}{y} = -b\,dt$ and $\int \frac{dy}{y} = -\int b\,dt$ ...(2)

Integrating Eqs. (1) and (2) we obtain

$\ln x = at + \ln c_1$ or $\frac{x}{c_1} = e^{at}$ and $x = c_1 e^{at}$

$\ln y = -bt + \ln c_2$ or $\frac{y}{c_2} = e^{-bt}$ and $y = c_2 e^{-bt}$ ← Q.E.D.

(b) To obtain the equation of the pathline we eliminate t from the parametric equations.

$x = c_1 e^{at}$ $\therefore \ln\frac{x}{c_1} = at$ or $t = \frac{1}{a}\ln\frac{x}{c_1}$

$y = c_2 e^{-bt}$ $\therefore \ln\frac{y}{c_2} = -bt$ or $t = -\frac{1}{b}\ln\frac{y}{c_2}$

Equating expressions for t, we obtain

$\frac{1}{a}\ln\frac{x}{c_1} = -\frac{1}{b}\ln\frac{y}{c_2}$ or $-\frac{b}{a}\ln\frac{x}{c_1} = \ln\frac{y}{c_2}$

Thus $\left(\frac{x}{c_1}\right)^{-b/a} = \frac{y}{c_2}$ or $y\left(\frac{x}{c_1}\right)^{b/a} = c_2$

At $t=0$ $x = 1 = c_1$, $y = 2 = c_2$. Since $a = b$, then the pathline of the particle is $xy = 2$. ← Pathline

(c) The streamline in the x-y plane has slope $\frac{dy}{dx} = \frac{v}{u} = -\frac{b}{a}\frac{y}{x}$

Thus $\frac{dy}{y} + \frac{b}{a}\frac{dx}{x} = 0$. This can be integrated to obtain

$\ln y + \frac{b}{a}\ln x = \text{constant} = \ln c$

Simplifying we obtain $y x^{b/a} = c$. With $b=a$, the equation of the streamline through point (1,2) is then

$xy = 2$ ← Streamline

Problem 2.14

[2]

2.14 A velocity field is given by $\vec{V} = ayt\hat{i} - bx\hat{j}$, where $a = 1$ s^{-2} and $b = 4$ s^{-1}. Find the equation of the streamlines at any time t. Plot several streamlines at $t = 0$ s, $t = 1$ s, and $t = 20$ s.

Given: Velocity field

Find: Equation of streamlines; Plot streamlines

Solution:

Streamlines are given by $\frac{v}{u} = \frac{dy}{dx} = \frac{-b \cdot x}{a \cdot y \cdot t}$

So, separating variables $a \cdot t \cdot y \cdot dy = -b \cdot x \cdot dx$

Integrating $\frac{1}{2} \cdot a \cdot t \cdot y^2 = -\frac{1}{2} \cdot b \cdot x^2 + C$

The solution is $y = \sqrt{C - \frac{b \cdot x^2}{a \cdot t}}$

For $t = 0$ s $\quad x = c \qquad$ For $t = 1$ s $\quad y = \sqrt{C - 4 \cdot x^2} \qquad$ For $t = 20$ s $\quad y = \sqrt{C - \frac{x^2}{5}}$

t = 0

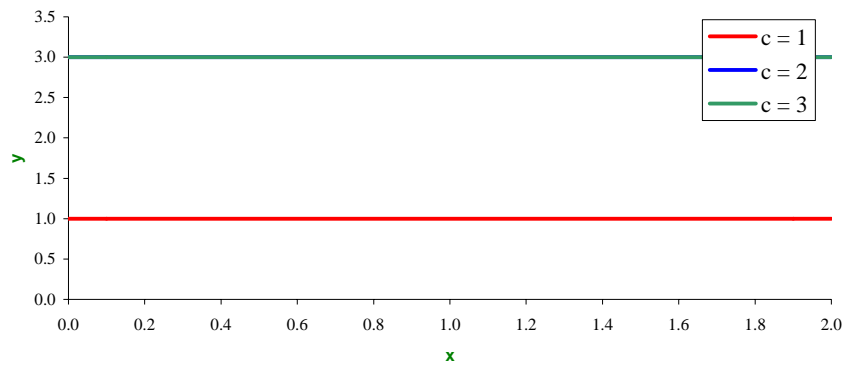
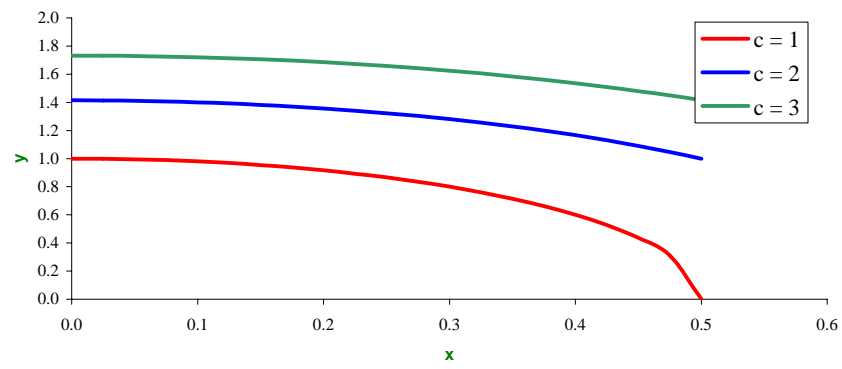
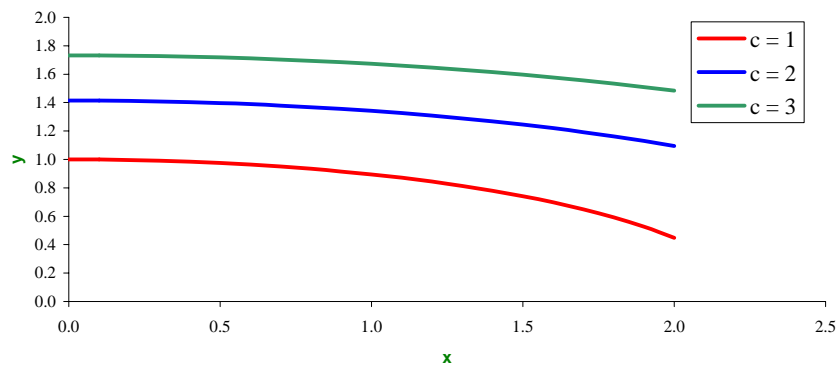
	C = 1	C = 2	C = 3
x	**y**	**y**	**y**
0.00	1.00	2.00	3.00
0.10	1.00	2.00	3.00
0.20	1.00	2.00	3.00
0.30	1.00	2.00	3.00
0.40	1.00	2.00	3.00
0.50	1.00	2.00	3.00
0.60	1.00	2.00	3.00
0.70	1.00	2.00	3.00
0.80	1.00	2.00	3.00
0.90	1.00	2.00	3.00
1.00	1.00	2.00	3.00
1.10	1.00	2.00	3.00
1.20	1.00	2.00	3.00
1.30	1.00	2.00	3.00
1.40	1.00	2.00	3.00
1.50	1.00	2.00	3.00
1.60	1.00	2.00	3.00
1.70	1.00	2.00	3.00
1.80	1.00	2.00	3.00
1.90	1.00	2.00	3.00
2.00	1.00	2.00	3.00

t =1 s

	C = 1	C = 2	C = 3
x	**y**	**y**	**y**
0.000	1.00	1.41	1.73
0.025	1.00	1.41	1.73
0.050	0.99	1.41	1.73
0.075	0.99	1.41	1.73
0.100	0.98	1.40	1.72
0.125	0.97	1.39	1.71
0.150	0.95	1.38	1.71
0.175	0.94	1.37	1.70
0.200	0.92	1.36	1.69
0.225	0.89	1.34	1.67
0.250	0.87	1.32	1.66
0.275	0.84	1.30	1.64
0.300	0.80	1.28	1.62
0.325	0.76	1.26	1.61
0.350	0.71	1.23	1.58
0.375	0.66	1.20	1.56
0.400	0.60	1.17	1.54
0.425	0.53	1.13	1.51
0.450	0.44	1.09	1.48
0.475	0.31	1.05	1.45
0.500	0.00	1.00	1.41

t = 20 s

	C = 1	C = 2	C = 3
x	**y**	**y**	**y**
0.00	1.00	1.41	1.73
0.10	1.00	1.41	1.73
0.20	1.00	1.41	1.73
0.30	0.99	1.41	1.73
0.40	0.98	1.40	1.72
0.50	0.97	1.40	1.72
0.60	0.96	1.39	1.71
0.70	0.95	1.38	1.70
0.80	0.93	1.37	1.69
0.90	0.92	1.36	1.68
1.00	0.89	1.34	1.67
1.10	0.87	1.33	1.66
1.20	0.84	1.31	1.65
1.30	0.81	1.29	1.63
1.40	0.78	1.27	1.61
1.50	0.74	1.24	1.60
1.60	0.70	1.22	1.58
1.70	0.65	1.19	1.56
1.80	0.59	1.16	1.53
1.90	0.53	1.13	1.51
2.00	0.45	1.10	1.48

Streamline Plot (t = 0)
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
y
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
x
c = 1
c = 2
c = 3
Streamline Plot (t = 1s)
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
y
0.0
0.1
0.2
0.3
0.4
0.5
0.6
x
c = 1
c = 2
c = 3
Streamline Plot (t = 20s)
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
y
0.0
0.5
1.0
1.5
2.0
2.5
x
c = 1
c = 2
c = 3

Problem 2.15

2.15 Verify that $x_p = -a\sin(\omega t)$, $y_p = a\cos(\omega t)$ is the equation for the pathlines of particles for the flow field of Problem 2.10. Find the frequency of motion ω as a function of the amplitude of motion, a, and K. Verify that $x_p = -a\sin(\omega t)$, $y_p = a\cos(\omega t)$ is also the equation for the pathlines of particles for the flow field of Problem 2.11, except that ω is now a function of M. Plot typical pathlines for both flow fields and discuss the difference.

Given: Pathlines of particles

Find: Conditions that make them satisfy Problem 2.10 flow field; Also Problem 2.11 flow field; Plot pathlines

Solution:

The given pathlines are $\quad x_p = -a\cdot\sin(\omega\cdot t) \qquad y_p = a\cdot\cos(\omega\cdot t)$

The velocity field of Problem 2.10 is $\quad u = -\dfrac{K\cdot y}{2\cdot\pi\cdot\left(x^2 + y^2\right)} \qquad v = \dfrac{K\cdot x}{2\cdot\pi\cdot\left(x^2 + y^2\right)}$

If the pathlines are correct we should be able to substitute x_p and y_p into the velocity field to find the velocity as a function of time:

$$u = -\frac{K\cdot y}{2\cdot\pi\cdot\left(x^2 + y^2\right)} = -\frac{K\cdot a\cdot\cos(\omega\cdot t)}{2\cdot\pi\cdot\left(a^2\cdot\sin(\omega\cdot t)^2 + a^2\cdot\cos(\omega\cdot t)^2\right)} = -\frac{K\cdot\cos(\omega\cdot t)}{2\cdot\pi\cdot a} \qquad (1)$$

$$v = \frac{K\cdot x}{2\cdot\pi\cdot\left(x^2 + y^2\right)} = -\frac{K\cdot(-a\cdot\sin(\omega\cdot t))}{2\cdot\pi\cdot\left(a^2\cdot\sin(\omega\cdot t)^2 + a^2\cdot\cos(\omega\cdot t)^2\right)} = -\frac{K\cdot\sin(\omega\cdot t)}{2\cdot\pi\cdot a} \qquad (2)$$

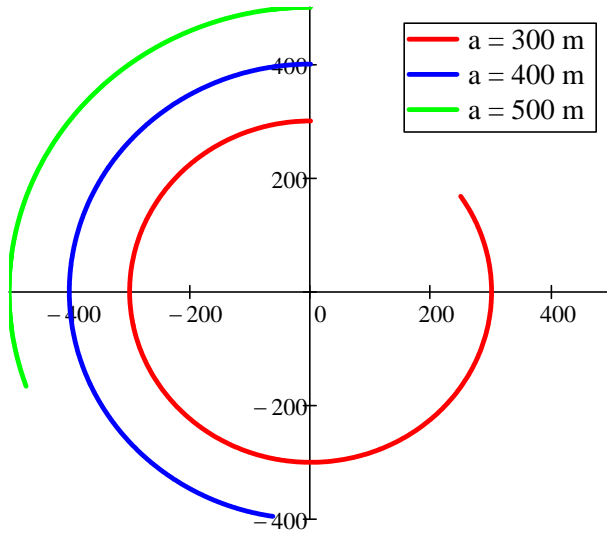
We should also be able to find the velocity field as a function of time from the pathline equations (Eq. 2.9):

$$\frac{dx_p}{dt} = u \qquad \frac{dx_p}{dt} = v \qquad (2.9)$$

$$u = \frac{dx_p}{dt} = -a\cdot\omega\cdot\cos(\omega\cdot t) \qquad v = \frac{dy_p}{dt} = -a\cdot\omega\cdot\sin(\omega\cdot t) \qquad (3)$$

Comparing Eqs. 1, 2 and 3 $\quad u = -a\cdot\omega\cdot\cos(\omega\cdot t) = -\dfrac{K\cdot\cos(\omega\cdot t)}{2\cdot\pi\cdot a} \qquad v = -a\cdot\omega\cdot\sin(\omega\cdot t) = -\dfrac{K\cdot\sin(\omega\cdot t)}{2\cdot\pi\cdot a}$

Hence we see that $\quad a\cdot\omega = \dfrac{K}{2\cdot\pi\cdot a} \quad$ or $\quad \omega = \dfrac{K}{2\cdot\pi\cdot a^2} \quad$ for the pathlines to be correct.

The pathlines are

To plot this in Excel, compute x_p and y_p for t ranging from 0 to 60 s, with ω given by the above formula. Plot y_p versus x_p. Note that outer particles travel much slower!

This is the free vortex flow discussed in Example 5.6

The velocity field of Problem 2.11 is $u = -\frac{M \cdot y}{2 \cdot \pi}$ $v = \frac{M \cdot x}{2 \cdot \pi}$

If the pathlines are correct we should be able to substitute x_p and y_p into the velocity field to find the velocity as a function of time:

$$u = -\frac{M \cdot y}{2 \cdot \pi} = -\frac{M \cdot (a \cdot \cos(\omega \cdot t))}{2 \cdot \pi} = -\frac{M \cdot a \cdot \cos(\omega \cdot t)}{2 \cdot \pi} \qquad (4)$$

$$v = \frac{M \cdot x}{2 \cdot \pi} = \frac{M \cdot (-a \cdot \sin(\omega \cdot t))}{2 \cdot \pi} = -\frac{M \cdot a \cdot \sin(\omega \cdot t)}{2 \cdot \pi} \qquad (5)$$

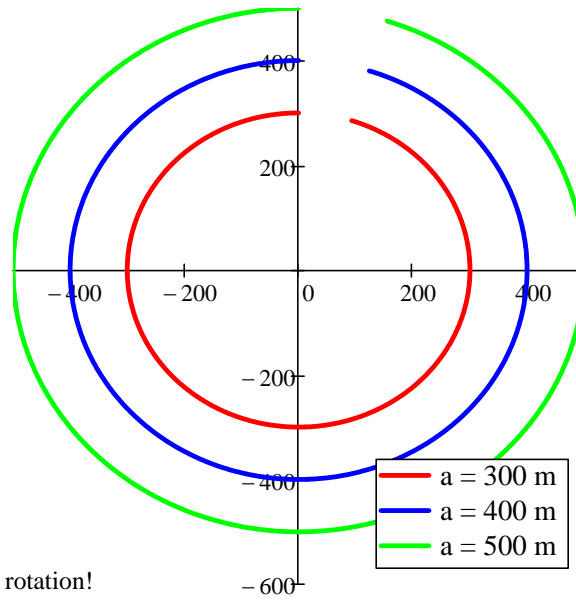
Recall that $u = \frac{dx_p}{dt} = -a \cdot \omega \cdot \cos(\omega \cdot t)$ $v = \frac{dy_p}{dt} = -a \cdot \omega \cdot \sin(\omega \cdot t)$ (3)

Comparing Eqs. 1, 4 and 5 $u = -a \cdot \omega \cdot \cos(\omega \cdot t) = -\frac{M \cdot a \cdot \cos(\omega \cdot t)}{2 \cdot \pi}$ $v = -a \cdot \omega \cdot \sin(\omega \cdot t) = -\frac{M \cdot a \cdot \sin(\omega \cdot t)}{2 \cdot \pi}$

Hence we see that $\omega = \frac{M}{2 \cdot \pi}$ for the pathlines to be correct.

The pathlines

To plot this in Excel, compute x_p and y_p for t ranging from 0 to 75 s, with ω given by the above formula. Plot y_p versus x_p. Note that outer particles travel faster!

This is the forced vortex flow discussed in Example 5.6

Note that this is rigid body rotation!

Problem 2.16

[2]

2.16 Air flows downward toward an infinitely wide horizontal flat plate. The velocity field is given by $\vec{V} = (ax\hat{i} - ay\hat{j})(2 + \cos\ \omega t)$, where $a = 5\ \text{s}^{-1}$, $\omega = 2\pi\ \text{s}^{-1}$, x and y (measured in meters) are horizontal and vertically upward, respectively, and t is in s. Obtain an algebraic equation for a streamline at $t = 0$. Plot the streamline that passes through point $(x, y) = (3, 3)$ at this instant. Will the streamline change with time? Explain briefly. Show the velocity vector on your plot at the same point and time. Is the velocity vector tangent to the streamline? Explain.

Given: Time-varying velocity field

Find: Streamlines at t = 0 s; Streamline through (3,3); velocity vector; will streamlines change with time

Solution:

For streamlines $\quad \frac{v}{u} = \frac{dy}{dx} = -\frac{a\cdot y\cdot(2+\cos(\omega\cdot t))}{a\cdot x\cdot(2+\cos(\omega\cdot t))} = -\frac{y}{x}$

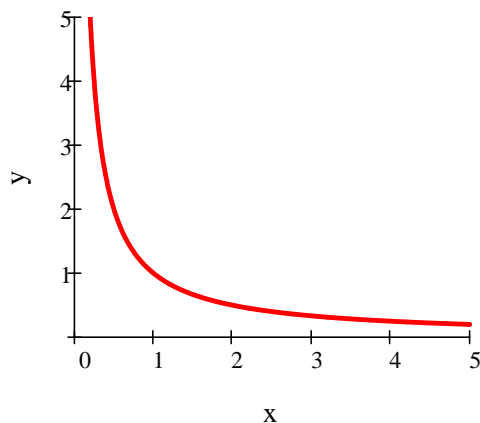
At t = 0 (actually all times!) $\quad \frac{dy}{dx} = -\frac{y}{x}$

So, separating variables $\quad \frac{dy}{y} = -\frac{dx}{x}$

Integrating $\quad \ln(y) = -\ln(x) + c$

The solution is $\quad y = \frac{C}{x}$ $\quad$ which is the equation of a hyperbola.

For the streamline through point (3,3) $\quad C = \frac{3}{3} \quad C = 1 \quad$ and $\quad y = \frac{1}{x}$

The streamlines will not change with time since dy/dx does not change with time.

At t = 0 $\quad u = a\cdot x\cdot(2+\cos(\omega\cdot t)) = 5\cdot\frac{1}{s}\times 3\cdot m\times 3$

$u = 45\cdot\frac{m}{s}$

$v = -a\cdot y\cdot(2+\cos(\omega\cdot t)) = 5\cdot\frac{1}{s}\times 3\cdot m\times 3$

$v = -45\cdot\frac{m}{s}$

The velocity vector is tangent to the curve;

Tangent of curve at (3,3) is $\quad \frac{dy}{dx} = -\frac{y}{x} = -1$

Direction of velocity at (3,3) is $\quad \frac{v}{u} = -1$

This curve can be plotted in Excel.

Problem 2.17

Given: Velocity field $\vec{V} = Bx(1+At)\hat{i} + Cy\hat{j}$, with $A = 0.5\,s^{-1}$, $B = C = 1\,s^{-1}$; coordinates measured in meters.

Plot: the pathline of the particle that passed through the point (1, 1, 0) at time t=0.
Compare with the streamlines through the same point at the instants t = 0, 1, and 2s

Solution:

For a particle, $u = \frac{dx}{dt}$ and $v = dy/dt$
then

$$u = Bx(1+At) = \frac{dx}{dt}, \quad \int_{x_0}^{x} \frac{dx}{x} = \int_0^t B(1+At)\,dt$$

$$\ln \frac{x}{x_0} = B\left[t + \tfrac{1}{2}At^2\right]_0^t = B\left[t + \tfrac{1}{2}At^2\right] \quad \therefore x = x_0 e^{B(t+\frac{1}{2}At^2)}$$

$$v = Cy = dy/dt, \quad \int_0^t C\,dt = \int_{y_0}^{y} \frac{dy}{y} \quad \therefore y = y_0 e^{Ct}$$ } x, y

The pathline may be plotted by varying t as shown below

The streamline is found (at given t) from $\left.\frac{dy}{dx}\right)_{streamline} = \frac{v}{u}$
Then

$$\frac{dy}{dx} = \frac{Cy}{Bx(1+At)} \quad \text{and} \quad (1+At)\frac{dy}{y} = \frac{C}{B}\frac{dx}{x}$$

and

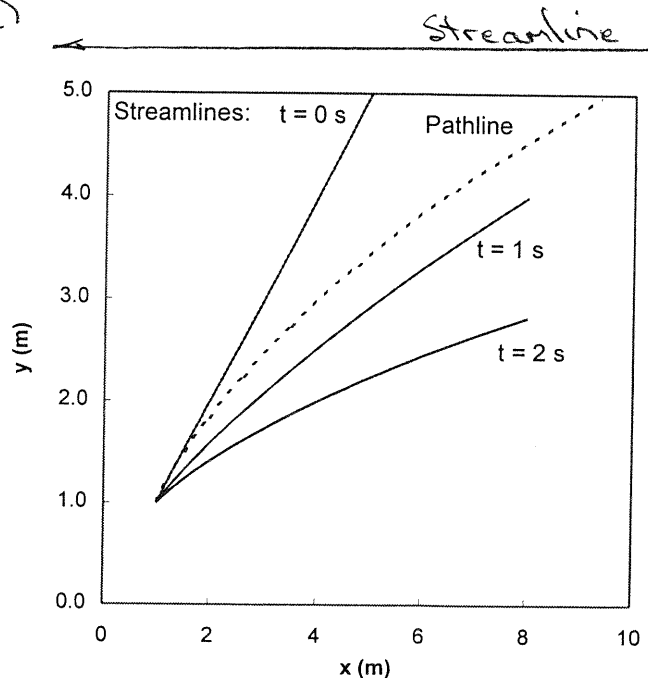
$$(1+At)\ln y = \frac{C}{B}\ln x + \ln c_1, \quad c_1 x^{C/B} = y^{(1+At)}$$

Streamline through point (1, 1, 0) gives $c_1 = 1$. Then on substituting for A, B, and C we obtain

$$x = y^{(1+0.5t)}$$

At $t=0$, $x = y$
$t = 1s$, $x = y^{1.5}$
$t = 2s$, $x = y^2$

Streamline

Problem 2.18

Given: Velocity field $\vec{V} = A\hat{i} + Bt\hat{j}$, where $A = 2$ m/s, $B = 0.6$ m/s², and coordinates are in meters.

Find: (a) position functions for particle located at $(x_0, y_0) = 1,1$ at time $t = 0$

(b) algebraic expression for pathline of particle of part (a).

Plot: the pathline and compare with streamline through the same point at $t = 0, 1, 2$ s.

Solution:

For a particle $u = \frac{dx}{dt}$ and $v = \frac{dy}{dt}$

Then,

$u = A = dx/dt, \quad \int_{x_0}^{x} dx = \int_0^t A\,dt \quad$ and $\quad x = x_0 + At \qquad$ (1a)

$v = Bt = dy/dt, \quad \int_{y_0}^{y} dy = \int_0^t Bt\,dt \quad$ and $\quad y = y_0 + \frac{1}{2}Bt^2 \qquad$ (1b)

Substituting values for A, B, x_0, and y_0, then

$x = 1 + 2t$ and $y = 1 + 0.30\,t^2$ ← x, y

(b) To determine the pathline for the particle, we eliminate t from the parametric equations of part (a).

From Eq. 1a, $t = (x - x_0)/A$. Substituting into Eq. (1b), then

$$y - y_0 = \frac{B(x - x_0)^2}{2A^2} \qquad (2)$$

Substituting numerical values,

$y = 1 + 0.075\,(x - 1)^2$ ← pathline

(c) The streamline is found (at given t) from $\left.\frac{dy}{dx}\right)_s = \frac{v}{u}$

$$\left.\frac{dy}{dx}\right)_{\text{streamline}} = \frac{v}{u} = \frac{Bt}{A}$$

$\therefore\ y = \frac{Bt}{A}x + C$

Through point (1,1)

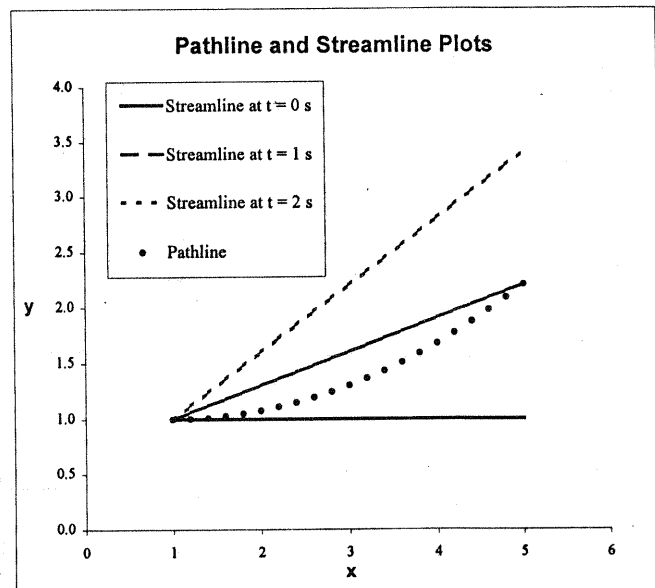
$C = 1 - \frac{0.6}{2}t = 1 - 0.3t$

$y = 1 + 0.3\,t\,(x - 1)$ ←

Streamline through (1,1)

@ $t = 0$, $y = 1$

$t = 1$ s, $y = 1 + 0.3\,(x - 1)$

$t = 2$ s, $y = 1 + 0.6\,(x - 1)$

Problem 2.19

2.19 A velocity field is given by $\vec{V} = axt\hat{i} - by\hat{j}$, where $a = 0.1$ s^{-2} and $b = 1$ s^{-1}. For the particle that passes through the point $(x, y) = (1, 1)$ at instant $t = 0$ s, plot the pathline during the interval from $t = 0$ to $t = 3$ s. Compare with the streamlines plotted through the same point at the instants $t = 0$, 1, and 2 s.

Given: Velocity field

Find: Plot pathlines and streamlines

Solution:

Pathlines are given by $\dfrac{dx}{dt} = u = a \cdot x \cdot t$ $\quad$ $\dfrac{dy}{dt} = v = -b \cdot y$

So, separating variables $\dfrac{dx}{x} = a \cdot t \cdot dt$ $\quad$ $\dfrac{dy}{y} = -b \cdot dt$

Integrating $\ln(x) = \dfrac{1}{2} \cdot a \cdot t^2 + c_1$ $\quad$ $\ln(y) = -b \cdot t + c_2$

For initial position (x_0, y_0) $\quad$ $x = x_0 \cdot e^{\frac{a}{2} \cdot t^2}$ $\quad$ $y = y_0 \cdot e^{-b \cdot t}$

Using the given data, and IC $(x_0, y_0) = (1,1)$ at $t = 0$

$$x = e^{0.05 \cdot t^2} \qquad y = e^{-t}$$

Streamlines are given by $\dfrac{v}{u} = \dfrac{dy}{dx} = \dfrac{-b \cdot y}{a \cdot x \cdot t}$

So, separating variables $\dfrac{dy}{y} = -\dfrac{b}{a \cdot t} \cdot \dfrac{dx}{x}$

Integrating $\ln(y) = -\dfrac{b}{a \cdot t} \cdot \ln(x) + C$

The solution is $y = C \cdot x^{-\frac{b}{a \cdot t}}$

For streamline at (1,1) at $t = 0$ s $\quad$ $x = c$

For streamline at (1,1) at $t = 1$ s $\quad$ $y = x^{-10}$

For streamline at (1,1) at $t = 2$ s $\quad$ $y = x^{-5}$

Pathline

t	x	y
0.00	1.00	1.00
0.25	1.00	0.78
0.50	1.01	0.61
0.75	1.03	0.47
1.00	1.05	0.37
1.25	1.08	0.29
1.50	1.12	0.22
1.75	1.17	0.17
2.00	1.22	0.14
2.25	1.29	0.11
2.50	1.37	0.08
2.75	1.46	0.06
3.00	1.57	0.05
3.25	1.70	0.04
3.50	1.85	0.03
3.75	2.02	0.02
4.00	2.23	0.02
4.25	2.47	0.01
4.50	2.75	0.01
4.75	3.09	0.01
5.00	3.49	0.01

Streamlines

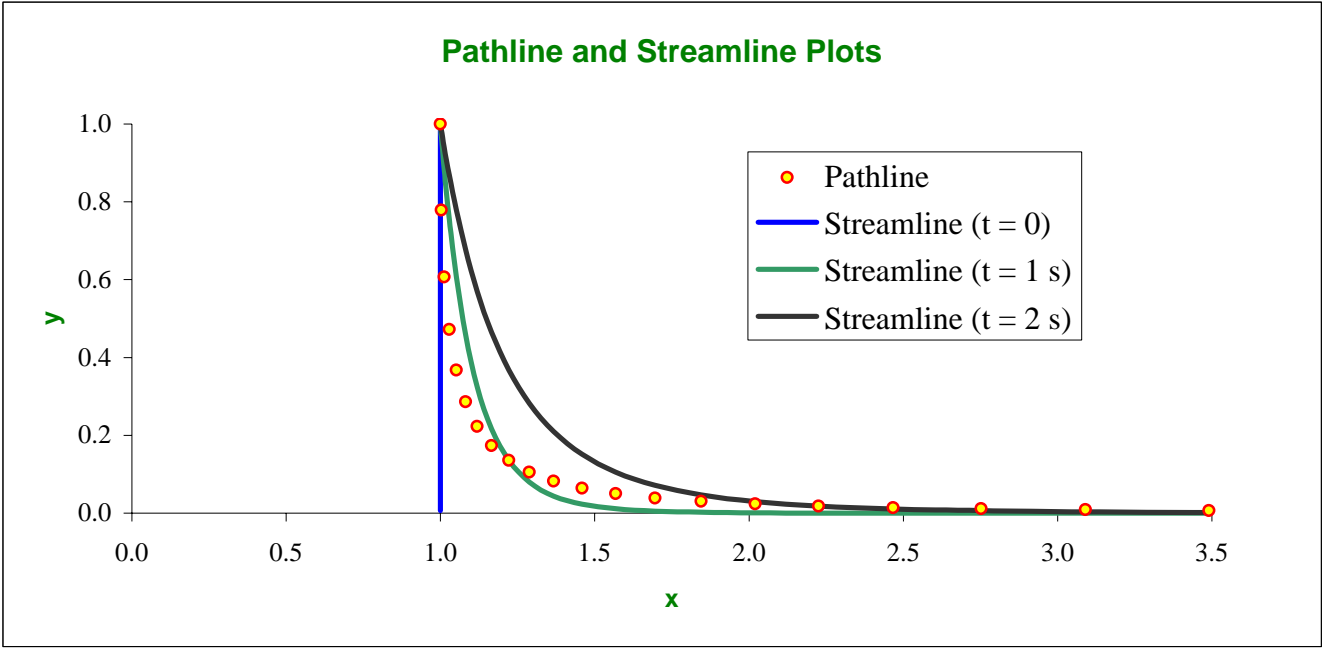
t = 0

x	y
1.00	1.00
1.00	0.78
1.00	0.61
1.00	0.47
1.00	0.37
1.00	0.29
1.00	0.22
1.00	0.17
1.00	0.14
1.00	0.11
1.00	0.08
1.00	0.06
1.00	0.05
1.00	0.04
1.00	0.03
1.00	0.02
1.00	0.02
1.00	0.01
1.00	0.01
1.00	0.01
1.00	0.01

t = 1 s

x	y
1.00	1.00
1.00	0.97
1.01	0.88
1.03	0.75
1.05	0.61
1.08	0.46
1.12	0.32
1.17	0.22
1.22	0.14
1.29	0.08
1.37	0.04
1.46	0.02
1.57	0.01
1.70	0.01
1.85	0.00
2.02	0.00
2.23	0.00
2.47	0.00
2.75	0.00
3.09	0.00
3.49	0.00

t = 2 s

x	y
1.00	1.00
1.00	0.98
1.01	0.94
1.03	0.87
1.05	0.78
1.08	0.68
1.12	0.57
1.17	0.47
1.22	0.37
1.29	0.28
1.37	0.21
1.46	0.15
1.57	0.11
1.70	0.07
1.85	0.05
2.02	0.03
2.23	0.02
2.47	0.01
2.75	0.01
3.09	0.00
3.49	0.00

Problem 2.20

Given: Velocity field $\vec{V} = ax\hat{i} + by(1+ct)\hat{j}$, where $a = b = 2\,s^{-1}$, $c = 0.4\,s^{-1}$, and coordinates are measured in meters

Plot: the pathline (during the interval $0 \le t \le 1.5\,s$) of the particle that passed through the point $(x_0, y_0) = (1,1)$ at time $t = 0$.
Compare with the streamline plotted through the same point at $t = 0, 1$, and $1.5\,s$

Solution:

For a particle, $u = dx/dt$ and $v = dy/dt$

then $u = dx/dt = ax$, $\int_{x_0}^{x} \frac{dx}{x} = \int_0^t a\,dt$, $\ln\frac{x}{x_0} = at$, $x = x_0 e^{at}$

Also $v = dy/dt = by(1+ct)$, $\int_{y_0}^{y} \frac{dy}{y} = \int_0^t b(1+ct)\,dt$

$\ln y/y_0 = b(t + \frac{1}{2}ct^2)$ $\qquad y = y_0 e^{b(t+\frac{1}{2}ct^2)}$

Substituting for a, b, c, x_0, and y_0

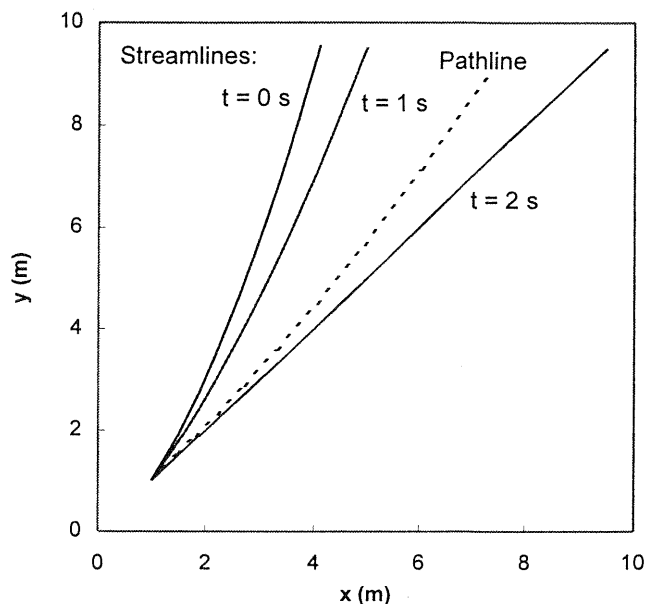
$x = e^{2t}$, $y = e^{(2t+0.4t^2)}$ ← (x,y) pathline

The streamline is found (at given t) from $dy/dx)_s = v/u$

then $\frac{dy}{dx} = \frac{by(1+ct)}{ax}$, $\int_{y_0}^{y} \frac{dy}{y} = \int_{x_0} \frac{b(1+ct)}{a}\frac{dx}{x}$, $\ln\frac{y}{y_0} = \frac{b(1+ct)}{a}\ln\frac{x}{x_0}$

$y = y_0\left(\frac{x}{x_0}\right)^{\frac{b(1+ct)}{a}}$. Substituting for a, b, c, x_0, and y_0

$y = x^{(1+0.4t)}$ ← streamline

At $t=0$, $y = x$
$t = 1\,s$, $y = x^{1.4}$
$t = 1.5\,s$, $y = x^{1.6}$

Problem 2.21

2.21 Consider the flow field $\vec{V} = axt\hat{i} + b\hat{j}$, where $a = 0.1\ \text{s}^{-2}$ and $b = 4$ m/s. Coordinates are measured in meters. For the particle that passes through the point $(x, y) = (3, 1)$ at the instant $t = 0$, plot the pathline during the interval from $t = 0$ to 3 s. Compare this pathline with the streamlines plotted through the same point at the instants $t = 1$, 2, and 3 s.

Given: Flow field

Find: Pathline for particle starting at (3,1); Streamlines through same point at t = 1, 2, and 3 s

Solution:

For particle paths $\quad \dfrac{dx}{dt} = u = a\cdot x\cdot t \quad$ and $\quad \dfrac{dy}{dt} = v = b$

Separating variables and integrating $\quad \dfrac{dx}{x} = a\cdot t\cdot dt \quad$ or $\quad \ln(x) = \dfrac{1}{2}\cdot a\cdot t^2 + c_1$

$dy = b\cdot dt \quad$ or $\quad y = b\cdot t + c_2$

Using initial condition (x,y) = (3,1) and the given values for a and b

$c_1 = \ln(3\cdot m) \quad$ and $\quad c_2 = 1\cdot m$

The pathline is then $\quad x = 3\cdot e^{0.05\cdot t^2} \quad$ and $\quad y = 4\cdot t + 1$

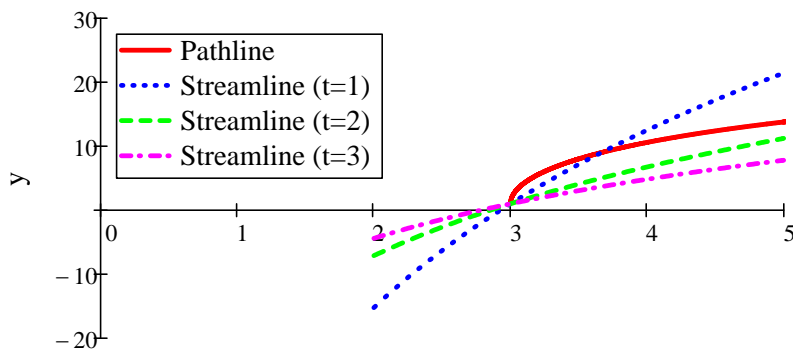
For streamlines (at any time t) $\quad \dfrac{v}{u} = \dfrac{dy}{dx} = \dfrac{b}{a\cdot x\cdot t}$

So, separating variables $\quad dy = \dfrac{b}{a\cdot t}\cdot\dfrac{dx}{x}$

Integrating $\quad y = \dfrac{b}{a\cdot t}\cdot\ln(x) + c$

We are interested in instantaneous streamlines at various times that always pass through point (3,1). Using a and b values:

$$c = y - \frac{b}{a\cdot t}\cdot\ln(x) = 1 - \frac{4}{0.1\cdot t}\cdot\ln(3)$$

The streamline equation is $\quad y = 1 + \dfrac{40}{t}\cdot\ln\left(\dfrac{x}{3}\right)$

These curves can be plotted in Excel.

Problem 2.22

2.22 Consider the garden hose of Fig. 2.5. Suppose the velocity field is given by $\vec{V} = u_0\hat{i} + v_0 \sin[\omega(t - x/u_0)]\hat{j}$, where the x direction is horizontal and the origin is at the mean position of the hose, $u_0 = 10$ m/s, $v_0 = 2$ m/s, and $\omega = 5$ cycle/s. Find and plot on one graph the instantaneous streamlines that pass through the origin at $t = 0$ s, 0.05 s, 0.1 s, and 0.15 s. Also find and plot on one graph the pathlines of particles that left the origin at the same four times.

Given: Velocity field

Find: Plot streamlines that are at origin at various times and pathlines that left origin at these times

Solution:

For streamlines
$$\frac{v}{u} = \frac{dy}{dx} = \frac{v_0 \cdot \sin\left[\omega\cdot\left(t - \frac{x}{u_0}\right)\right]}{u_0}$$

So, separating variables (t=const)
$$dy = \frac{v_0 \cdot \sin\left[\omega\cdot\left(t - \frac{x}{u_0}\right)\right]}{u_0}\cdot dx$$

Integrating
$$y = \frac{v_0 \cdot \cos\left[\omega\cdot\left(t - \frac{x}{u_0}\right)\right]}{\omega} + c$$

Using condition y = 0 when x = 0
$$y = \frac{v_0 \cdot \left[\cos\left[\omega\cdot\left(t - \frac{x}{u_0}\right)\right] - \cos(\omega\cdot t)\right]}{\omega}$$

This gives streamlines y(x) at each time t

For particle paths, first find x(t)
$$\frac{dx}{dt} = u = u_0$$

Separating variables and integrating $\quad dx = u_0\cdot dt \quad$ or $\quad x = u_0\cdot t + c_1$

Using initial condition x = 0 at t = τ $\quad c_1 = -u_0\cdot\tau \qquad x = u_0\cdot(t - \tau)$

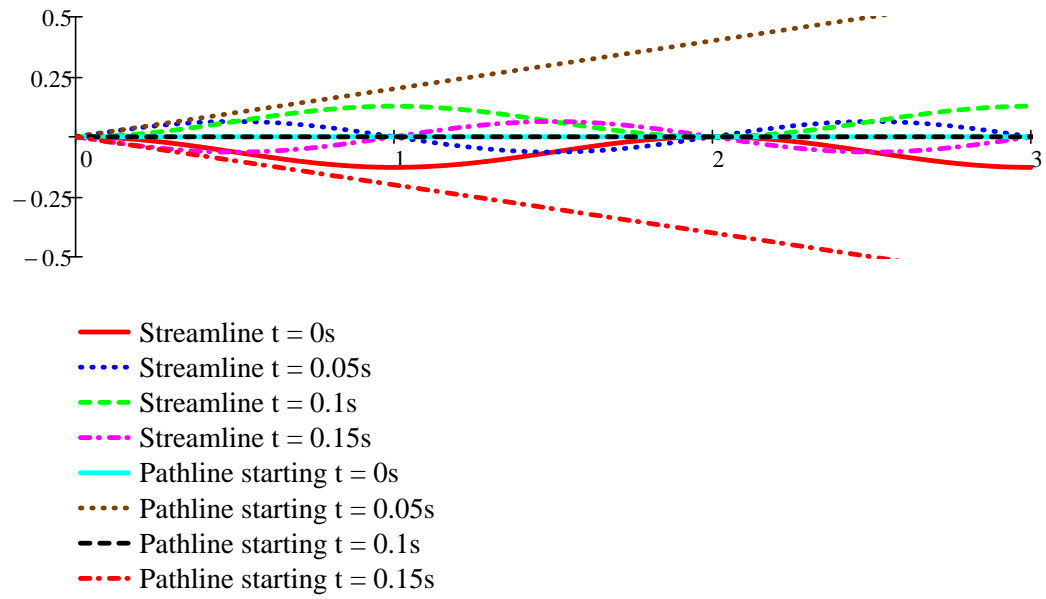
For y(t) we have
$$\frac{dy}{dt} = v = v_0\cdot\sin\left[\omega\cdot\left(t - \frac{x}{u_0}\right)\right] \quad \text{so} \quad \frac{dy}{dt} = v = v_0\cdot\sin\left[\omega\cdot\left[t - \frac{u_0\cdot(t-\tau)}{u_0}\right]\right]$$

and
$$\frac{dy}{dt} = v = v_0\cdot\sin(\omega\cdot\tau)$$

Separating variables and integrating $\quad dy = v_0\cdot\sin(\omega\cdot\tau)\cdot dt \qquad y = v_0\cdot\sin(\omega\cdot\tau)\cdot t + c_2$

Using initial condition y = 0 at t = τ $\quad c_2 = -v_0\cdot\sin(\omega\cdot\tau)\cdot\tau \qquad y = v_0\cdot\sin(\omega\cdot\tau)\cdot(t-\tau)$

The pathline is then

$x(t,\tau) = u_0\cdot(t-\tau) \qquad y(t,\tau) = v_0\cdot\sin(\omega\cdot\tau)\cdot(t-\tau)$ These terms give the path of a particle (x(t),y(t)) that started at t = τ.

The streamlines are sinusoids; the pathlines are straight (once a water particle is fired it travels in a straight line).

These curves can be plotted in *Excel.*

Problem 2.23

2.23 Using the data of Problem 2.22, find and plot the streakline shape produced after the first second of flow.

Given: Velocity field

Find: Plot streakline for first second of flow

Solution:

Following the discussion leading up to Eq. 2.10, we first find equations for the pathlines in form

$$x_p(t) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_p(t) = y(t, x_0, y_0, t_0)$$

where x_0, y_0 is the position of the particle at $t = t_0$, and re-interprete the results as streaklines

$$x_{st}(t_0) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_{st}(t_0) = y(t, x_0, y_0, t_0)$$

which gives the streakline at t, where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

For particle paths, first find x(t) $\qquad \dfrac{dx}{dt} = u = u_0$

Separating variables and integrating $\qquad dx = u_0 \cdot dt \quad$ or $\quad x = x_0 + u_0 \cdot (t - t_0)$

For y(t) we have $\qquad \dfrac{dy}{dt} = v = v_0 \cdot \sin\left[\omega \cdot \left(t - \dfrac{x}{u_0}\right)\right] \quad$ so $\quad \dfrac{dy}{dt} = v = v_0 \cdot \sin\left[\omega \cdot \left[t - \dfrac{x_0 + u_0 \cdot (t - t_0)}{u_0}\right]\right]$

and $\qquad \dfrac{dy}{dt} = v = v_0 \cdot \sin\left[\omega \cdot \left(t_0 - \dfrac{x_0}{u_0}\right)\right]$

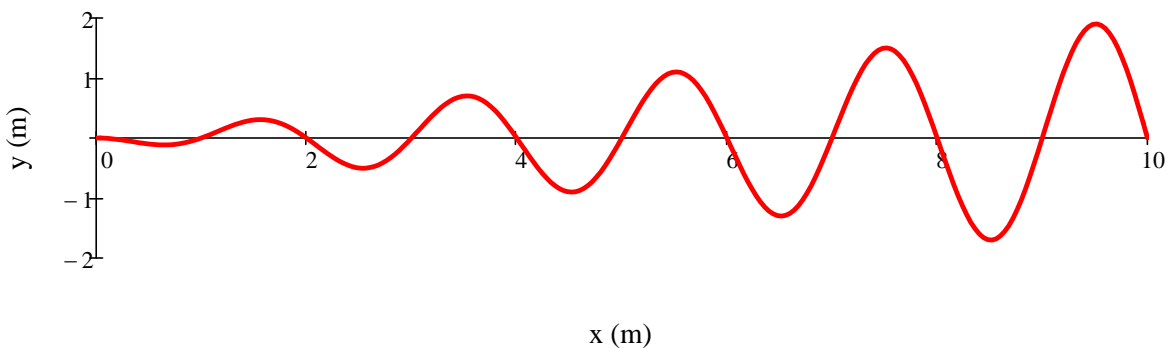
Separating variables and integrating $\qquad dy = v_0 \cdot \sin\left[\omega \cdot \left(t_0 - \dfrac{x_0}{u_0}\right)\right] \cdot dt \qquad y = y_0 + v_0 \cdot \sin\left[\omega \cdot \left(t_0 - \dfrac{x_0}{u_0}\right)\right] \cdot (t - t_0)$

The streakline is then $\qquad x_{st}(t_0) = x_0 + u_0 (t - t_0) \qquad y_{st}(t_0) = y_0 + v_0 \cdot \sin\left[\omega \cdot \left(t_0 - \dfrac{x_0}{u_0}\right)\right] \cdot (t - t_0)$

With $\qquad x_0 = y_0 = 0$

$$x_{st}(t_0) = u_0 \cdot (t - t_0) \qquad y_{st}(t_0) = v_0 \cdot \sin[\omega \cdot (t_0)] \cdot (t - t_0)$$

Streakline for First Second

x (m)

This curve can be plotted in *Excel*. For t = 1, t_0 ranges from 0 to t.

Problem 2.24

Given: Velocity field $\vec{V} = Bx(1+At)\hat{i} + Cy\hat{j}$, with $A = 0.5\,s^{-1}$, $B = C = 1\,s^{-1}$; coordinates measured in meters

Plot: the streakline formed by particles that passed through point $(x_0, y_0, z_0) = (1,1,0)$ during interval from $t=0$ to $t = 3\,s$.
Compare with streamlines through point at $t = 0, 1,$ and $2\,s$

Solution

Streakline at $t = 3\,s$ connects particles that passed through point $(1,1,0)$ at earlier times $t_0 = 0, 1,$ and $2\,s$

For a particle, $u = \frac{dx}{dt}$ and $v = dy/dt$

then

$$u = Bx(1+At) = \frac{dx}{dt}, \quad \int_{x_0}^{x} \frac{dx}{x} = \int_{t_0}^{t} B(1+At)\,dt$$

$$\therefore \ln\frac{x}{x_0} = B\left[t + \tfrac{1}{2}At^2\right]_{t_0}^{t} = B\left[(t-t_0) + \tfrac{1}{2}AB(t^2-t_0^2)\right]$$

$$x = x_0 e^{B[(t-t_0) + \frac{1}{2}AB(t^2-t_0^2)]} \qquad (1a)$$

Also $v = Cy = dy/dt$, $\int_{t_0}^{t} C\,dt = \int_{y_0}^{y} \frac{dy}{y}$, $\therefore y = y_0 e^{C(t-t_0)}$ (1b)

The velocity vector is tangent to the streamline

$$\left.\frac{dy}{dx}\right)_{\text{streamline}} = \frac{v}{u} = \frac{Cy}{Bx(1+At)} \quad \text{and} \quad (1+At)\frac{dy}{y} = \frac{C}{B}\frac{dx}{x}$$

Then

$$(1+At)\ln y = \frac{C}{B}\ln x + \ln c_1 \quad \text{and} \quad c_1 x^{C/B} = y^{(1+At)}$$

Streamline through point $(1,1,0)$ gives $c_1 = 1$. Then on substituting for $A, B,$ and C we obtain

$$x = y^{(1+0.5t)} \qquad \text{Streamline}$$

At $t = 0$: $x = y$
$t = 1\,s$: $x = y^{1.5}$
$t = 2\,s$: $x = y^2$

} These streamlines through $(1,1,0)$ are shown on the plot

Points on the streakline have coordinates given by Eqs 1a & b

$$x = x_0 e^{B[(t-t_0) + \frac{1}{2}AB(t^2-t_0^2)]} \qquad y = y_0 e^{C(t-t_0)}$$

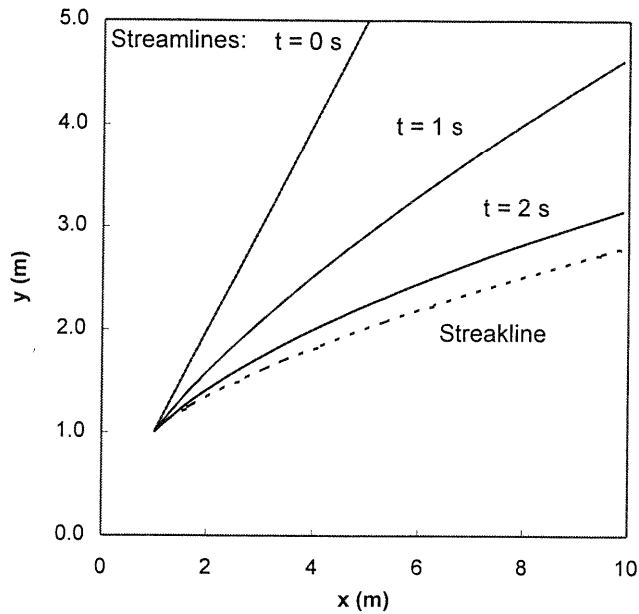
Substituting for $A, B,$ and C

$$x = x_0 e^{[(t-t_0) + 0.25(t^2-t_0^2)]} \qquad y = y_0 e^{(t-t_0)}$$

The streakline through $(x_0, y_0) = (1,1)$ at time $t = 3\,s$ is obtained by substituting $x_0 = 1$, $y_0 = 1$, $t = 3\,s$ and varying t_0 in these equations.

Thus, $x = e^{[(3-t_0)+0.25(9-t_0^2)]}$

$y = e^{(3-t_0)}$

give points (obtained by varying t_0) on the streakline through (1,1,0) at $t = 3$ s

Given: Velocity field $\vec{V} = ax(1+bt)\hat{i} + cy\hat{j}$, where $a = c = 1\,s^{-1}$, $b = 0.2\,s^{-1}$, and coordinates are measured in meters.

Plot: the streakline that passes through the point $(x_0, y_0) = (1,1)$ during the interval $0 \le t \le 3\,s$.
Compare with the streamlines plotted through the same point at $t = 0$, 1, and 2 s

Solution:

Streakline at $t = 3\,s$ connects particles that passed through point (x_0, y_0) at earlier times $\tau = 0, 1, 2$, and 3 s.

For a particle, $u = dx/dt$ and $v = dy/dt$

Then

$$u = ax(1+bt) = \frac{dx}{dt} \quad \text{and} \quad \int_{x_0}^{x} \frac{dx}{x} = \int_{\tau}^{t} a(1+bt)\,dt$$

$$\ln\frac{x}{x_0} = a\left(t + \frac{b}{2}t^2\right)\Big]_{\tau}^{t} = a\left[(t-\tau) + \frac{b}{2}(t^2-\tau^2)\right]$$

$$x = x_0\, e^{a\left[(t-\tau) + \frac{b}{2}(t^2-\tau^2)\right]}$$

Also $v = \dfrac{dy}{dt} = cy$, $\displaystyle\int_{y_0}^{y}\frac{dy}{y} = \int_{\tau}^{t} c\,dt$, $\ln\dfrac{y}{y_0} = c(t-\tau)$, $y = y_0\, e^{c(t-\tau)}$

Substituting for a, b, c, x_0, and y_0, gives:

$$x = e^{[t-\tau] + 0.1(t^2-\tau^2)}, \quad y = e^{(t-\tau)} \qquad \longleftarrow (x,y) \text{ streakline}$$

The streakline may be plotted by substituting values for τ in the range $0 \le \tau \le 3\,s$ as shown below.

The streamline is found (at given t) from $dy/dx)_s = \dfrac{v}{u}$

Thus

$$dy/dx = \frac{cy}{ax(1+bt)} \quad \text{and} \quad \int_{y_0}^{y}\frac{dy}{y} = \int_{x_0}^{x}\frac{c}{a(1+bt)}\frac{dx}{x}$$

$$\ln\frac{y}{y_0} = \frac{c}{a(1+bt)}\ln\frac{x}{x_0} \quad \text{or} \quad y = y_0\left[\frac{x}{x_0}\right]^{c/a(1+bt)}$$

Substituting values for x_0, y_0, a, b, c, then

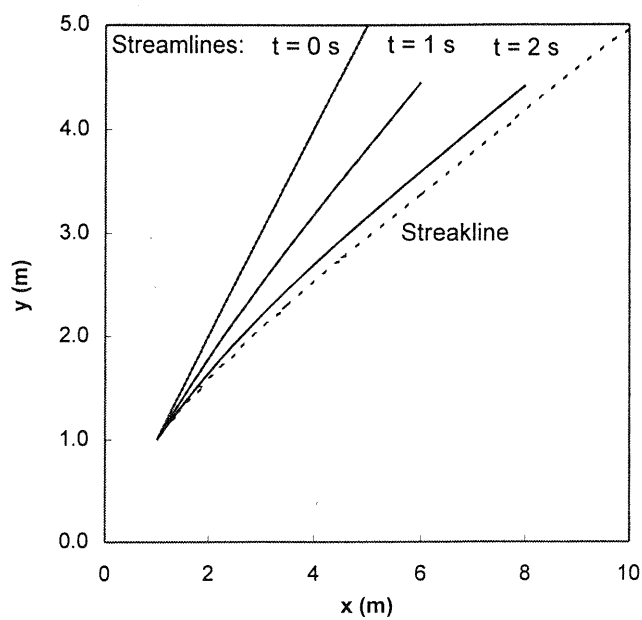
$$y = x^{1/(1+0.2t)} \quad \text{or} \quad x = y^{(1+0.2t)} \qquad \longleftarrow \text{streamline}$$

At $t = 0$, $x = y$

$t = 1\,s$, $x = y^{1.2}$

$t = 2\,s$, $x = y^{1.4}$

Problem 2.25

Streamlines:
t = 0 s
t = 1 s
t = 2 s
Streakline
y (m)
x (m)
5.0
4.0
3.0
2.0
1.0
0.0
0
2
4
6
8
10

Given: Velocity field $\vec{V} = axt\hat{i} + b\hat{j}$, where $a = 0.2\ s^{-1}$, $b = 1\ m/s$, and coordinates are in meters.

Plot: the pathline (during the interval $0 \le t \le 3s$) of the particle that passed through the point $(x_0, y_0) = (1, 2)$ at time $t = 0$.
Compare with the streakline through the same point at the instant $t = 3s$.

Solution:

The pathline and streakline are based on parametric equations for a particle.

For a particle $u = dx/dt$ and $v = dy/dt$

then $u = \frac{dx}{dt} = axt$, $\int_{x_0}^{x} \frac{dx}{x} = \int_{t_0}^{t} at\,dt$, $\ln\frac{x}{x_0} = \frac{1}{2}a(t^2 - t_0^2)$

$$x = x_0 e^{\frac{1}{2}a(t^2 - t_0^2)}$$

Also $v = dy/dt = b$, $\int_{y_0}^{y} dy = \int_{t_0}^{t} b\,dt$, $y = y_0 + b(t - t_0)$

In the above equations, x_0, y_0 are coordinates of particle at t_0

(a) The pathline is obtained by following the particle that passed through the point $(x_0, y_0) = (1, 2)$ at time $t_0 = 0$

Thus
$$x = x_0 e^{\frac{1}{2}at^2} = e^{0.1t^2}$$
$$y = y_0 + bt = 2 + t$$
$\leftarrow$ (x, y) pathline

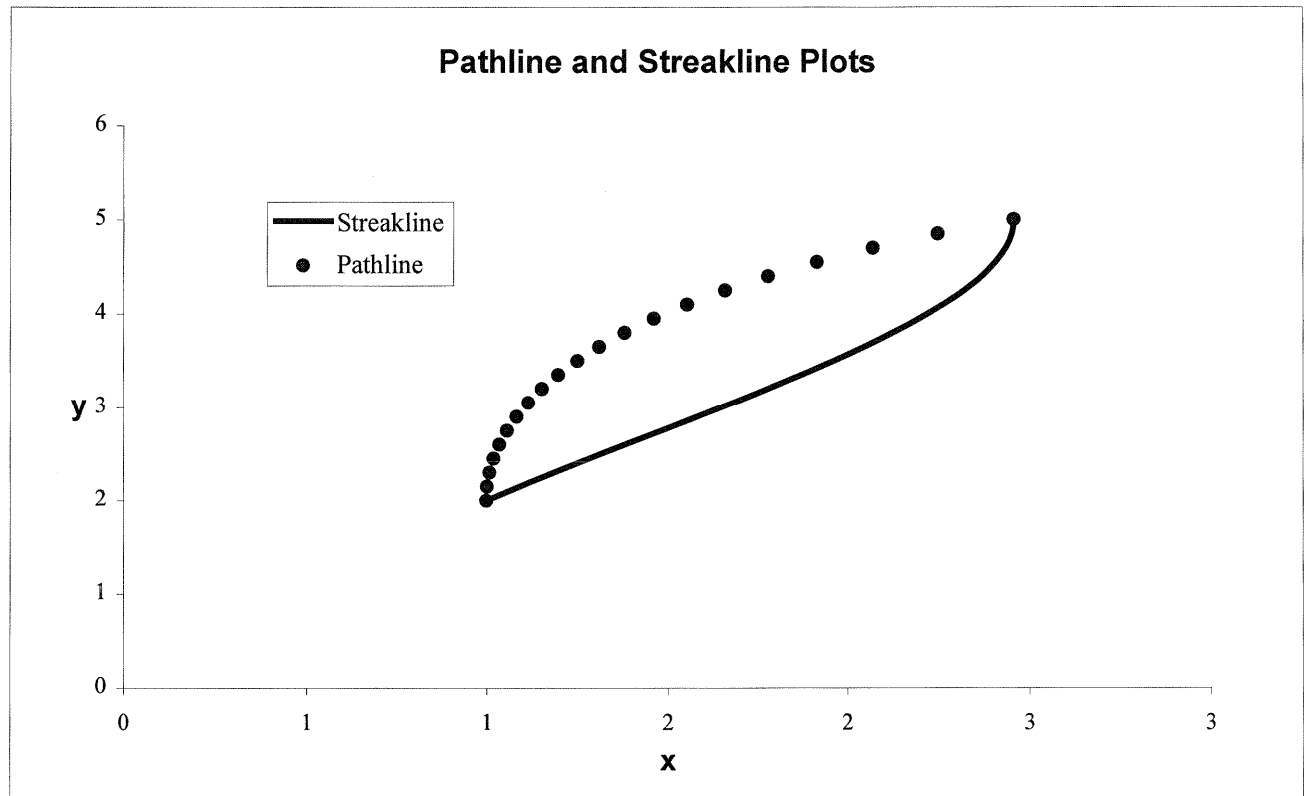
The pathline may be plotted by varying t $(0 \le t \le 3s)$ as shown below

(b) The streakline is obtained by locating (and connecting) at time $t = 3s$, all the particles that passed through the point $(x_0, y_0) = (1, 2)$ at some earlier time t_0.

Thus
$$x = x_0 e^{\frac{1}{2}a(9 - t_0^2)} = e^{0.1(9 - t_0^2)}$$
$$y = y_0 + b(t - t_0) = 2 + (3 - t_0) = 5 - t_0$$
$\leftarrow$ (x, y) streakline

The streakline may be plotted by varying t_0 $(0 \le t_0 \le 3s)$ as shown below.

Problem 2.26

Pathline and Streakline Plots
Streakline
Pathline
y
x
6
5
4
3
2
1
0
0
1
1
2
2
3
3

Problem 2.27

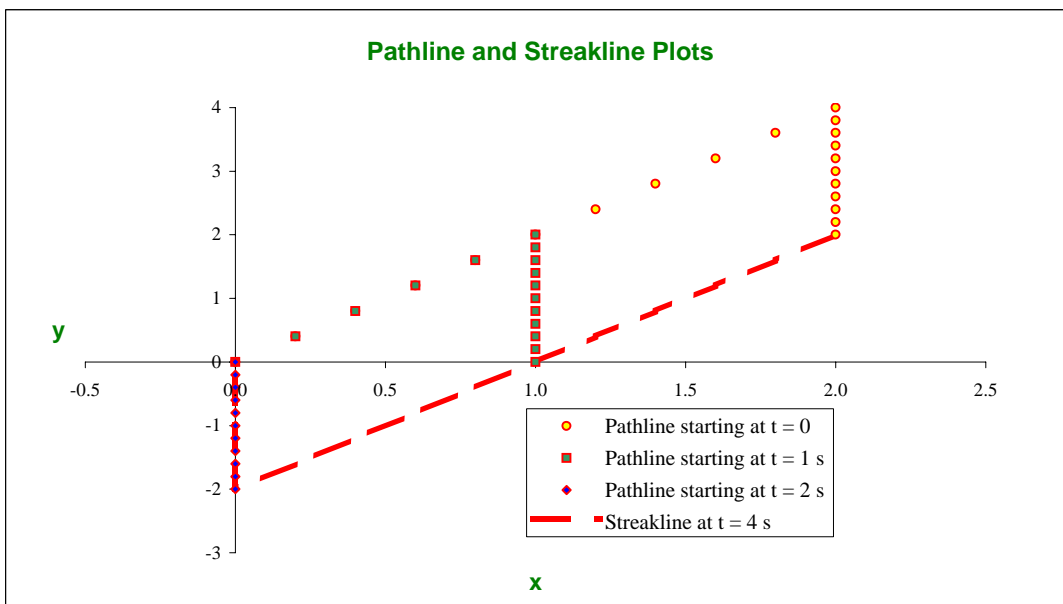
2.27 Tiny hydrogen bubbles are being used as tracers to visualize a flow. All the bubbles are generated at the origin ($x = 0$, $y = 0$). The velocity field is unsteady and obeys the equations:

$$u = 1 \text{ m/s} \qquad v = 2 \text{ m/s} \qquad 0 \le t < 2 \text{ s}$$
$$u = 0 \qquad v = -1 \text{ m/s} \qquad 0 \le t \le 4 \text{ s}$$

Plot the pathlines of bubbles that leave the origin at $t = 0$, 1, 2, 3, and 4 s. Mark the locations of these five bubbles at $t = 4$ s. Use a dashed line to indicate the position of a streakline at $t = 4$ s.

Solution The particle starting at t = 3 s follows the particle starting at t = 2 s;
The particle starting at t = 4 s doesn't move!

Pathlines: Starting at t = 0 | Starting at t = 1 s | Starting at t = 2 s | Streakline at t = 4 s

t	x (t = 0)	y (t = 0)	x (t = 1 s)	y (t = 1 s)	x (t = 2 s)	y (t = 2 s)	x (Streakline)	y (Streakline)
0.00	0.00	0.00					2.00	2.00
0.20	0.20	0.40					1.80	1.60
0.40	0.40	0.80					1.60	1.20
0.60	0.60	1.20					1.40	0.80
0.80	0.80	1.60					1.20	0.40
1.00	1.00	2.00	0.00	0.00			1.00	0.00
1.20	1.20	2.40	0.20	0.40			0.80	-0.40
1.40	1.40	2.80	0.40	0.80			0.60	-0.80
1.60	1.60	3.20	0.60	1.20			0.40	-1.20
1.80	1.80	3.60	0.80	1.60			0.20	-1.60
2.00	2.00	4.00	1.00	2.00	0.00	0.00	0.00	-2.00
2.20	2.00	3.80	1.00	1.80	0.00	-0.20	0.00	-1.80
2.40	2.00	3.60	1.00	1.60	0.00	-0.40	0.00	-1.60
2.60	2.00	3.40	1.00	1.40	0.00	-0.60	0.00	-1.40
2.80	2.00	3.20	1.00	1.20	0.00	-0.80	0.00	-1.20
3.00	2.00	3.00	1.00	1.00	0.00	-1.00	0.00	-1.00
3.20	2.00	2.80	1.00	0.80	0.00	-1.20	0.00	-0.80
3.40	2.00	2.60	1.00	0.60	0.00	-1.40	0.00	-0.60
3.60	2.00	2.40	1.00	0.40	0.00	-1.60	0.00	-0.40
3.80	2.00	2.20	1.00	0.20	0.00	-1.80	0.00	-0.20
4.00	2.00	2.00	1.00	0.00	0.00	-2.00	0.00	0.00

Problem 2.28

[4]

2.28 A flow is described by velocity field, $\vec{V} = ay^2\hat{i} + b\hat{j}$, where $a = 1\ \text{m}^{-1}\,\text{s}^{-1}$ and $b = 2$ m/s. Coordinates are measured in meters. Obtain the equation for the streamline passing through point (6, 6). At $t = 1$ s, what are the coordinates of the particle that passed through point (1, 4) at $t = 0$? At $t = 3$ s, what are the coordinates of the particle that passed through point (−3, 0) 2 s earlier? Show that pathlines, streamlines, and streaklines for this flow coincide.

Given: 2D velocity field

Find: Streamlines passing through (6,6); Coordinates of particle starting at (1,4); that pathlines, streamlines and streaklines coincide

Solution:

For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{b}{a \cdot y^2}$ or $\int a \cdot y^2\, dy = \int b\, dx$

Integrating $\frac{a \cdot y^3}{3} = b \cdot x + C$

For the streamline through point (6,6) $C = 60$ and $y^3 = 6 \cdot x + 180$

For particle that passed through (1,4) at t = 0 $u = \frac{dx}{dt} = a \cdot y^2$ $\int 1\, dx = x - x_0 = \int a \cdot y^2\, dt$ but we need y(t)

$v = \frac{dy}{dt} = b$ $\int 1\, dy = \int b\, dt$ $y = y_0 + b \cdot t = y_0 + 2 \cdot t$

Then $x - x_0 = \int_0^t a \cdot (y_0 + b \cdot t)^2\, dt$ $x = x_0 + a \cdot \left(y_0^2 \cdot t + b \cdot y_0 \cdot t^2 + \frac{b^2 \cdot t^3}{3} \right)$

Hence, with $x_0 = 1$ $y_0 = 4$ $x = 1 + 16 \cdot t + 8 \cdot t^2 + \frac{4}{3} \cdot t^3$ At t = 1 s $x = 26.3 \cdot m$

$y = 4 + 2 \cdot t$ $y = 6 \cdot m$

For particle that passed through (-3,0) at t = 1 $\int 1\, dy = \int b\, dt$ $y = y_0 + b \cdot (t - t_0)$

$x - x_0 = \int_{t_0}^t a \cdot (y_0 + b \cdot t)^2\, dt$ $x = x_0 + a \cdot \left[y_0^2 \cdot (t - t_0) + b \cdot y_0 \cdot (t^2 - t_0^2) + \frac{b^2}{3} \cdot (t^3 - t_0^3) \right]$

Hence, with $x_0 = -3$, $y_0 = 0$ at $t_0 = 1$ $x = -3 + \frac{4}{3} \cdot (t^3 - 1) = \frac{1}{3} \cdot (4 \cdot t^3 - 13)$ $y = 2 \cdot (t - 1)$

Evaluating at t = 3 $x = 31.7 \cdot m$ $y = 4 \cdot m$

This is a steady flow, so pathlines, streamlines and streaklines always coincide

Given: Velocity field in xy plane, $\vec{V} = a\hat{i} + bx\hat{j}$, where $a = 2$ m/s and $b = 1\ s^{-1}$.

Find: (a) Equation for streamline through $(x,y) = (2,5)$.

(b) At $t = 2$ s, coordinates of particle $(0,4)$ at $t = 0$.

(c) At $t = 3$ s, coordinates of particle $(1, 4.25)$ at $t = 1$ s.

(d) compare pathline, streamline, streakline.

Solution: For a streamline $\dfrac{dx}{u} = \dfrac{dy}{v}$

For $\vec{V} = a\hat{i} + bx\hat{j}$, $u = a$ and $v = bx$, so $\dfrac{dx}{a} = \dfrac{dy}{bx}$ or

$$x\,dx = \frac{a}{b}\,dy$$

Integrating

$$\frac{x^2}{2} = \frac{a}{b}y + c' \quad \text{or} \quad y = \frac{b}{2a}x^2 + c$$

Evaluating c at $(x,y) = (2,5)$,

$$c = y - \frac{b}{2a}x^2 = 5\,m - \frac{1}{2} \times \frac{1}{s} \times \frac{s}{2m}(2m)^2 = 4\,m$$

Streamline through $(x,y) = (2,5)$ is $\quad y = \dfrac{x^2}{4} + 4 \quad$ (a)

To locate particles, derive parametric equations

$$u_p = \frac{dx}{dt} = a, \quad dx = a\,dt, \quad \text{and} \quad x - x_0 = a(t - t_0)$$

$$v_p = \frac{dy}{dt} = bx, \quad dy = bx\,dt = b(x_0 + at - at_0)$$

$$y - y_0 = bx_0(t - t_0) + \frac{a}{2}(t^2 - t_0^2) - at_0(t - t_0)$$

For the particle at $(x_0, y_0) = (0,4)$ at $t = 0$,

$x = 0 + at \qquad$ so at $t = 2$ s, $x = 2\frac{m}{s} \times 2\,s = 4\,m$

$y = 4 + \dfrac{at^2}{2} \qquad$ so at $t = 2$ s, $y = 4 + \frac{1}{2} \times 2\frac{m}{s} \times (2)^2 s^2$

$y = 8\,m \quad$ (b)

Problem 2.29

For the particle at $(x,y) = (1, 4.25)$ at $t = 1\,s$,

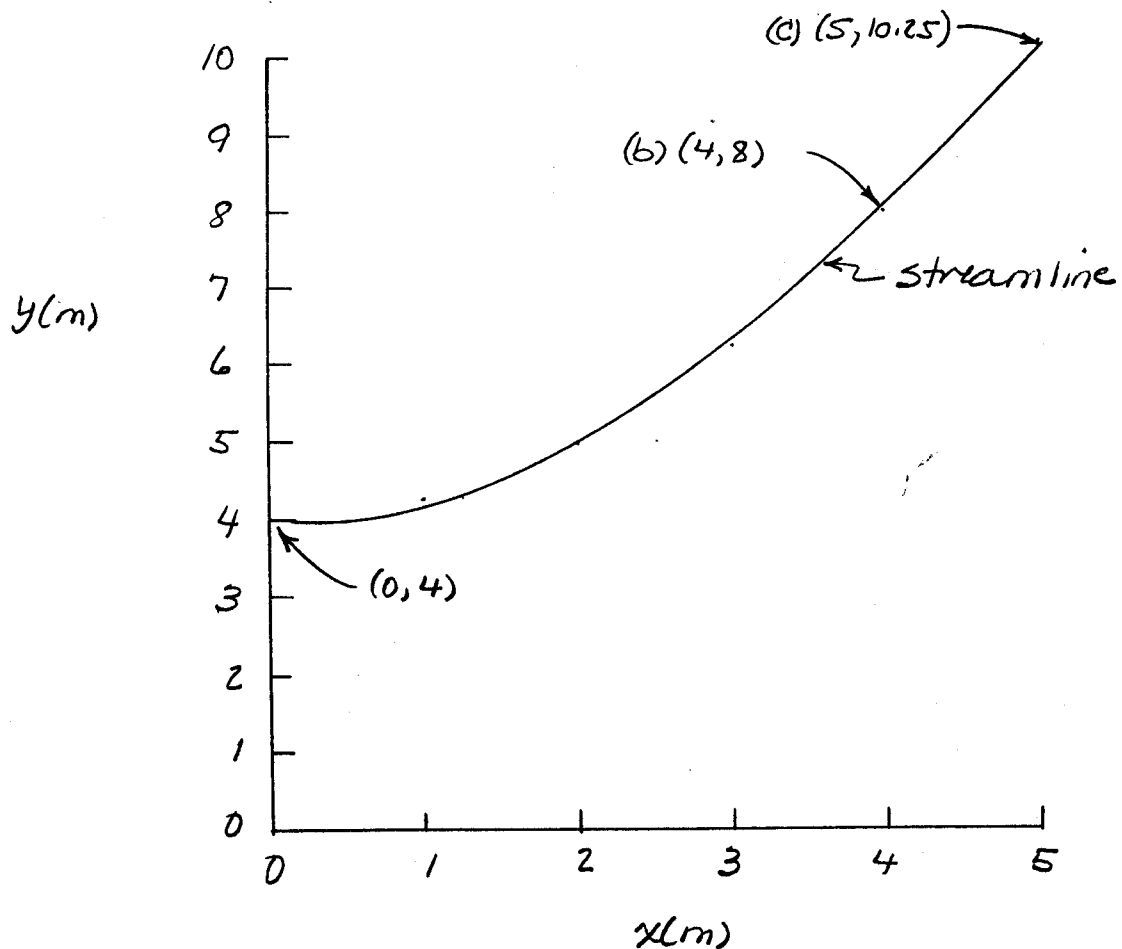
$$x = x_0 + a(t - t_0) = 1 + a(t-1)$$

so at $t = 3\,s$, $x = 1 + 2\frac{m}{s}(3-1)s = 5\,m$

$$y = y_0 + bx_0(t - t_0) + \frac{a}{2}(t^2 - t_0^2) - at_0(t - t_0)$$

$$= 4.25 + \frac{1}{s} \times 1m \times (t-1) + \frac{1}{2} \times 2\frac{m}{s}(t^2 - 1) - 2\frac{m}{s} \times 1s(t-1)$$

so at $t = 3\,s$, $y = 4.25 + 2 + 8 - 4 = 10.25\,m$ (c)

All these points lie on the same streamline, as shown below:

For this steady flow, streamlines, pathlines, and streaklines coincide, as expected.

Problem 2.30

Given: Velocity field $\vec{V} = ay\hat{i} + bt\hat{j}$, where $a = 1\ s^{-1}$; $b = 0.5\ m/s^2$; t in s.

Find: (a) At $t = 2s$, particle that passed $(1, 2)$ at $t = 0\ s$

(b) At $t = 3s$, particle that passed $(1, 2)$ at $t = 2s$

(c) Plot pathline and streakline through $(1, 2)$; compare with streamlines at $t = 0, 1, 2\ s$.

Solution: Pathline and streakline are based on parametric equations for a particle. Thus

$$v = \frac{dy}{dt} = bt, \text{ so } dy = bt\,dt, \text{ and } y - y_0 = \frac{b}{2}(t^2 - t_0^2)$$

and $$u = \frac{dx}{dt} = ay = a\left[y_0 + \frac{b}{2}(t^2 - t_0^2)\right]$$

so

$$x\Big]_{x_0}^{x} = a\left[y_0 t + \frac{b}{2}\left(\frac{t^3}{3} - t_0^2 t\right)\right]_{t_0}^{t} \;;\; x = x_0 + ay_0(t - t_0) + \frac{ab}{2}\left(\frac{t^3 - t_0^3}{3} + t_0^2(t_0 - t)\right)$$

where x_0, y_0 are coordinates of particle at t_0.

For (a), $t_0 = 0$, and $(x_0, y_0) = (1, 2)$. Thus at $t = 2s$, $y = y_0 + \frac{bt^2}{2}$

$$y = 2\,m + \frac{1}{2} \times 0.5\frac{m}{s^2} \times (2)^2 s^2 = 3.00\,m$$

$$x = 1\,m + \frac{1}{s} \times 2\,m\,(2-0)s + \frac{1}{2} \times \frac{1}{s} \times 0.5\frac{m}{s^2}\left(\frac{(2)^3 - 0}{3} + 0\right)s^3 = 5.67\,m$$

$t_0 = 0$ (a)
At $t = 2s$, $(x, y) = (5.67, 3.00)\,m$

For (b), $t_0 = 2s$, and $(x_0, y_0) = (1, 2)$. Thus at $t = 3s$, the particle is at

$$y(3) = 2\,m + \frac{1}{2} \times 0.5\frac{m}{s^2}\left[(3)^2 - (2)^2\right]s^2 = 3.25\,m$$

$$x(3) = 1\,m + \frac{1}{s} \times 2\,m\,(3-2)s + \frac{1}{2} \times \frac{1}{s} \times 0.5\frac{m}{s^2}\left(\frac{(3)^3 - (2)^3}{3} + (2)^2(2-3)\right)s^3 = 3.58\,m$$

At $t = 3s$, $t_0 = 2s$ (b)
$(x, y) = (3.58, 3.25)\,m$

For (c), the streakline may be plotted at any t by varying t_0, as shown on the next page.

The streamline is found (at given t) from $\frac{dx}{u} = \frac{dy}{v}$

Substituting $u = ay$ and $v = bt$, $dx = \frac{ay}{bt}dy$ or $y^2 = \frac{2bt}{a}x + c$

Thus $c = y_0^2 - \frac{2bt}{a}x_0$

For $t = 0$, $y^2 = c$; at $(x_0, y_0) = (1, 2)$, then $c = 4$

$t = 1$, $y^2 = \frac{2b}{a}x + c$; at $(x_0, y_0) = (1, 2)$, then $c = 3$

$t = 2$, $y^2 = \frac{4b}{a}x + c$; at $(x, y) = (1, 2)$, $c = 2$; for $t = 3s$, $c = 1$

(c)

Problem 2.30

Recall $\vec{V} = ay\hat{\imath} + bt\hat{\jmath}$, where $a = 1\,s^{-1}$; $b = 0.5\ m/s^2$, $(x_0, y_0) = (1, 2)\ m$.

Part (a): Pathline of particle located at (x_0, y_0) at $t_0 = 0\,s$:

t_0 (s)	t (s)	x (m)	y (m)
0	0	1.00	2.00
0	1	3.08	2.25
0	2	5.67	3.00
0	3	9.25	4.25

Part (b): Pathline of particle located at (x_0, y_0) at $t_0 = 2\,s$:

t_0 (s)	t (s)	x (m)	y (m)
2	2	1.00	2.00
2	3	3.58	3.25
2	4	7.67	5.00

Part (c): Streamlines through point (x_0, y_0) at $t = 0, 1, 2,$ and $3\,s$:

	t (s)	0	1	2	3
	c =	4.0	3.0	2.0	1.0
t_0 (s)	x (m)	y (m)	y (m)	y (m)	y (m)
0	1	2.00	2.00	2.00	2.00
0	2	2.00	2.24	2.45	2.65
0	3	2.00	2.45	2.83	3.16
0	4	2.00	2.65	3.16	3.61
0	5	2.00	2.83	3.46	4.00
0	6	2.00	3.00	3.74	4.36
0	7	2.00	3.16	4.00	4.69
0	8	2.00	3.32	4.24	5.00
0	9	2.00	3.46	4.47	5.29
0	10	2.00	3.61	4.69	5.57

Streakline at $t = 3\,s$ of particles that passed thru point (x_0, y_0):

t_0 (s)	t (s)	x (m)	y (m)
0	3	9.25	4.25
1	3	6.67	4.00
2	3	3.58	3.25
3	3	1.00	2.00

Given: Velocity field $\vec{V} = at\hat{i} + b\hat{j}$, where $a = 0.4\ m/s^2$, $b = 2\ m/s$, and coordinates are measured in meters

Find: (a) At $t = 2s$, coordinates of particle that passed through $(x_0, y_0) = (2,1)$ at $t = 0$
(b) At $t = 3s$, coordinates of the particle that passed through (x_0, y_0) at $t = 2s$

Plot: the pathline and streakline through point $(2,1)$; compare with the streamlines through the same point at $t = 0, 1, 2s$

Solution:

The pathline and streakline are based on parametric equations for a particle.

For a particle $u = dx/dt$ and $v = dy/dt$

Thus

$$u = \frac{dx}{dt} = at, \quad \int_{x_0}^{x} dx = \int_{t_0}^{t} at\,dt, \quad x = x_0 + \frac{1}{2}a(t^2 - t_0^2) \qquad (1a)$$

$$v = \frac{dy}{dt} = b, \quad \int_{y_0}^{y} dy = \int_{t_0}^{t} b\,dt, \quad y = y_0 + b(t - t_0) \qquad (1b)$$

In the above equations, x_0, y_0 are coordinates of the particle at time t_0

(a) The pathline is obtained by following the particle that passed through the point $(x_0, y_0) = (2,1)$ at time $t_0 = 0$

Thus

$$\left.\begin{aligned} x &= x_0 + \tfrac{1}{2}at^2 = 2 + 0.2t^2 \\ y &= y_0 + bt = 1 + 2t \end{aligned}\right\} \longleftarrow (x,y) \text{ pathline}$$

At $t = 2s$, particle is at $(x,y) = (2.8, 5)\ m$ ← (a)

The pathline may be plotted by varying t $(0 \le t \le 3s)$ as shown below

(b) The streakline is obtained by locating (and connecting) at time $t = 3s$, all the particles that passed through the point $(x_0, y_0) = (2,1)$ at some earlier time t_0

Thus

$$\left.\begin{aligned} x &= x_0 + \tfrac{1}{2}a(9 - t_0^2) = 2 + 0.2(9 - t_0^2) \\ y &= y_0 + b(t - t_0) = 1 + 2(3 - t_0) \end{aligned}\right\} \longleftarrow (x,y) \text{ streakline}$$

At $t = 2s$, particle is at $(x,y) = (3,3)$ ← (b)

The streakline may be plotted by varying t_0 $(0 \le t_0 \le 3s)$ as shown below.

The streamline is found (at given t) from $\left.\frac{dy}{dx}\right)_s = \frac{v}{u}$

Problem 2.31

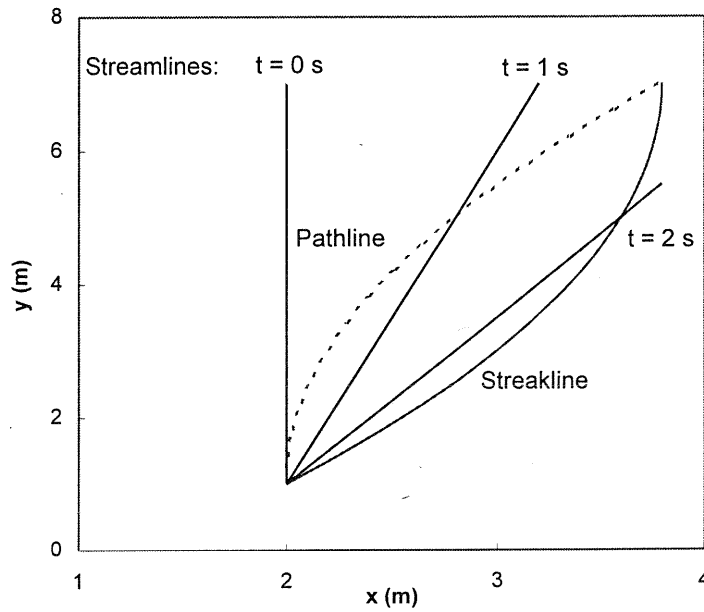
Then, $dy/dx = \frac{b}{at}$, $\int_{y_0}^{y} dy = \int_{x_0}^{x} \frac{b}{at} dx$, $y - y_0 = \frac{b}{at}(x - x_0)$

Streamline through point (2,1) gives $y - 1 = \frac{b}{at}(x-2)$

$y = 1 + \frac{5(x-2)}{t}$ or $x = 2 + \frac{t}{5}(y-1)$ ← Streamline

At $t=0$, $x=2$

$t=1$, $y = 5x-9$

$t=2$, $y = 2.5x-4$

Problem 2.32

Given: Variation of air viscosity with temperature (absolute) is

$$\mu = \frac{bT^{1/2}}{1+S/T}$$

where $b = 1.458 \times 10^{-6}\ kg/m.s.K^{1/2}$, $S = 110.4K$

Find: Equation for calculating air viscosity in British Gravitational units as a function of absolute temperature in degrees Rankine. Check result using data from Appendix A

Solution:

Convert constants.

$$b = 1.458 \times 10^{-6} \frac{kg}{m.s.K^{1/2}} \times \frac{lbm}{0.4536kg} \times \frac{slug}{32.17\ lbm} \times \frac{lbf.s^2}{slug.ft} \times \frac{0.3048\ m}{ft} \times \left(\frac{5K}{9°R}\right)^{1/2}$$

$$b = 2.27 \times 10^{-8}\ lbf.s/ft^2.°R^{1/2}$$

$$S = 110.4K \times \frac{9°R}{5K} = 198.7°R$$

Then in British Gravitational Units

$$\mu = \frac{2.27 \times 10^{-8}\ T^{1/2}}{1 + 198.7/T}$$ ← μ

where units of T are °R; μ is in $lbf.s/ft^2$

Evaluate at T = 80°F (539.7°R)

$$\mu = \frac{2.27 \times 10^{-8} \times (539.7)^{1/2}}{1 + 198.7/539.7} = 3.855 \times 10^{-7}\ lbf.s/ft^2$$

From Table A.9 (Appendix A) at T = 80°F

$\mu = 3.86 \times 10^{-7}\ lbf.s/ft^2$ ✓ check.

Problem 2.33

Given: Variation of air viscosity with temperature (absolute) is

$$\mu = \frac{bT^{1/2}}{1+S/T}$$

where $b = 1.458 \times 10^{-6} \; \frac{kg}{m \cdot s \cdot K^{1/2}}$

$S = 110.4 \, K$

Find: Equation for kinematic viscosity of air (in SI units) as a function of temperature at atmospheric pressure. Assume ideal gas behavior. Check result using data from Appendix A.

Solution:

For an ideal gas, $p = \rho R T$. From Table A.6, $R = 286.9 \; N \cdot m / kg \cdot K$

The kinematic viscosity, $\nu \equiv \mu/\rho$

$$\therefore \nu = \frac{\mu}{\rho} = \frac{\mu R T}{p} = \frac{RT}{p} \frac{bT^{1/2}}{1+S/T} = \frac{Rb}{p} \frac{T^{3/2}}{1+S/T} = \frac{b' T^{3/2}}{1+S/T}$$

where $b' = \frac{Rb}{p} = 286.9 \frac{N \cdot m}{kg \cdot K} \times 1.458 \times 10^{-6} \frac{kg}{m \cdot s \cdot K^{1/2}} \times \frac{m^2}{101.3 \times 10^3 \, N}$

$$b' = 4.129 \times 10^{-9} \; m^2/s \cdot K^{3/2}$$

$$\therefore \nu = \frac{b' T^{3/2}}{1+S/T} \qquad \nu$$

where $b' = 4.129 \times 10^{-9} \; m^2/s \cdot K^{3/2}$, $S = 110.4 \, K$
units of T are (K); ν is in m^2/s

Evaluate at $T = 20°C = 293.2 \, K$

$$\nu = \frac{4.129 \times 10^{-9} \, (293.2)^{3/2}}{1 + 110.4/293.2} = 1.506 \times 10^{-5} \; m^2/s$$

From Table A.10 (Appendix A) at $T = 20°C$

$\nu = 1.51 \times 10^{-5} \; m^2/s$ ✓ Check.

Problem 2.34

2.34 Some experimental data for the viscosity of helium at 1 atm are

T, °C	0	100	200	300	400
μ, N·s/m^2(× 10^5)	1.86	2.31	2.72	3.11	3.46

Using the approach described in Appendix A-3, correlate these data to the empirical Sutherland equation

$$\mu = \frac{bT^{1/2}}{1 + S/T}$$

(where T is in kelvin) and obtain values for constants b and S.

Given: Viscosity data

Find: Obtain values for coefficients in Sutherland equation

Solution:

Data:

T (°C)	T (K)	μ(x10^5)
0	273	1.86E-05
100	373	2.31E-05
200	473	2.72E-05
300	573	3.11E-05
400	673	3.46E-05

Using procedure of Appendix A.3:

T (K)	$T^{3/2}/\mu$
273	2.43E+08
373	3.12E+08
473	3.78E+08
573	4.41E+08
673	5.05E+08

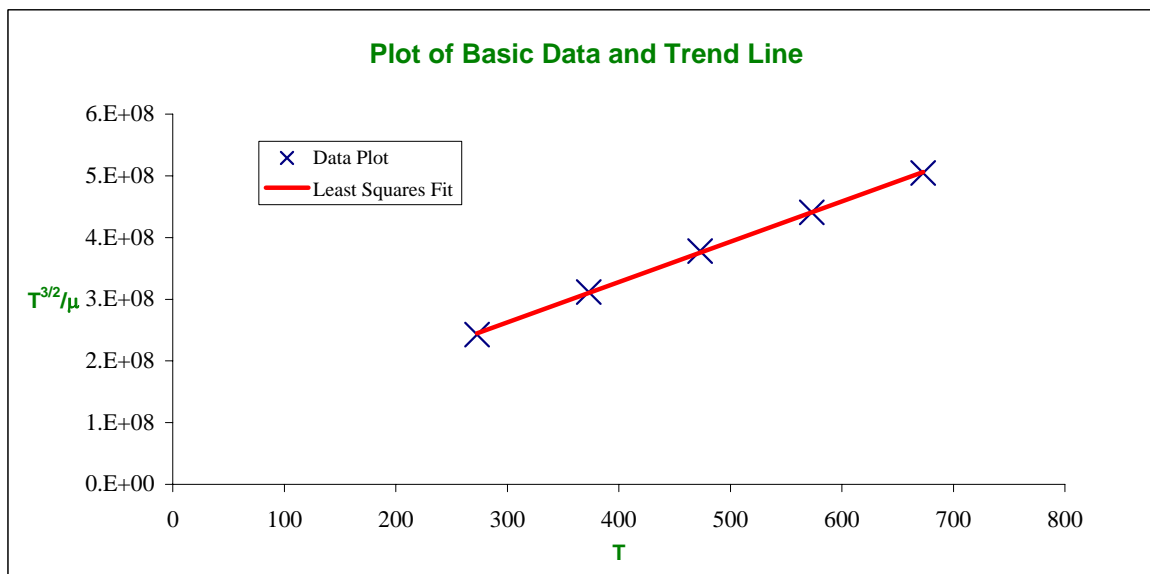
The equation to solve for coefficients S and b is

$$\frac{T^{3/2}}{\mu} = \left(\frac{1}{b}\right)T + \frac{S}{b}$$

From the built-in *Excel Linear Regression* functions:

Slope = 6.534E+05
Intercept = 6.660E+07
R^2 = 0.9996

Hence:

b = 1.531E-06 kg/m·s·K$^{1/2}$
S = 101.9 K

Problem 2.35

2.35 The velocity distribution for laminar flow between parallel plates is given by

$$\frac{u}{u_{max}} = 1 - \left(\frac{2y}{h}\right)^2$$

where h is the distance separating the plates and the origin is placed midway between the plates. Consider a flow of water at 15°C, with $u_{max} = 0.10$ m/s and $h = 0.1$ mm. Calculate the shear stress on the upper plate and give its direction. Sketch the variation of shear stress across the channel.

Given: Velocity distribution between flat plates

Find: Shear stress on upper plate; Sketch stress distribution

Solution:

Basic equation $\quad \tau_{yx} = \mu \cdot \frac{du}{dy} \qquad \frac{du}{dy} = \frac{d}{dy} u_{max} \cdot \left[1 - \left(\frac{2 \cdot y}{h}\right)^2\right] = u_{max} \cdot \left(-\frac{4}{h^2}\right) \cdot 2 \cdot y = -\frac{8 \cdot u_{max} \cdot y}{h^2}$

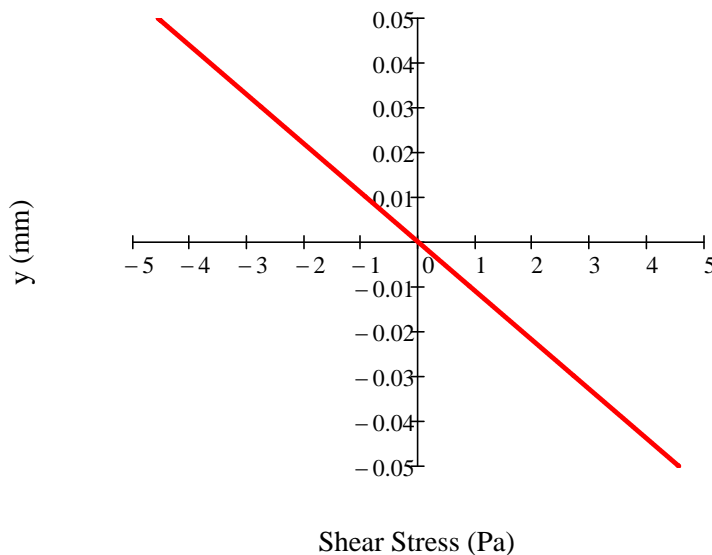
$$\tau_{yx} = -\frac{8 \cdot \mu \cdot u_{max} \cdot y}{h^2}$$

At the upper surface $\quad y = \frac{h}{2} \quad$ and $\quad h = 0.1 \cdot mm \qquad u_{max} = 0.1 \cdot \frac{m}{s} \qquad \mu = 1.14 \times 10^{-3} \cdot \frac{N \cdot s}{m^2}$ (Table A.8)

Hence $\quad \tau_{yx} = -8 \times 1.14 \times 10^{-3} \cdot \frac{N \cdot s}{m^2} \times 0.1 \cdot \frac{m}{s} \times \frac{0.1}{2} \cdot mm \times \frac{1 \cdot m}{1000 \cdot mm} \times \left(\frac{1}{0.1 \cdot mm} \times \frac{1000 \cdot mm}{1 \cdot m}\right)^2$

$$\tau_{yx} = -4.56 \cdot \frac{N}{m^2}$$

The upper plate is a minus y surface. Since $\tau_{yx} < 0$, the shear stress on the upper plate must act in the plus x direction.

The shear stress varies linearly with y $\qquad \tau_{yx}(y) = -\left(\frac{8 \cdot \mu \cdot u_{max}}{h^2}\right) \cdot y$

Problem 2.36

[2]

2.36 The velocity distribution for laminar flow between parallel plates is given by

$$\frac{u}{u_{\max}} = 1 - \left(\frac{2y}{h}\right)^2$$

where h is the distance separating the plates and the origin is placed midway between the plates. Consider flow of water at 15°C with maximum speed of 0.05 m/s and $h = 0.1$ mm. Calculate the force on a 1m^2 section of the lower plate and give its direction.

Given: Velocity distribution between parallel plates

Find: Force on lower plate

Solution:

Basic equations $\quad F = \tau_{yx} \cdot A \qquad \tau_{yx} = \mu \cdot \frac{du}{dy}$

$$\frac{du}{dy} = \frac{d}{dy} u_{max} \cdot \left[1 - \left(\frac{2 \cdot y}{h}\right)^2\right] = u_{max} \cdot \left(-\frac{4}{h^2}\right) \cdot 2 \cdot y = -\frac{8 \cdot u_{max} \cdot y}{h^2}$$

so $\quad \tau_{yx} = -\frac{8 \cdot \mu \cdot u_{max} \cdot y}{h^2} \quad$ and $\quad F = -\frac{8 \cdot A \cdot \mu \cdot u_{max} \cdot y}{h^2}$

At the lower surface $\quad y = -\frac{h}{2} \quad$ and $\quad h = 0.1 \cdot mm \qquad A = 1 \cdot m^2$

$$u_{max} = 0.05 \cdot \frac{m}{s} \qquad \mu = 1.14 \times 10^{-3} \cdot \frac{N \cdot s}{m^2} \qquad \text{(Table A.8)}$$

Hence

$$F = -8 \times 1 \cdot m^2 \times 1.14 \times 10^{-3} \cdot \frac{N \cdot s}{m^2} \times 0.05 \cdot \frac{m}{s} \times \frac{-0.1}{2} \cdot mm \times \frac{1 \cdot m}{1000 \cdot mm} \times \left(\frac{1}{0.1} \cdot \frac{1}{mm} \times \frac{1000 \cdot mm}{1 \cdot m}\right)^2$$

$F = 2.28 \cdot N \qquad$ (to the right)

Problem 2.37

[2]

Explain how an ice skate interacts with the ice surface. What mechanism acts to reduce sliding friction between skate and ice?

Open-Ended Problem Statement: Explain how an ice skate interacts with the ice surface. What mechanism acts to reduce sliding friction between skate and ice?

Discussion: The normal freezing and melting temperature of ice is 0°C (32°F) at atmospheric pressure. The melting temperature of ice decreases as pressure is increased. Therefore ice can be caused to melt at a temperature below the normal melting temperature when the ice is subjected to increased pressure.

A skater is supported by relatively narrow blades with a short contact against the ice. The blade of a typical skate is less than 3 mm wide. The length of blade in contact with the ice may be just ten or so millimeters. With a 3 mm by 10 mm contact patch, a 75 kg skater is supported by a pressure between skate blade and ice on the order of tens of megaPascals (hundreds of atmospheres). Such a pressure is enough to cause ice to melt rapidly.

When pressure is applied to the ice surface by the skater, a thin surface layer of ice melts to become liquid water and the skate glides on this thin liquid film. Viscous friction is quite small, so the effective friction coefficient is much smaller than for sliding friction.

The magnitude of the viscous drag force acting on each skate blade depends on the speed of the skater, the area of contact, and the thickness of the water layer on top of the ice.

The phenomenon of static friction giving way to viscous friction is similar to the hydroplaning of a pneumatic tire caused by a layer of water on the road surface.

Problem 2.38

2.38 Crude oil, with specific gravity SG = 0.85 and viscosity $\mu = 2.15 \times 10^{-3}$ lbf • s/ft², flows steadily down a surface inclined $\theta = 45$ degrees below the horizontal in a film of thickness $h = 0.1$ in. The velocity profile is given by

$$u = \frac{\rho g}{\mu}\left(hy - \frac{y^2}{2}\right)\sin\theta$$

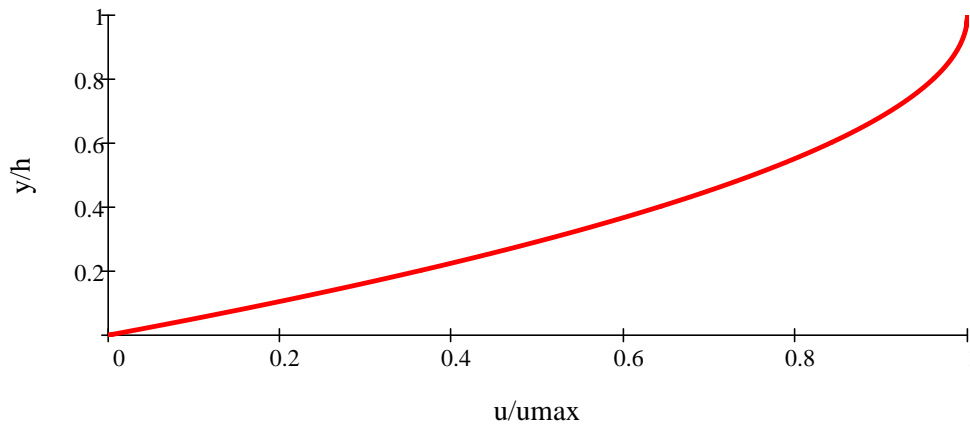
(Coordinate x is along the surface and y is normal to the surface.) Plot the velocity profile. Determine the magnitude and direction of the shear stress that acts on the surface.

Given: Velocity profile

Find: Plot of velocity profile; shear stress on surface

Solution:

The velocity profile is $u = \frac{\rho \cdot g}{\mu} \cdot \left(h \cdot y - \frac{y^2}{2}\right) \cdot \sin(\theta)$ so the maximum velocity is at y = h $u_{max} = \frac{\rho \cdot g}{\mu} \cdot \frac{h^2}{2} \cdot \sin(\theta)$

Hence we can plot $\frac{u}{u_{max}} = 2 \cdot \left[\frac{y}{h} - \frac{1}{2} \cdot \left(\frac{y}{h}\right)^2\right]$

This graph can be plotted in *Excel*

The given data is $h = 0.1 \cdot in$ $\mu = 2.15 \times 10^{-3} \cdot \frac{lbf \cdot s}{ft^2}$ $\theta = 45 \cdot deg$

Basic equation $\tau_{yx} = \mu \cdot \frac{du}{dy}$ $\tau_{yx} = \mu \cdot \frac{du}{dy} = \mu \cdot \frac{d}{dy} \frac{\rho \cdot g}{\mu} \cdot \left(h \cdot y - \frac{y^2}{2}\right) \cdot \sin(\theta) = \rho \cdot g \cdot (h - y) \cdot \sin(\theta)$

At the surface y = 0 $\tau_{yx} = \rho \cdot g \cdot h \cdot \sin(\theta)$

Hence $\tau_{yx} = 0.85 \times 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times 0.1 \cdot in \times \frac{1 \cdot ft}{12 \cdot in} \times \sin(45 \cdot deg) \times \frac{lbf \cdot s^2}{slug \cdot ft}$ $\tau_{yx} = 0.313 \cdot \frac{lbf}{ft^2}$

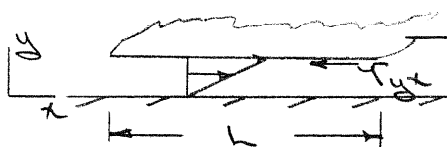
The surface is a positive y surface. Since $\tau_{yx} > 0$, the shear stress on the surface must act in the plus x direction.

Problem 2.39

Given: Skater, of weight $W = 100$ lbf, glides on one skate at speed $V = 20$ ft/s. Skate blade, of length $L = 11.5$ in and width $w = 0.125$ in. glides on thin film of water of height $h = 5.75 \times 10^{-5}$ in.

Find: the deceleration of the skater due to viscous shear.

Solution:

Model flow as one-dimensional shear flow

$V = 20$ ft/s

Basic equation: $\tau_{yx} = \mu \dfrac{du}{dy}$

Assumptions: 1. Newtonian fluid
2. Linear velocity profile
3. Neglect end effects.

From Table A.7, Appendix A, at 32°F

$\mu = 3.66 \times 10^{-5}$ lbf·s/ft²

$$\tau_{yx} = \mu \frac{du}{dy} = \mu \frac{V}{h} = 3.66 \times 10^{-5} \frac{\text{lbf} \cdot \text{s}}{\text{ft}^2} \times 20 \frac{\text{ft}}{\text{s}} \times \frac{1}{5.75 \times 10^{-5}\,\text{in}} \times \frac{12\,\text{in}}{\text{ft}}$$

$\tau_{yx} = 153$ lbf/ft²

$\Sigma F_x = m a_x \qquad \therefore \ \tau_{yx} A = -\dfrac{W}{g} a_x$

$$a_x = -\frac{\tau_{yx} A g}{W} = -\frac{\tau_{yx} L w g}{W}$$

$$= -153 \frac{\text{lbf}}{\text{ft}^2} \times 11.5\,\text{in} \times 0.125\,\text{in} \times 32.2 \frac{\text{ft}}{\text{s}^2} \times \frac{1}{100\,\text{lbf}} \times \frac{\text{ft}^2}{144\,\text{in}^2}$$

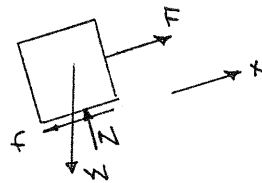
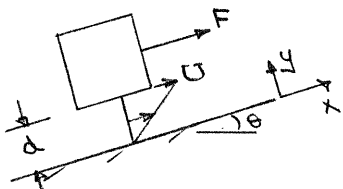
$a_x = -0.491$ ft/s² ← a_x

Problem 2.40

Given: Block of weight 10 lbf, 10 in. on each edge, is pulled up a plane, inclined at 25° to the horizontal, over a film of SAE 10W oil at 100°F. The speed of the block is constant at 2 ft/s and the oil film thickness is 0.001 in. Velocity profile in film is linear.

Find: Force required.

Solution:

Since the block is moving at constant velocity, U, then $\Sigma \vec{F}_{ext} = 0$
Consider the forces along the direction of motion and look at a free body diagram of the block.

Since $\Sigma F_x = 0$, then $F - f - W\sin\theta = 0$

Now the friction force, $f = \tau A$

where $\tau = \mu \frac{du}{dy}$

For small gap (linear velocity profile) $\tau = \mu \frac{U}{d}$

Hence

$$f = \tau A = \mu \frac{U}{d} A$$

and

$$F - \mu \frac{U}{d} A - W\sin\theta = 0$$

Thus

$$F = \mu \frac{U}{d} A + W\sin\theta$$

From Fig A.2, Appendix A, for SAE 10W oil @ 100°F (38°C), $\mu = 3.7\times10^{-2}\ \text{N}\cdot\text{s}/\text{m}^2$

$$F = \mu \frac{U}{d} A + W\sin\theta$$

$$= 3.7\times10^{-2}\ \frac{\text{N}\cdot\text{s}}{\text{m}^2} \times 2.09\times10^{-2}\ \frac{\text{lbf}\cdot\text{s}}{\text{ft}^2}\cdot\frac{\text{m}^2}{\text{N}\cdot\text{s}} \times 2\ \frac{\text{ft}}{\text{s}} \times (10)^2\ \text{in}^2 \times \frac{1}{0.001\ \text{in}} \times \frac{\text{ft}}{12\ \text{in}} + 10\ \text{lbf}\sin25°$$

$F = 17.1$ lbf ← F

Problem 2.41

[2]

2.41 Tape is to be coated on both sides with glue by drawing it through a narrow gap. The tape is 0.015 in. thick and 1.00 in. wide. It is centered in the gap with a clearance of 0.012 in. on each side. The glue, of viscosity $\mu = 0.02$ slug/(ft • s), completely fills the space between the tape and gap. If the tape can withstand a maximum tensile force of 25 lbf, determine the maximum gap region through which it can be pulled at a speed of 3 ft/s.

Given: Data on tape mechanism

Find: Maximum gap region that can be pulled without breaking tape

Solution:

Basic equation $\tau_{yx} = \mu \cdot \frac{du}{dy}$ and $F = \tau_{yx} \cdot A$

Here F is the force on each side of the tape; the total force is then $F_T = 2 \cdot F = 2 \cdot \tau_{yx} \cdot A$

The velocity gradient is linear as shown $\frac{du}{dy} = \frac{V - 0}{c} = \frac{V}{c}$

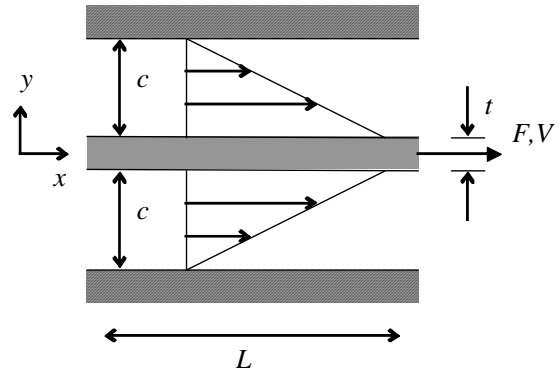
The area of contact is $A = w \cdot L$

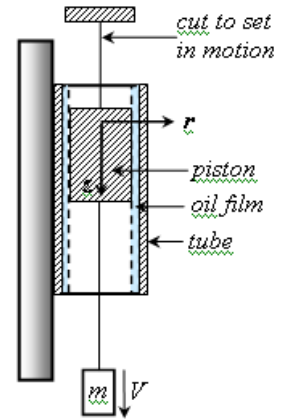
Combining these results

$$F_T = 2 \cdot \mu \cdot \frac{V}{c} \cdot w \cdot L$$

Solving for L $L = \frac{F_T \cdot c}{2 \cdot \mu \cdot V \cdot w}$

The given data is $F_T = 25 \cdot \text{lbf}$ $c = 0.012 \cdot \text{in}$ $\mu = 0.02 \cdot \frac{\text{slug}}{\text{ft} \cdot \text{s}}$ $V = 3 \cdot \frac{\text{ft}}{\text{s}}$ $w = 1 \cdot \text{in}$

Hence $L = 25 \cdot \text{lbf} \times 0.012 \cdot \text{in} \times \frac{1 \cdot \text{ft}}{12 \cdot \text{in}} \times \frac{1}{2} \times \frac{1}{0.02} \cdot \frac{\text{ft} \cdot \text{s}}{\text{slug}} \times \frac{1}{3} \cdot \frac{\text{s}}{\text{ft}} \times \frac{1}{1} \frac{1}{\text{in}} \times \frac{12 \cdot \text{in}}{1 \cdot \text{ft}} \times \frac{\text{slug} \cdot \text{ft}}{\text{s}^2 \cdot \text{lbf}}$ $L = 2.5\,\text{ft}$

Problem 2.42

2.42 A 73-mm-diameter aluminum (SG = 2.64) piston of 100-mm length resides in a stationary 75-mm-inner-diameter steel tube lined with SAE 10W-30 oil at 25°C. A mass $m = 2$ kg is suspended from the free end of the piston. The piston is set into motion by cutting a support cord. What is the terminal velocity of mass m? Assume a linear velocity profile within the oil.

Given: Flow data on apparatus

Find: The terminal velocity of mass *m*

Solution:

Given data: $D_{piston} = 73\cdot mm$ $\quad D_{tube} = 75\cdot mm$ $\quad Mass = 2\cdot kg$ $\quad L = 100\cdot mm$ $\quad SG_{Al} = 2.64$

Reference data: $\rho_{water} = 1000\cdot\frac{kg}{m^3}$ (maximum density of water)

From Fig. A.2:, the dynamic viscosity of SAE 10W-30 oil at 25°C is: $\mu = 0.13\cdot\frac{N\cdot s}{m^2}$

The terminal velocity of the mass *m* is equivalent to the terminal velocity of the piston. At that terminal speed, the acceleration of the piston is zero. Therefore, all forces acting on the piston must be balanced. This means that the force driving the motion (i.e. the weight of mass *m* and the piston) balances the viscous forces acting on the surface of the piston. Thus, at $r = R_{piston}$:

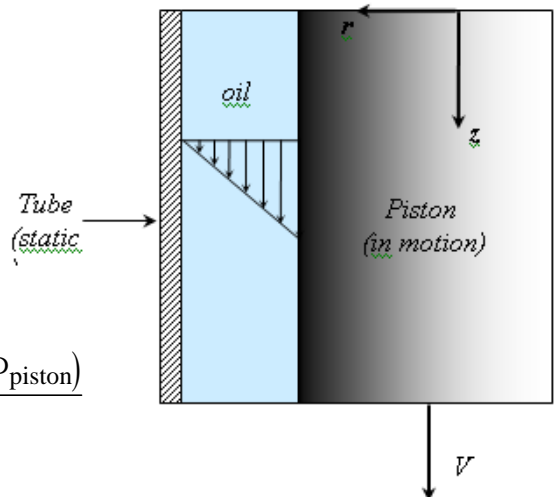
$$\left[Mass + SG_{Al}\cdot\rho_{water}\cdot\left(\frac{\pi\cdot D_{piston}^{2}\cdot L}{4}\right)\right]\cdot g = \tau_{rz}\cdot A = \left(\mu\cdot\frac{d}{dr}V_z\right)\cdot\left(\pi\cdot D_{piston}\cdot L\right)$$

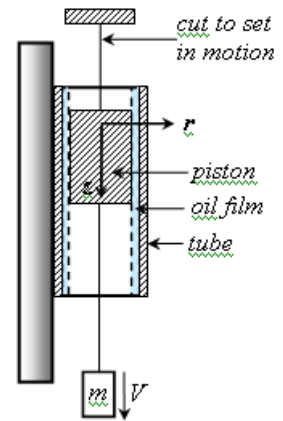
The velocity profile within the oil film is linear ...

Therefore $$\frac{d}{dr}V_z = \frac{V}{\left(\frac{D_{tube} - D_{piston}}{2}\right)}$$

Thus, the terminal velocity of the piston, *V*, is:

$$V = \frac{g\cdot\left(SG_{Al}\cdot\rho_{water}\cdot\pi\cdot D_{piston}^{2}\cdot L + 4\cdot Mass\right)\cdot\left(D_{tube} - D_{piston}\right)}{8\cdot\mu\cdot\pi\cdot D_{piston}\cdot L}$$

or $$V = 10.2\frac{m}{s}$$

Problem 2.43

2.43 The piston in Problem 2.42 is traveling at terminal speed. The mass m now disconnects from the piston. Plot the piston speed vs. time. How long does it take the piston to come within 1 percent of its new terminal speed?

Given: Flow data on apparatus

Find: Sketch of piston speed vs time; the time needed for the piston to reach 99% of its new terminal speed.

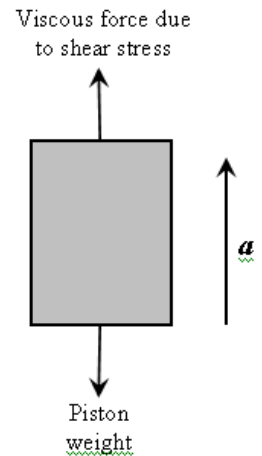
Solution:

Given data: $D_{piston} = 73 \cdot mm$ $\quad D_{tube} = 75 \cdot mm$ $\quad L = 100 \cdot mm$ $\quad SG_{Al} = 2.64$ $\quad V_0 = 10.2 \cdot \frac{m}{s}$

Reference data: $\rho_{water} = 1000 \cdot \frac{kg}{m^3}$ (maximum density of water) (From Problem 2.42)

From Fig. A.2, the dynamic viscosity of SAE 10W-30 oil at 25°C is: $\mu = 0.13 \cdot \frac{N \cdot s}{m^2}$

The free body diagram of the piston after the cord is cut is:

Piston weight: $$W_{piston} = SG_{Al} \cdot \rho_{water} \cdot g \cdot \left(\frac{\pi \cdot D_{piston}^2}{4}\right) \cdot L$$

Viscous force: $F_{viscous}(V) = \tau_{rz} \cdot A$ or $$F_{viscous}(V) = \mu \cdot \left[\frac{V}{\frac{1}{2} \cdot (D_{tube} - D_{piston})}\right] \cdot (\pi \cdot D_{piston} \cdot L)$$

Applying Newton's second law: $$m_{piston} \cdot \frac{dV}{dt} = W_{piston} - F_{viscous}(V)$$

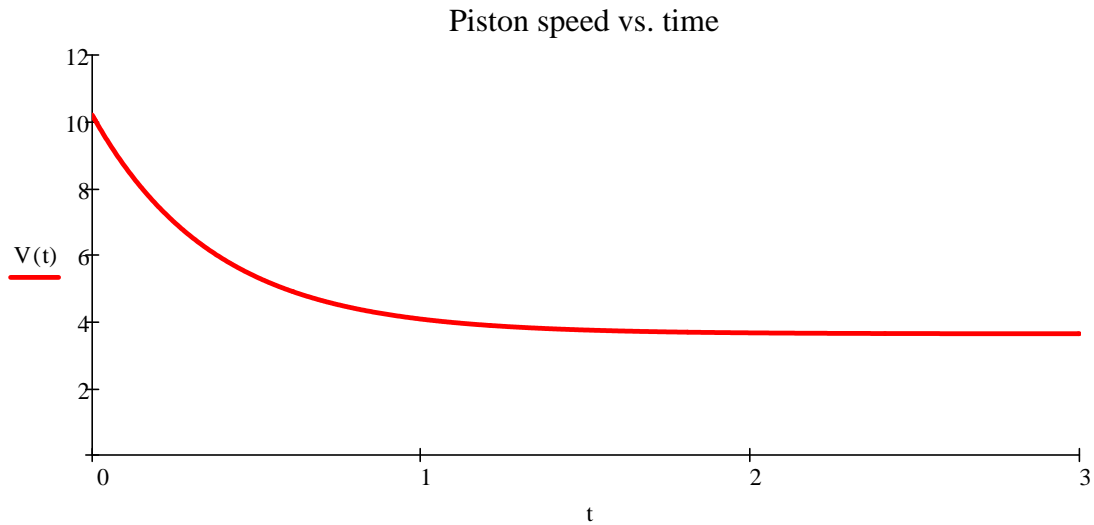
Therefore $\frac{dV}{dt} = g - a \cdot V$ where $$a = \frac{8 \cdot \mu}{SG_{Al} \cdot \rho_{water} \cdot D_{piston} \cdot (D_{tube} - D_{piston})}$$

If $V = g - a \cdot V$ then $\frac{dX}{dt} = -a \cdot \frac{dV}{dt}$

The differential equation becomes $\frac{dX}{dt} = -a \cdot X$ where $X(0) = g - a \cdot V_0$

The solution to this differential equation is: $X(t) = X_0 \cdot e^{-a \cdot t}$ or $g - a \cdot V(t) = (g - a \cdot V_0) \cdot e^{-a \cdot t}$

Therefore $V(t) = \left(V_0 - \frac{g}{a}\right) \cdot e^{(-a \cdot t)} + \frac{g}{a}$

Plotting piston speed vs. time (which can be done in Excel)

The terminal speed of the piston, V_t, is evaluated as t approaches infinity

$$V_t = \frac{g}{a} \qquad \text{or} \qquad V_t = 3.63 \frac{m}{s}$$

The time needed for the piston to slow down to within 1% of its terminal velocity is:

$$t = \frac{1}{a} \cdot \ln\left(\frac{V_0 - \frac{g}{a}}{1.01 \cdot V_t - \frac{g}{a}}\right) \qquad \text{or} \qquad t = 1.93\,s$$

Given: Block of mass M slides on thin film of oil of thickness h. Contact area of block is A. At time t=0, mass m is released from rest.

Block, Cord, Oil film (viscosity, μ), Mass, h, V, $\vec{g}$, M, m

$M = 5\,kg$, $m = 1\,kg$, $A = 25\,cm^2$, $h = 0.5\,m$

Find: (a) Expression for viscous force on block when moving at speed V
(b) Differential equation governing block speed as a function of time
(c) Expression for block speed $V = V(t)$; plot
(d) If $V = 1\,m/s$ at $t = 1\,s$, find μ

Solution:

Basic equations: $\tau_{yx} = \mu \frac{du}{dy}$ $\qquad \Sigma \vec{F} = m\vec{a}$

Assumptions: (1) Newtonian fluid
(2) Linear velocity profile in oil film

Then, $F_v = \tau A = \mu \frac{du}{dy} A = \mu \frac{\Delta u}{\Delta y} A = \mu \frac{V}{h} A$ ←— F_v

For the block, $\Sigma F_x = F_t - F_v = M \frac{dV_b}{dt}$ (1)

For the falling mass $\Sigma F_y = mg - F_t = m \frac{dV_m}{dt}$, or

$F_t = mg - m\frac{dV_m}{dt}$ (2)

Since $V_b = V_m = V$, then substituting from Eq.(2) into (1) gives

$$mg - m\frac{dV}{dt} - F_v = M\frac{dV}{dt} = mg - m\frac{dV}{dt} - \mu\frac{V}{h}A$$

Finally, $mg - \mu \frac{V}{h} A = (M+m)\frac{dV}{dt}$ ←— Diff. Eq.

To solve we separate variables and integrate

$$t = \int_0^t dt = \int_0^V \frac{(M+m)}{mg - \mu\frac{V}{h}A}\,dV = -(M+m)\frac{h}{\mu A}\ln\left(mg - \frac{\mu V A}{h}\right)\Bigg]_0^V$$

$$t = -\frac{(M+m)h}{\mu A}\ln\left(1 - \frac{\mu V A}{mgh}\right)$$

Taking antilogarithms,

$$1 - \frac{\mu V A}{mgh} = e^{-\frac{\mu A t}{(M+m)h}}$$

Solving for V,

$$V = \frac{mgh}{\mu A}\left(1 - e^{-\frac{\mu A t}{(M+m)h}}\right)$$ ←— V

[Plot of V versus t]

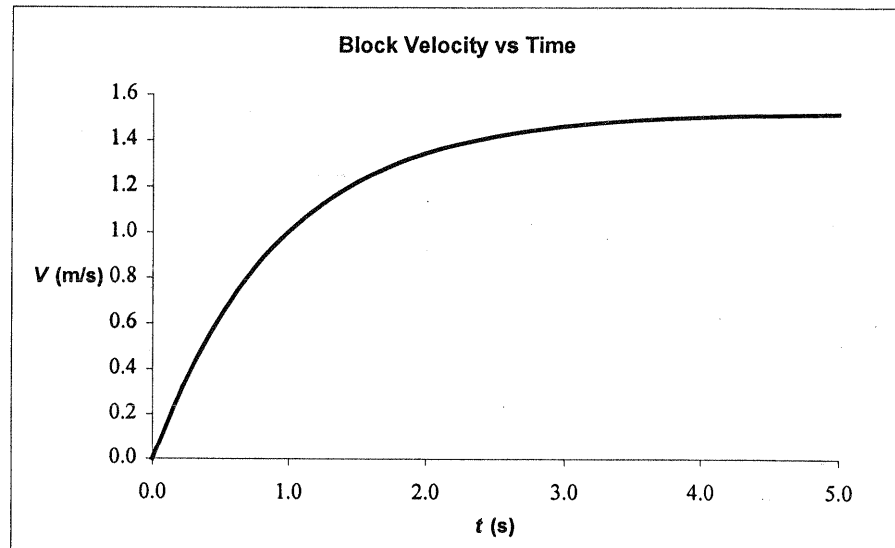
The velocity increases exponentially to $V_{max} = \frac{mgh}{\mu A}$

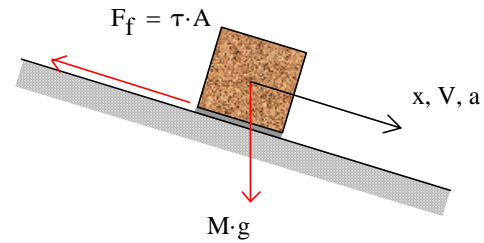
Using Excel's solver, with $V = 1$ m/s at $t = 1$ s, the oil viscosity is found to be $\mu = 1.29$ N.s/m² ←── μ

The V(t) plot, with M = 5 kg, m = 1 kg, A = 25 cm², h = 0.5 mm and μ = 1.29 N.s/m², is generated from

$$V = \frac{Mgh}{\mu A}\left[1 - e^{-\frac{\mu A t}{(M+m)h}}\right]$$

t (s)	*V* (m/s)
0.00	0.00
0.25	0.36
0.50	0.63
0.75	0.84
1.00	1.00
1.25	1.12
1.50	1.22
1.75	1.29
2.00	1.34
2.25	1.39
2.50	1.42
2.75	1.44
3.00	1.46
3.25	1.47
3.50	1.49
3.75	1.49
4.00	1.50
4.25	1.51
4.50	1.51
4.75	1.51
5.00	1.51

M = 5 kg
m = 1 kg
A = 25 cm²
h = 0.5 mm
μ = **1.29** N.s/m² (From *Solver* or *Goal Seek*)

Problem 2.45

2.45 A block 0.1 m square, with 5 kg mass, slides down a smooth incline, 30° below the horizontal, on a film of SAE 30 oil at 20°C that is 0.20 mm thick. If the block is released from rest at $t = 0$, what is its initial acceleration? Derive an expression for the speed of the block as a function of time. Plot the curve for $V(t)$. Find the speed after 0.1 s. If we want the mass to instead reach a speed of 0.3 m/s at this time, find the viscosity μ of the oil we would have to use.

Given:
Data on the block and incline

Find:
Initial acceleration; formula for speed of block; plot; find speed after 0.1 s. Find oil viscosity if speed is 0.3 m/s after 0.1 s

Solution:

Given data $M = 5\cdot kg$ $A = (0.1\cdot m)^2$ $d = 0.2\cdot mm$ $\theta = 30\cdot deg$

From Fig. A.2 $\mu = 0.4\cdot\frac{N\cdot s}{m^2}$

Applying Newton's 2nd law to initial instant (no frictic $M\cdot a = M\cdot g\cdot\sin(\theta) - F_f = M\cdot g\cdot\sin(\theta)$

so $a_{init} = g\cdot\sin(\theta) = 9.81\cdot\frac{m}{s^2}\times\sin(30\cdot deg)$ $a_{init} = 4.9\frac{m}{s^2}$

Applying Newton's 2nd law at any instant $M\cdot a = M\cdot g\cdot\sin(\theta) - F_f$ and $F_f = \tau\cdot A = \mu\cdot\frac{du}{dy}\cdot A = \mu\cdot\frac{V}{d}\cdot A$

so $M\cdot a = M\cdot\frac{dV}{dt} = M\cdot g\cdot\sin(\theta) - \frac{\mu\cdot A}{d}\cdot V$

Separating variables

$$\frac{dV}{g\cdot\sin(\theta) - \frac{\mu\cdot A}{M\cdot d}\cdot V} = dt$$

Integrating and using limits

$$-\frac{M\cdot d}{\mu\cdot A}\cdot\ln\left(1 - \frac{\mu\cdot A}{M\cdot g\cdot d\cdot\sin(\theta)}\cdot V\right) = t$$

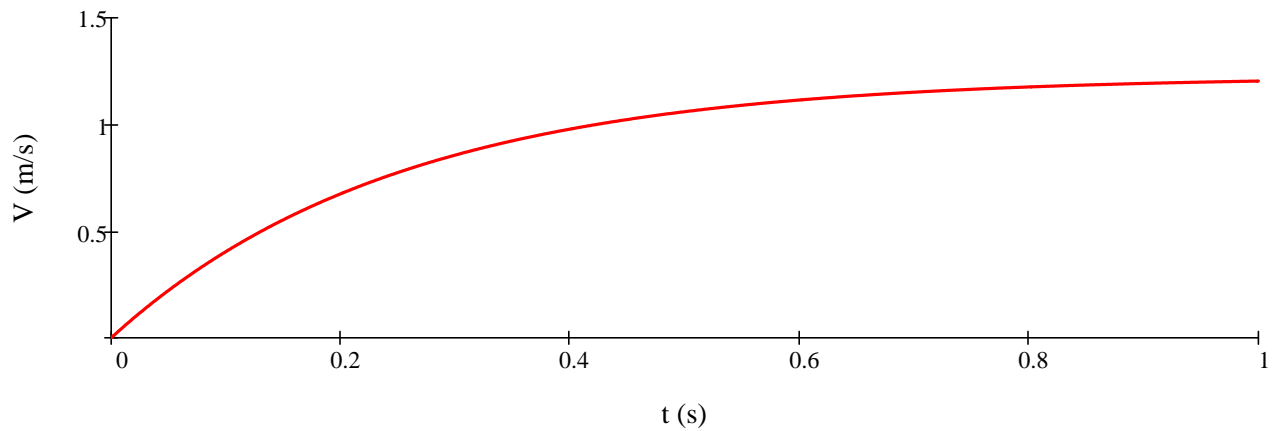
or

$$V(t) = \frac{M\cdot g\cdot d\cdot\sin(\theta)}{\mu\cdot A}\cdot\left(1 - e^{\frac{-\mu\cdot A}{M\cdot d}\cdot t}\right)$$

At t = 0.1 s

$$V = 5\cdot kg\times 9.81\cdot\frac{m}{s^2}\times 0.0002\cdot m\cdot\sin(30\cdot deg)\times\frac{m^2}{0.4\cdot N\cdot s\cdot(0.1\cdot m)^2}\times\frac{N\cdot s^2}{kg\cdot m}\times\left[1 - e^{-\left(\frac{0.4\cdot 0.01}{5\cdot 0.0002}\cdot 0.1\right)}\right]$$

$$V(0.1\cdot s) = 0.404\cdot\frac{m}{s}$$

The plot looks like

To find the viscosity for which V(0.1 s) = 0.3 m/s, we must solve

$$V(t = 0.1\cdot s) = \frac{M\cdot g\cdot d\cdot \sin(\theta)}{\mu\cdot A}\cdot\left[1 - e^{\frac{-\mu\cdot A}{M\cdot d}\cdot(t=0.1\cdot s)}\right]$$

The viscosity μ is implicit in this equation, so solution must be found by manual iteration, or by any of a number of classic root-finding numerical methods, or by using *Excel*'s *Goal Seek*

Using *Excel*: $\mu = 1.08\cdot\frac{N\cdot s}{m^2}$

Problem 2.46

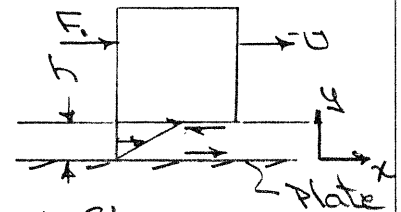
Given: Block of mass M moves at steady speed V under influence of constant force F_1 on a thin film of oil of thickness h and viscosity μ; block is square, a mm on a side.

Find: (a) Magnitude and direction of shear stress acting on bottom of block and supporting plate.
(b) Expression for time required to lose 95 % of its initial speed when force is suddenly removed.
(c) Expect shape of speed vs time curve.

Solution:

Basic equations: $\tau_{yx} = \mu \frac{du}{dy}$ $\quad \Sigma \vec{F} = m\vec{a}$

Assumptions: (1) Newtonian fluid
(2) Linear velocity profile in oil film

$$\tau_{yx} = \mu \frac{du}{dy} = \mu \frac{\Delta u}{\Delta y} = \mu \frac{V}{h} \qquad \tau_{yx}$$

Bottom of block is $-y$ surface, so τ_{yx} acts to left
Plate surface is $+y$ surface, so τ_{yx} acts to right

Viscous shear force on block is $F_v = \tau A = \tau a^2 = \frac{\mu V a^2}{h}$

When F_1 is removed, block slows under action of F_v

$$\Sigma F_x = m \frac{dV}{dt} = -F_v = -\frac{\mu V a^2}{h}$$

Separating variables and integrating we have

$$\int_{V_i}^{V} \frac{dV}{V} = -\int_0^t \frac{\mu a^2}{mh}\, dt$$

Then

$$\ln \frac{V}{V_i} = -\frac{\mu a^2}{mh} t \quad \text{----(1)}$$

and

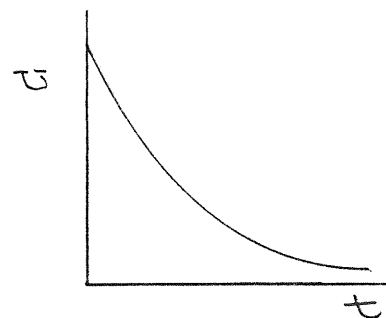
$$t = -\frac{mh}{\mu a^2} \ln \frac{V}{V_i}$$

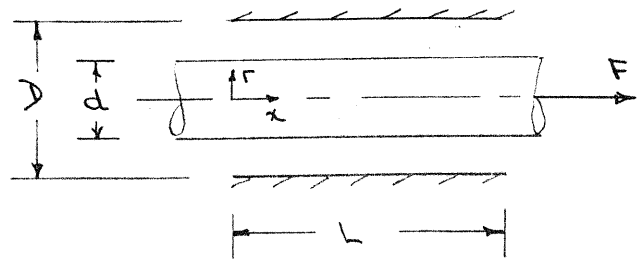
For $V/V_i = 0.05$

$$t = 3.0 \frac{mh}{\mu a^2} \qquad t$$

From Eq.(1) we can write

$$V = V_i e^{-\frac{\mu a^2 t}{h}}$$

The speed thus decreases exponentially with time.

Problem 2.47

Given: Wire, of diameter d, is to be coated with varnish by drawing it through a circular die of diameter, D, and length, L.

$d = 0.9\ mm$, $D = 1.0\ mm$, $L = 50\ mm$

Varnish, $\mu = 20$ centipoise fills the space between wire and die
Wire is drawn through at speed, $V = 50\ m/s$

Find: Force required to pull the wire

Solution

$$\Sigma F_x = ma_x$$

Since V_{wire} = constant, applied force must be sufficient to balance friction force, F_f

$F_f = \tau A$ where $\tau = \mu \frac{du}{dr}$ and $A = \pi d L$

Assuming a linear velocity distribution in varnish

$$\tau_s = \mu \left.\frac{du}{dr}\right)_s = \mu \frac{V_{D/2} - V_{d/2}}{D/2 - d/2} = -\mu \frac{V}{(D-d)/2}$$

(negative stress on positive r surface must act in negative x direction)

$$F - F_f = 0$$

$$F = \tau A = \mu \frac{2V}{(D-d)} \times \pi d L$$

$$F = 20\ cp \times \frac{gm}{100\ cm.s.cp} \times 2\pi \times 50 \frac{m}{s} \times 0.9\ mm \times 50\ mm \times \frac{1}{0.1\ mm} \times \frac{cm}{10\ mm} \times \frac{kg}{1000\ gm} \times \frac{N.s^2}{kg.m}$$

$F = 2.83\ N$ ← F

Problem 2.48

2.48 A double-pipe heat exchanger consists of two concentric fluid-carrying pipes used to transfer heat between nonmixing fluids. The figure shown below is a full-section view of a 0.85-m length of the double-pipe apparatus.

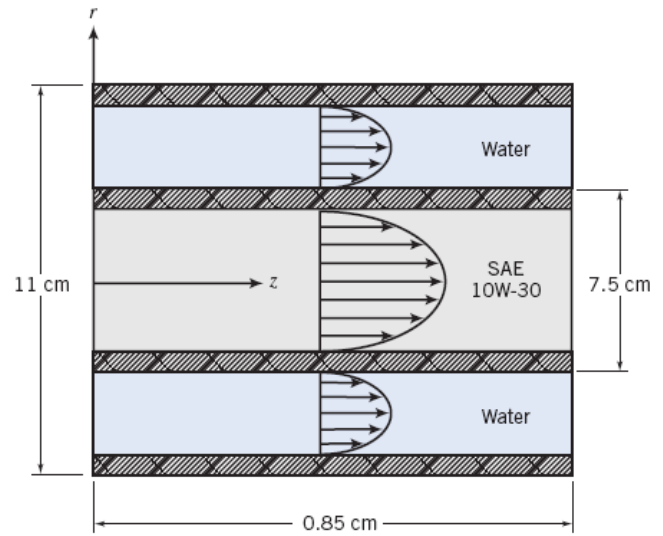
SAE10W-30 oil at 100°C flows through the 7.5-cm-outer-diameter inside pipe. Water at 10°C flows through the annulus between the inside pipe and the 11-cm-outer-diameter outside pipe. The wall thickness of each pipe is 3 mm. The theoretical velocity profiles for laminar flow through a pipe and annulus are:

Inner pipe: $u_z(r) = u_{max}\left[1 - \left(\frac{r}{R_{i,\text{ inside}}}\right)^2\right]$

where: $u_{max} = \frac{R_{i,\text{ inside}}^2 \Delta P}{4\mu L}$

Annulus: $u_z(r) = \frac{1}{4\mu}\left(\frac{\Delta P}{L}\right)$

$$\times \left[R_{i,\text{ outside}}^2 - r^2 - \frac{R_{o,\text{ inside}}^2 - R_{i,\text{ outside}}^2}{\ln\left(\frac{R_{i,\text{ outside}}}{R_{o,\text{ inside}}}\right)} \bullet \ln\left(\frac{r}{R_{i,\text{ outside}}}\right)\right]$$

Show that the no-slip condition is satisfied by these expressions. The pressure drop across the given length is 2.5 Pa and 8 Pa for the water and oil flows, respectively. If both flows are in the same direction (along the $+z$ axis), what is the net viscous force acting on the inner pipe?

NOTE: Figure is wrong - length is 0.85 m

Given: Data on double pipe heat exchanger

Find: Whether no-slip is satisfied; net viscous force on inner pipe

Solution:

For the oil, the velocity profile is $u_z(r) = u_{max}\cdot\left[1 - \left(\frac{r}{R_{ii}}\right)^2\right]$ where $u_{max} = \frac{R_{ii}^2\cdot\Delta p}{4\cdot\mu\cdot L}$

Check the no-slip condition. When $r = R_{ii}$ $u_z(R_{ii}) = u_{max}\cdot\left[1 - \left(\frac{R_{ii}}{R_{ii}}\right)^2\right] = 0$

For the water, the velocity profile is $u_z(r) = \frac{1}{4\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left(R_{io}^2 - r^2 - \frac{R_{oi}^2 - R_{io}^2}{\ln\left(\frac{R_{io}}{R_{oi}}\right)}\cdot\ln\left(\frac{r}{Rio}\right)\right)$

Check the no-slip condition. When $r = R_{oi}$ $u_z(R_{oi}) = \frac{1}{4\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left(R_{io}^2 - R_{oi}^2 - \frac{R_{oi}^2 - R_{io}^2}{\ln\left(\frac{R_{io}}{R_{oi}}\right)}\cdot\ln\left(\frac{R_{oi}}{Rio}\right)\right)$

$$u_z(R_{oi}) = \frac{1}{4\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left[R_{io}^2 - R_{oi}^2 + \left(R_{oi}^2 - R_{io}^2\right)\right] = 0$$

When $r = R_{io}$ $$u_z(R_{io}) = \frac{1}{4\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left(R_{io}^2 - R_{io}^2 - \frac{R_{oi}^2 - R_{io}^2}{\ln\left(\frac{R_{io}}{R_{oi}}\right)}\cdot\ln\left(\frac{R_{io}}{R_{io}}\right)\right) = 0$$

The no-slip condition holds on all three surfaces.

The given data is $R_{ii} = \frac{7.5\cdot cm}{2} - 3\cdot mm$ $R_{ii} = 3.45\cdot cm$ $R_{io} = \frac{7.5\cdot cm}{2}$ $R_{io} = 3.75\cdot cm$ $R_{oi} = \frac{11\cdot cm}{2} - 3\cdot mm$ $R_{oi} = 5.2\cdot cm$

$\Delta p_w = 2.5\cdot Pa$ $\Delta p_{oil} = 8\cdot Pa$ $L = 0.85\cdot m$

The viscosity of water at 10ºC is (Fig. A.2) $\mu_w = 1.25\times 10^{-3}\cdot\frac{N\cdot s}{m^2}$

The viscosity of SAE 10-30 oil at 100ºC is (Fig. A.2) $\mu_{oil} = 1\times 10^{-2}\cdot\frac{N\cdot s}{m^2}$

For each, shear stress is given by $\tau_{rx} = \mu\cdot\frac{du}{dr}$

For water $$\tau_{rx} = \mu\cdot\frac{du_z(r)}{dr} = \mu_w\cdot\frac{d}{dr}\left[\frac{1}{4\cdot\mu_w}\cdot\frac{\Delta p_w}{L}\cdot\left(R_{io}^2 - r^2 - \frac{R_{oi}^2 - R_{io}^2}{\ln\left(\frac{R_{io}}{R_{oi}}\right)}\cdot\ln\left(\frac{r}{Rio}\right)\right)\right]$$

$$\tau_{rx} = \frac{1}{4}\cdot\frac{\Delta p_w}{L}\cdot\left(-2\cdot r - \frac{R_{oi}^2 - R_{io}^2}{\ln\left(\frac{R_{io}}{R_{oi}}\right)\cdot r}\right)$$

so on the pipe surface $$F_w = \tau_{rx}\cdot A = \frac{1}{4}\cdot\frac{\Delta p_w}{L}\cdot\left(-2\cdot R_{io} - \frac{R_{oi}^2 - R_{io}^2}{\ln\left(\frac{R_{io}}{R_{oi}}\right)\cdot R_{io}}\right)\cdot 2\cdot\pi\cdot R_{io}\cdot L$$

$$F_w = \Delta p_w\cdot\pi\cdot\left(-R_{io}^2 - \frac{R_{oi}^2 - R_{io}^2}{2\cdot\ln\left(\frac{R_{io}}{R_{oi}}\right)}\right)$$

Hence $$F_w = 2.5\cdot\frac{N}{m^2}\times\pi\times\left[-\left(3.75\cdot cm\times\frac{1\cdot m}{100\cdot cm}\right)^2 - \frac{\left[(5.2\cdot cm)^2 - (3.75\cdot cm)^2\right]\times\left(\frac{1\cdot m}{100\cdot cm}\right)^2}{2\cdot\ln\left(\frac{3.75}{5.2}\right)}\right]$$

$$F_w = 0.00454\,N$$

This is the force on the r-negative surface of the fluid; on the outer pipe itself we also have $F_w = 0.00454\,N$

For oil $$\tau_{rx} = \mu\cdot\frac{du_z(r)}{dr} = \mu_{oil}\cdot\frac{d}{dr}\,u_{max}\cdot\left[1 - \left(\frac{r}{R_{ii}}\right)^2\right] = -\frac{2\cdot\mu_{oil}\cdot u_{max}\cdot r}{R_{ii}^2} = -\frac{\Delta p_{oil}\cdot r}{2\cdot L}$$

so on the pipe surface $$F_{oil} = \tau_{rx}\cdot A = -\frac{\Delta p_{oil}\cdot Rii}{2\cdot L}\cdot 2\cdot\pi\cdot R_{ii}\cdot L = -\Delta p_{oil}\cdot\pi\cdot R_{ii}^2$$

This should not be a surprise: the pressure drop just balances the friction!

Hence $F_{oil} = -8 \cdot \frac{N}{m^2} \times \pi \times \left(3.45 \cdot cm \times \frac{1 \cdot m}{100 \cdot cm}\right)^2$ $F_{oil} = -0.0299\,N$

This is the force on the r-positive surface of the fluid; on the pipe it is equal and opposite $F_{oil} = 0.0299\,N$

The total force is $F = F_w + F_{oil}$ $F = 0.0345\,N$

Note we didn't need the viscosities because all quantities depend on the Δp's!

Problem 2.49

2.49 Repeat Problem 2.48 assuming a counterflow arrangement, where the oil flows in the $+z$ direction and the water flows in the $-z$ direction.

SAE10W-30 oil at 100°C flows through the 7.5-cm-outer-diameter inside pipe. Water at 10°C flows through the annulus between the inside pipe and the 11-cm-outer-diameter outside pipe. The wall thickness of each pipe is 3 mm. The theoretical velocity profiles for laminar flow through a pipe and annulus are:

Inner pipe: $$u_z(r) = u_{max}\left[1 - \left(\frac{r}{R_{i,\,\text{inside}}}\right)^2\right]$$

where: $$u_{max} = \frac{R^2_{i,\,\text{inside}}\Delta P}{4\mu L}$$

Annulus: $$u_z(r) = \frac{1}{4\mu}\left(\frac{\Delta P}{L}\right) \times \left[R^2_{i,\,\text{outside}} - r^2 - \frac{R^2_{o,\,\text{inside}} - R^2_{i,\,\text{outside}}}{\ln\left(\frac{R_{i,\,\text{outside}}}{R_{o,\,\text{inside}}}\right)} \cdot \ln\left(\frac{r}{R_{i,\,\text{outside}}}\right)\right]$$

Show that the no-slip condition is satisfied by these expressions. The pressure drop across the given length is 2.5 Pa and 8 Pa for the water and oil flows, respectively. If both flows are in the same direction (along the $+z$ axis), what is the net viscous force acting on the inner pipe?

NOTE: Figure is wrong - length is 0.85 m

Given: Data on counterflow heat exchanger

Find: Whether no-slip is satisfied; net viscous force on inner pipe

Solution:

The analysis for Problem 2.48 is repeated, except the oil flows in reverse, so the pressure drop is -2.5 Pa not 2.5 Pa.

For the oil, the velocity profile is $$u_z(r) = u_{max}\cdot\left[1 - \left(\frac{r}{R_{ii}}\right)^2\right]$$ where $$u_{max} = \frac{R_{ii}^2\cdot\Delta p}{4\cdot\mu\cdot L}$$

Check the no-slip condition. When $r = R_{ii}$ $$u_z(R_{ii}) = u_{max}\cdot\left[1 - \left(\frac{R_{ii}}{R_{ii}}\right)^2\right] = 0$$

For the water, the velocity profile is $$u_z(r) = \frac{1}{4\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left(R_{io}^2 - r^2 - \frac{R_{oi}^2 - R_{io}^2}{\ln\left(\frac{R_{io}}{R_{oi}}\right)}\cdot\ln\left(\frac{r}{R_{io}}\right)\right)$$

Check the no-slip condition. When $r = R_{oi}$ $$u_z(R_{oi}) = \frac{1}{4\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left(R_{io}^2 - R_{oi}^2 - \frac{R_{oi}^2 - R_{io}^2}{\ln\left(\frac{R_{io}}{R_{oi}}\right)}\cdot\ln\left(\frac{R_{oi}}{R_{io}}\right)\right)$$

$$u_z(R_{oi}) = \frac{1}{4\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left[R_{io}^2 - R_{oi}^2 + \left(R_{oi}^2 - R_{io}^2\right)\right] = 0$$

When $r = R_{io}$ $$u_z(R_{io}) = \frac{1}{4\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left(R_{io}^2 - R_{io}^2 - \frac{R_{oi}^2 - R_{io}^2}{\ln\left(\frac{R_{io}}{R_{oi}}\right)}\cdot\ln\left(\frac{R_{io}}{R_{io}}\right)\right) = 0$$

The no-slip condition holds on all three surfaces.

The given data is $R_{ii} = \frac{7.5\cdot cm}{2} - 3\cdot mm$ $R_{ii} = 3.45\cdot cm$ $R_{io} = \frac{7.5\cdot cm}{2}$ $R_{io} = 3.75\cdot cm$ $R_{oi} = \frac{11\cdot cm}{2} - 3\cdot mm$ $R_{oi} = 5.2\cdot cm$

$\Delta p_w = -2.5\cdot Pa$ $\Delta p_{oil} = 8\cdot Pa$ $L = 0.85\cdot m$

The viscosity of water at 10ºC is (Fig. A.2) $\mu_w = 1.25\times 10^{-3}\cdot\frac{N\cdot s}{m^2}$

The viscosity of SAE 10-30 oil at 100ºC is (Fig. A.2) $\mu_{oil} = 1\times 10^{-2}\cdot\frac{N\cdot s}{m^2}$

For each, shear stress is given by $\tau_{rx} = \mu\cdot\frac{du}{dr}$

For water $$\tau_{rx} = \mu\cdot\frac{du_z(r)}{dr} = \mu_w\cdot\frac{d}{dr}\left[\frac{1}{4\cdot\mu_w}\cdot\frac{\Delta p_w}{L}\cdot\left(R_{io}^2 - r^2 - \frac{R_{oi}^2 - R_{io}^2}{\ln\left(\frac{R_{io}}{R_{oi}}\right)}\cdot\ln\left(\frac{r}{R_{io}}\right)\right)\right]$$

$$\tau_{rx} = \frac{1}{4}\cdot\frac{\Delta p_w}{L}\cdot\left(-2\cdot r - \frac{R_{oi}^2 - R_{io}^2}{\ln\left(\frac{R_{io}}{R_{oi}}\right)\cdot r}\right)$$

so on the pipe surface $$F_w = \tau_{rx}\cdot A = \frac{1}{4}\cdot\frac{\Delta p_w}{L}\cdot\left(-2\cdot R_{io} - \frac{R_{oi}^2 - R_{io}^2}{\ln\left(\frac{R_{io}}{R_{oi}}\right)\cdot R_{io}}\right)\cdot 2\cdot\pi\cdot R_{io}\cdot L$$

$$F_w = \Delta p_w\cdot\pi\cdot\left(-R_{io}^2 - \frac{R_{oi}^2 - R_{io}^2}{2\cdot\ln\left(\frac{R_{io}}{R_{oi}}\right)}\right)$$

Hence $$F_w = -2.5\cdot\frac{N}{m^2}\times\pi\times\left[-\left[(3.75\cdot cm)\times\frac{1\cdot m}{100\cdot cm}\right]^2 - \frac{\left[(5.2\cdot cm)^2 - (3.75\cdot cm)^2\right]\times\left(\frac{1\cdot m}{100\cdot cm}\right)^2}{2\cdot\ln\left(\frac{3.75}{5.2}\right)}\right]$$

$$F_w = -0.00454\,N$$

This is the force on the r-negative surface of the fluid; on the outer pipe itself we also have $F_w = -0.00454\,N$

For oil $$\tau_{rx} = \mu\cdot\frac{du_z(r)}{dr} = \mu_{oil}\cdot\frac{d}{dr}u_{max}\cdot\left[1 - \left(\frac{r}{R_{ii}}\right)^2\right] = -\frac{2\cdot\mu_{oil}\cdot u_{max}\cdot r}{R_{ii}^2} = -\frac{\Delta p_{oil}\cdot r}{2\cdot L}$$

so on the pipe surface $$F_{oil} = \tau_{rx}\cdot A = -\frac{\Delta p_{oil}\cdot Rii}{2\cdot L}\cdot 2\cdot\pi\cdot R_{ii}\cdot L = -\Delta p_{oil}\cdot\pi\cdot R_{ii}^2$$

This should not be a surprise: the pressure drop just balances the friction!

Hence $F_{oil} = -8 \cdot \frac{N}{m^2} \times \pi \times \left(3.45 \cdot cm \times \frac{1 \cdot m}{100 \cdot cm}\right)^2$ $F_{oil} = -0.0299\,N$

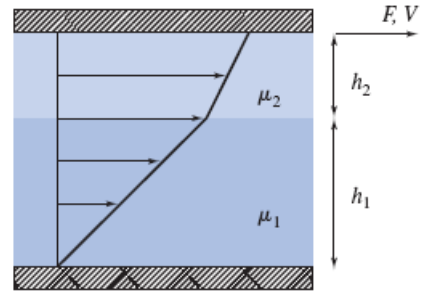
This is the force on the r-positive surface of the fluid; on the pipe it is equal and opposite $F_{oil} = 0.0299\,N$

The total force is $F = F_w + F_{oil}$ $F = 0.0254\,N$

Note we didn't need the viscosities because all quantities depend on the Δp's!

Problem 2.50

[2]

2.50 Fluids of viscosities $\mu_1 = 0.1$ N•s/m^2 and $\mu_2 = 0.15$ N • s/m^2 are contained between two plates (each plate is 1 m^2 in area). The thicknesses are $h_1 = 0.5$ mm and $h_2 = 0.3$ mm, respectively. Find the force F to make the upper plate move at a speed of 1 m/s. What is the fluid velocity at the interface between the two fluids?

Given: Flow between two plates

Find: Force to move upper plate; Interface velocity

Solution:

The shear stress is the same throughout (the velocity gradients are linear, and the stresses in the fluid at the interface must be equal and opposite).

Hence $\tau = \mu_1 \cdot \frac{du_1}{dy} = \mu_2 \cdot \frac{du_2}{dy}$ or $\mu_1 \cdot \frac{V_i}{h_1} = \mu_2 \cdot \frac{(V - V_i)}{h_2}$ where V_i is the interface velocity

Solving for the interface velocity V_i

$$V_i = \frac{V}{1 + \frac{\mu_1}{\mu_2} \cdot \frac{h_2}{h_1}} = \frac{1 \cdot \frac{m}{s}}{1 + \frac{0.1}{0.15} \cdot \frac{0.3}{0.5}}$$

$$V_i = 0.714 \frac{m}{s}$$

Then the force required is

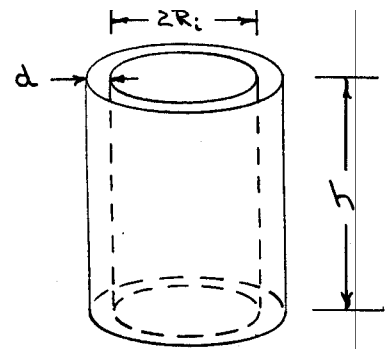
$$F = \tau \cdot A = \mu_1 \cdot \frac{V_i}{h_1} \cdot A = 0.1 \cdot \frac{N \cdot s}{m^2} \times 0.714 \cdot \frac{m}{s} \times \frac{1}{0.5 \cdot mm} \times \frac{1000 \cdot mm}{1 \cdot m} \times 1 \cdot m^2$$

$$F = 143 N$$

Problem 2.51

Given: Concentric cylinder viscometer

$R_i = 37.5$ mm, $d = 0.02$ mm, $h = 150$ mm

Inner cylinder rotates at $\omega = 100$ rpm, under torque, $T = 0.021$ N·m

Find: Viscosity of liquid in clearance gap.

Solution

The imposed torque must balance the resisting torque of the shear force.

The shear force is given by $F = \tau A$ where $A = 2\pi R_i h$

For a Newtonian fluid $\tau = \mu \frac{du}{dy}$

Since the velocity profile is assumed to be linear, $\tau = \mu \frac{V}{d}$

where V is the tangential velocity of the inner cylinder, $V = R_i \omega$

Thus,

$$F = \tau A = \mu \frac{V}{d} 2\pi R_i h = \frac{2\pi \mu R_i^2 \omega h}{d}$$

and the torque

$$T = R_i F = \frac{2\pi \mu R_i^3 \omega h}{d}$$

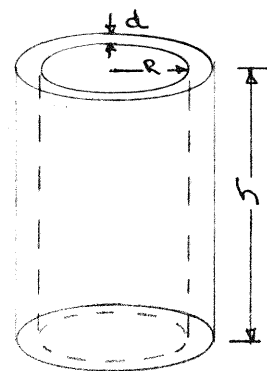
Solving for μ,

$$\mu = \frac{T d}{2\pi R_i^3 \omega h} = 0.021 \text{ N·m} \times 0.02 \text{ mm} \times \frac{1}{2\pi} \times \frac{1}{(37.5)^3 \text{ mm}^3} \times \frac{\text{min}}{100 \text{ rev}} \times \frac{1}{150 \text{ mm}}$$

$$\times \frac{\text{rev}}{2\pi \text{ rad}} \times \frac{60 \text{ s}}{\text{min}} \times (1000)^3 \frac{\text{mm}^3}{\text{m}^3}$$

$$\mu = 8.07 \times 10^{-4} \text{ N·s/m}^2 \longleftarrow \mu$$

Given: Concentric cylinder viscometer.
$R = 2.0$ in $\quad d = 0.001$ in $\quad h = 8$ in.
Inner cylinder rotates at 400 rpm
Gap filled with castor oil at 90°F.

Determine: Torque required to rotate the inner cylinder

Solution:

The required torque must balance the resisting torque of the shear force.

The shear force is given by $F = \tau A$ where $A = 2\pi R h$

For a Newtonian fluid $\tau = \mu \frac{du}{dy}$

For small gap (linear profile) $\tau = \mu \frac{V}{d}$

where V = tangential velocity of inner cylinder $= R\omega$

Hence
$$F = \tau A = \mu \frac{R\omega}{d} 2\pi R h = \frac{2\pi \mu R^2 \omega h}{d}$$

and the torque $T = RF = \dfrac{2\pi \mu R^3 \omega h}{d}$

From Fig A.2, for castor oil at 90°F (32°C), $\mu = 3.80 \times 10^{-1}$ N.s/m²

Substituting numerical values.

$$T = \frac{2\pi \mu R^3 \omega h}{d} = 2\pi \times 3.80 \times 10^{-1} \frac{\text{N.s}}{\text{m}^2} \times 2.09 \times 10^{-2} \frac{\text{lbf} \cdot \text{s} \cdot \text{m}^2}{\text{ft}^2 \cdot \text{N} \cdot \text{s}} \times (2.0)^3 \text{in}^3 \times 400 \frac{\text{rev}}{\text{min}} \times 8 \text{in} \times \frac{1}{10^{-3} \text{in}}$$
$$\times 2\pi \frac{\text{rad}}{\text{rev}} \times \frac{\text{min}}{60\,\text{s}} \times \frac{\text{ft}^3}{1728\,\text{in}^3}$$

$T = 77.4$ ft·lbf ← Torque

Problem 2.53

Given: Concentric-cylinder viscometer, driven by falling mass.

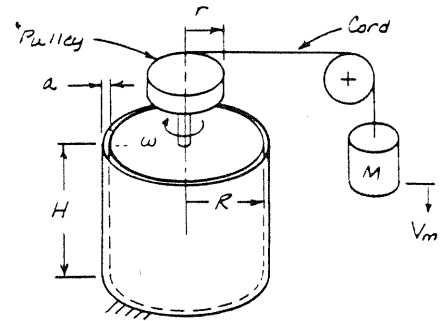
$M = 0.10$ kg $\quad r = 25$ mm

$R = 50$ mm $\quad a = 0.20$ mm

$H = 80$ mm $\quad V_m = 30$ mm/s

After starting transient, $V_m = \text{const.}$

Find: (a) An algebraic expression for viscosity of the liquid, in terms of $M, g, V_m, r, R, a,$ and H.

(b) Evaluate using the data given.

Solution: Apply Newton's law of viscosity.

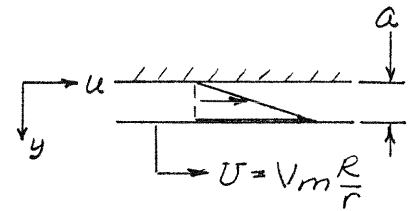
Basic equations: $\tau = \mu \dfrac{du}{dy} \qquad \Sigma M = 0 \qquad T = \tau A R$

Assumptions: (1) Newtonian liquid
(2) Narrow gap, so linear velocity profile
(3) Steady angular speed

Summing torques on the rotor

$$\Sigma M = Mgr - \tau A R = I\alpha \overset{=0(3)}{} = 0 \; ; \; A = 2\pi R H$$

Because $a \ll R$, treat the gap as plane. Then

$$\tau = \mu \frac{du}{dy} = \mu \frac{\Delta u}{\Delta y} = \mu \frac{U - 0}{a - 0} = \mu \frac{U}{a} = \frac{\mu V_m R}{a r}$$

Substituting,

$$Mgr - \frac{\mu V_m R}{a r} 2\pi R H \, R = Mgr - \frac{2\pi \mu V_m R^3 H}{a r} = 0$$

so

$$\mu = \frac{M g r^2 a}{2\pi V_m R^3 H} \qquad \mu$$

Evaluating for the given data

$$\mu = \frac{1}{2\pi} \times 0.10 \text{ kg} \times 9.81 \frac{\text{m}}{\text{s}^2} \times (0.025)^2 \text{m}^2 \times 0.0002 \text{ m} \times \frac{\text{s}}{0.030 \text{ m}} \times \frac{1}{(0.050)^3 \text{m}^3} \times \frac{1}{0.080 \text{ m}} \times \frac{\text{N} \cdot \text{s}^2}{\text{kg} \cdot \text{m}}$$

$$\mu = 0.0651 \text{ N} \cdot \text{s} / \text{m}^2 \quad (65.1 \text{ mPa} \cdot \text{s}) \qquad \mu$$

Problem 2.54

Given: Shaft turning inside stationary journal as shown, $N = 20$ rps.

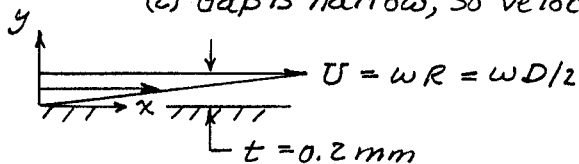
Torque, $T = 0.0036\ N \cdot m$

Find: Estimate viscosity of oil.

Solution: Basic equation $\tau_{yx} = \mu \frac{du}{dy}$

Assumptions: (1) Newtonian fluid

(2) Gap is narrow, so velocity profile is linear, $\frac{du}{dy} \approx \frac{\Delta u}{\Delta y}$

Then

Shear stress is

$$\tau_{yx} \approx \mu \frac{\Delta u}{\Delta y} = \mu \frac{U}{t} = \frac{\mu \omega D}{2t}$$

Neglecting end effects, torque is

$$T = FR = \tau_{yx} A R = \tau_{yx} (\pi D L) \frac{D}{2} = \frac{\mu \pi \omega D^3 L}{4t}$$

Solving for viscosity

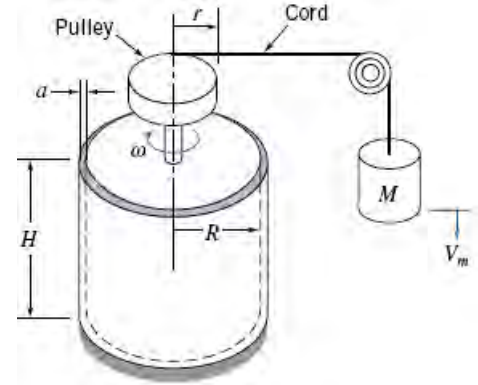
$$\mu = \frac{4tT}{\pi \omega D^3 L}$$

$$= \frac{4}{\pi} \times 0.2\ mm \times 0.0036\ N \cdot m \times \frac{s}{20\ rev} \times \frac{1}{(18)^3\ mm^3} \times \frac{1}{60\ mm} \times \frac{rev}{2\pi\ rad} \times (1000)^3 \frac{mm^3}{m^3}$$

$\mu = 0.0208\ N \cdot s / m^2$ ← μ

{ From Fig. A.2, this oil appears somewhat less viscous than SAE 10W, assuming the oil is at room temperature. }

Problem 2.55

2.55 The viscometer of Problem 2.53 is being used to verify that the viscosity of a particular fluid is $\mu = 0.1\ \text{N}\cdot\text{s/m}^2$. Unfortunately the cord snaps during the experiment. How long will it take the cylinder to lose 99% of its speed? The moment of inertia of the cylinder/pulley system is $0.0273\ \text{kg}\cdot\text{m}^2$.

Given:
Data on the viscometer

Find:
Time for viscometer to lose 99% of speed

Solution:

The given data is $R = 50\cdot\text{mm}$ $\quad H = 80\cdot\text{mm}$ $\quad a = 0.20\cdot\text{mm}$ $\quad I = 0.0273\cdot\text{kg}\cdot\text{m}^2$ $\quad \mu = 0.1\cdot\frac{\text{N}\cdot\text{s}}{\text{m}^2}$

The equation of motion for the slowing viscometer is $\quad I\cdot\alpha = \text{Torque} = -\tau\cdot A\cdot R$

where α is the angular acceleration and τ is the viscous stress, and A is the surface area of the viscometer

The stress is given by
$$\tau = \mu\cdot\frac{du}{dy} = \mu\cdot\frac{V-0}{a} = \frac{\mu\cdot V}{a} = \frac{\mu\cdot R\cdot\omega}{a}$$

where V and ω are the instantaneous linear and angular velocities.

Hence
$$I\cdot\alpha = I\cdot\frac{d\omega}{dt} = -\frac{\mu\cdot R\cdot\omega}{a}\cdot A\cdot R = \frac{\mu\cdot R^2\cdot A}{a}\cdot\omega$$

Separating variables
$$\frac{d\omega}{\omega} = -\frac{\mu\cdot R^2\cdot A}{a\cdot I}\cdot dt$$

Integrating and using IC $\omega = \omega_0$
$$\omega(t) = \omega_0\cdot e^{-\frac{\mu\cdot R^2\cdot A}{a\cdot I}\cdot t}$$

The time to slow down by 99% is obtained from solving
$$0.01\cdot\omega_0 = \omega_0\cdot e^{-\frac{\mu\cdot R^2\cdot A}{a\cdot I}\cdot t} \qquad \text{so} \qquad t = -\frac{a\cdot I}{\mu\cdot R^2\cdot A}\cdot\ln(0.01)$$

Note that $\quad A = 2\cdot\pi\cdot R\cdot H \quad$ so
$$t = -\frac{a\cdot I}{2\cdot\pi\cdot\mu\cdot R^3\cdot H}\cdot\ln(0.01)$$

$$t = -\frac{0.0002\cdot\text{m}\cdot 0.0273\cdot\text{kg}\cdot\text{m}^2}{2\cdot\pi}\cdot\frac{\text{m}^2}{0.1\cdot\text{N}\cdot\text{s}}\cdot\frac{1}{(0.05\cdot\text{m})^3}\cdot\frac{1}{0.08\cdot\text{m}}\cdot\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}\cdot\ln(0.01) \qquad t = 4.00\,\text{s}$$

Problem 2.56

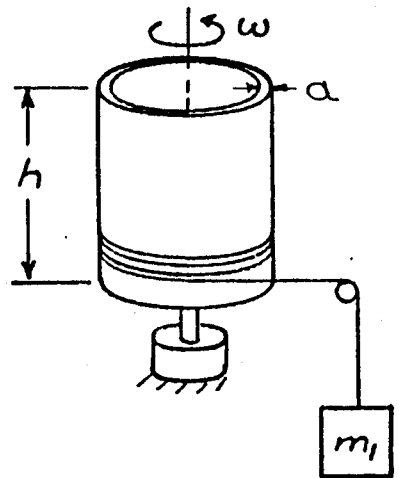
Given: Thin outer cylinder (mass, M_2, and radius R) of a concentric-cylinder viscometer is driven by the falling mass, M_1. Clearance between outer cylinder and stationary inner cylinder is a. Bearing friction, air resistance and mass of liquid in the viscometer may be neglected

Find: (a) algebraic expression for the torque due to viscous shear acting on cylinder at angular speed ω.
(b) differential equation and solution for $\omega(t)$
(c) expression for ω_{max}

Solution:

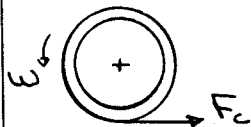
Basic equations: $\tau = \mu \frac{du}{dy}$

$\Sigma F = ma \quad , \Sigma M = I\alpha$

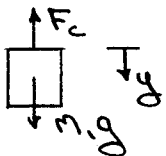
Assume: (1) Newtonian fluid
(2) linear velocity profile

In the gap, $\tau = \mu \frac{du}{dy} = \mu \frac{U}{a} = \frac{\mu R \omega}{a}$

$T = \tau A R = \frac{\mu R \omega}{a}(2\pi R h) R$

$T = \frac{2\pi R^3 \mu h}{a} \omega$ ← T

During acceleration, let the tension in the cord be F_c

For the cylinder $\Sigma M = F_c R - T = I\alpha = M_2 R^2 \frac{d\omega}{dt}$ ... (1)

For the mass $\Sigma F_y = M_1 g - F_c = ma = M_1 \frac{dV}{dt} = M_1 R \frac{d\omega}{dt}$ ... (2)

$\therefore F_c = M_1 g - M_1 R \frac{d\omega}{dt}$

Substituting into eq. (1)

$M_1 g R - \frac{2\pi R^3 \mu h}{a}\omega = (M_1 + M_2) R^2 \frac{d\omega}{dt}$

Let $M_1 g R = b$, $-2\pi R^3 \mu h / a = c$, $(M_1 + M_2)R^2 = f$

Then, $b + c\omega = f \frac{d\omega}{dt}$ or $\int_0^t \frac{1}{f} dt = \int_0^\omega \frac{d\omega}{(b + c\omega)}$

Integrating, $\frac{1}{f} t = \frac{1}{c} \ln(b + c\omega)\Big]_0^\omega = \frac{1}{c} \ln \frac{(b + c\omega)}{b} = \frac{1}{c} \ln\left(1 + \frac{c}{b}\omega\right)$.

$\frac{c}{f} t = \ln\left(1 + \frac{c}{b}\omega\right) \Rightarrow e^{\frac{c}{f}t} = \left(1 + \frac{c}{b}\omega\right) \Rightarrow \omega = \frac{b}{c}\left(e^{\frac{c}{f}t} - 1\right)$

Substituting for $b, c,$ and f

$$\omega = \frac{M_1 g R a}{2\pi R^3 \mu h}\left(1 - e^{\frac{-2\pi R^3 \mu h}{a(M_1 + M_2)R^2} t}\right) = \frac{M g a}{2\pi R^2 \mu h}\left[1 - e^{-\frac{2\pi R \mu h}{a(M_1 + M_2)} t}\right]$$ ← $\omega(t)$

Maximum ω occurs at $t \to \infty$

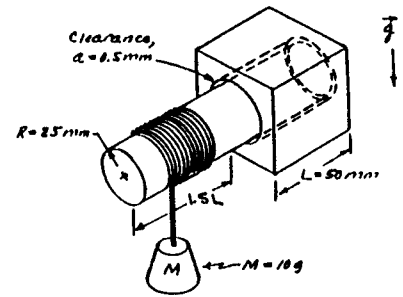
$$\omega_{max} = \frac{M g a}{2\pi R^2 \mu h}$$ ← ω_{max}

Problem 2.57

Given: Circular aluminum shaft in journal.

Symmetric clearance gap

filled with SAE 10W-30 at 30°C.

Shaft turned by mass and cord.

Find: (a) Develop and solve a differential equation for angular speed as a function of time.

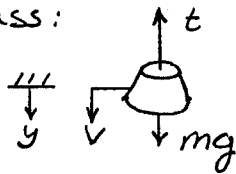
(b) Calculate maximum angular speed.

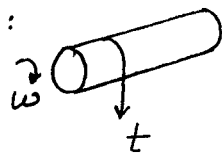
(c) Estimate time needed to reach 95 percent of maximum speed.

Solution: Apply summation of torques and Newton's second law.

Basic equations: $\Sigma T = I\frac{d\omega}{dt}$ $\qquad \Sigma F = m\frac{dV}{dt}$ $\qquad V = R\omega$

For the mass:

$$\Sigma F_y = mg - t = m\frac{dV}{dt} = mR\frac{d\omega}{dt} \qquad (1)$$

For the shaft:

$$\Sigma T = tR - T_{viscous} = I\frac{d\omega}{dt} \qquad (2)$$

$$T_{viscous} = \tau R A = \mu\frac{V}{a}R\,2\pi R L = \frac{2\pi\mu\omega R^3 L}{a}$$

Assume: (1) Newtonian liquid, (2) Small gap, (3) Linear profile

Then Eq. 2 becomes

$$tR - \frac{2\pi\mu R^3 L}{a}\omega = I\frac{d\omega}{dt} \;;\; I = \tfrac{1}{2}MR^2 \qquad (3)$$

Multiplying Eq. 1 by R and combining with Eq. 3 gives

$$mgR - mR^2\frac{d\omega}{dt} - \frac{2\pi\mu R^3 L}{a}\omega = I\frac{d\omega}{dt} \quad \text{or} \quad mgR - \frac{2\pi\mu R^3 L}{a}\omega = (I + mR^2)\frac{d\omega}{dt} \qquad (4)$$

This may be written $A - B\omega = C\frac{d\omega}{dt}$ where $A = mgR$, $B = \frac{2\pi\mu R^3 L}{a}$, $C = I + mR^2$

Separating variables $\quad \frac{d\omega}{A - B\omega} = \frac{dt}{C}$

Integrating

$$\int_0^\omega \frac{d\omega}{A - B\omega} = -\frac{1}{B}\ln(A - B\omega)\Big]_0^\omega = -\frac{1}{B}\ln\left(1 - \frac{B\omega}{A}\right) = \int_0^t \frac{dt}{C} = \frac{t}{C}$$

Simplifying

$$1 - \frac{B\omega}{A} = e^{-Bt/C} \quad \text{or} \quad \omega = \frac{A}{B}\left[1 - e^{-Bt/C}\right] \qquad (5)$$

$\omega(t)$

The maximum angular speed $(t \to \infty)$ is $\omega = A/B$.

$$A = mgR = 0.010\,\text{kg} \times 9.81\,\frac{\text{m}}{\text{s}^2} \times 0.025\,\text{m} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} = 2.45 \times 10^{-3}\,\text{N}\cdot\text{m}$$

$$B = \frac{2\pi\mu R^3 L}{a} = 2\pi \times 0.095\,\frac{\text{kg}}{\text{m}\cdot\text{s}} \times (0.025)^3\,\text{m}^3 \times 0.050\,\text{m} \times \frac{1}{0.0005\,\text{m}} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} = 9.33 \times 10^{-4}\,\text{N}\cdot\text{m}\cdot\text{s}$$

Problem 2.57

Evaluating, $\omega_{max} = \frac{A}{B} = 2.45 \times 10^{-5} \, N \cdot m \times \frac{1}{9.33 \times 10^{-4} \, N \cdot m \cdot sec} = 2.63 \, rad/s$.

Thus

$$\omega_{max} = 2.63 \frac{rad}{s} \times \frac{rev}{2\pi \, rad} \times \frac{60 \, s}{min} = 25.1 \, rpm$$

ω_{max}

From Eq. 5, $\omega = 0.95 \, \omega_{max}$ when $e^{-Bt/C} = 0.05$, or $Bt/C \simeq 3$; $t \simeq \frac{3C}{B}$

$$C = I + mR^2 = \frac{1}{2} MR^2 + mR^2 = (\frac{1}{2}M + m) R^2$$

$$M = \pi R^2 (1.5L + L)\rho = 2.5 \pi R^2 L \, SG \, \rho_w$$

$$M = 2.5 \pi \times (0.025)^2 m^2 \times 0.050 \, m \times (2.64) 1000 \frac{kg}{m^3} = 0.648 \, kg$$

$$C = (\frac{1}{2} \times 0.648 \, kg + 0.010 \, kg)(0.025)^2 m^2 = 2.09 \times 10^{-4} \, kg \cdot m^2$$

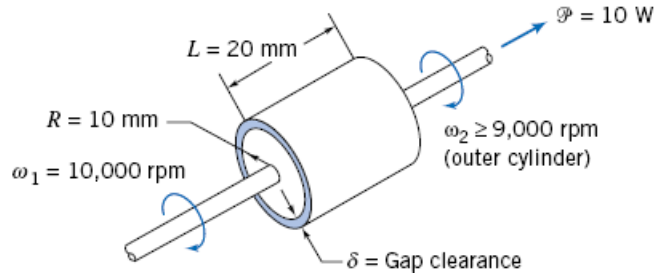
Thus

$$t_{95} = 3 \times 2.09 \times 10^{-4} \, kg \cdot m^2 \times \frac{1}{9.33 \times 10^{-4} \, N \cdot m \cdot s} \times \frac{N \cdot s^2}{kg \cdot m} = 0.671 \, s$$

t_{95}

{The terminal speed could have been computed from Eq. 4 by setting $d\omega/dt \to 0$, without solving the differential equation.}

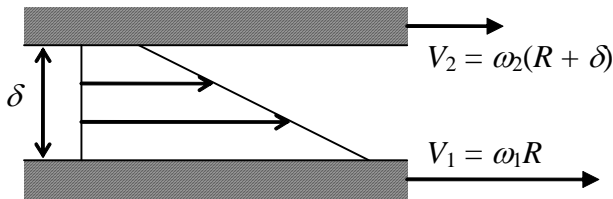
Problem 2.58

2.58 A shock-free coupling for a low-power mechanical drive is to be made from a pair of concentric cylinders. The annular space between the cylinders is to be filled with oil. The drive must transmit power, $\mathcal{P} = 10$ W. Other dimensions and properties are as shown. Neglect any bearing friction and end effects. Assume the minimum practical gap clearance δ for the device is $\delta = 0.25$ mm. Dow manufactures silicone fluids with viscosities as high as 10^6 centipoise. Determine the viscosity that should be specified to satisfy the requirement for this device.

Given: Shock-free coupling assembly

Find: Required viscosity

Solution:

Basic equation $\tau_{r\theta} = \mu \cdot \frac{du}{dr}$ Shear force $F = \tau \cdot A$ Torque $T = F \cdot R$ Power $P = T \cdot \omega$

Assumptions: Newtonian fluid, linear velocity profile

$$\tau_{r\theta} = \mu \cdot \frac{du}{dr} = \mu \cdot \frac{\Delta V}{\Delta r} = \mu \cdot \frac{\left[\omega_1 \cdot R - \omega_2 \cdot (R + \delta)\right]}{\delta}$$

$$\tau_{r\theta} = \mu \cdot \frac{(\omega_1 - \omega_2) \cdot R}{\delta}$$ Because $\delta << R$

Then $$P = T \cdot \omega_2 = F \cdot R \cdot \omega_2 = \tau \cdot A_2 \cdot R \cdot \omega_2 = \frac{\mu \cdot (\omega_1 - \omega_2) \cdot R}{\delta} \cdot 2 \cdot \pi \cdot R \cdot L \cdot R \cdot \omega_2$$

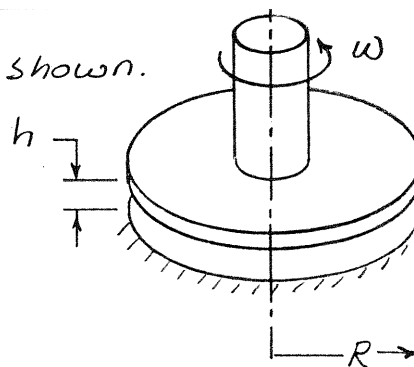
$$P = \frac{2 \cdot \pi \cdot \mu \cdot \omega_2 \cdot (\omega_1 - \omega_2) \cdot R^3 \cdot L}{\delta}$$

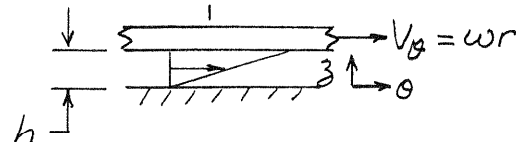
Hence $$\mu = \frac{P \cdot \delta}{2 \cdot \pi \cdot \omega_2 \cdot (\omega_1 - \omega_2) \cdot R^3 \cdot L}$$

$$\mu = \frac{10 \cdot W \times 2.5 \times 10^{-4} \cdot m}{2 \cdot \pi} \times \frac{1}{9000} \cdot \frac{min}{rev} \times \frac{1}{1000} \cdot \frac{min}{rev} \times \frac{1}{(.01 \cdot m)^3} \times \frac{1}{0.02 \cdot m} \times \frac{N \cdot m}{s \cdot W} \times \left(\frac{rev}{2 \cdot \pi \cdot rad}\right)^2 \times \left(\frac{60 \cdot s}{min}\right)^2$$

$\mu = 0.202 \cdot \frac{N \cdot s}{m^2}$ $\mu = 2.02\,\text{poise}$ which corresponds to SAE 30 oil at 30ºC.

Given: Parallel-disk apparatus as shown.

Find: (a) Algebraic expression for shear stress at any radial location.
(b) Expression for the torque needed to turn the upper disk.

Solution: Use r, θ, z coordinates at right:

$V_\theta = \omega r$, z, θ, h

Basic equations: $\tau_{z\theta} = \mu \dfrac{dV_\theta}{dz}$

$$dT = r\,dF = r\,\tau_{z\theta}\,dA$$

Assumptions: (1) Newtonian fluid
(2) No-slip condition
(3) Linear velocity profile (in narrow gap)

The velocity at any radial location on the rotating disk is $V_\theta = \omega r$.

Since the velocity profile is linear, then

$$\tau_{z\theta} = \mu \frac{dV_\theta}{dz} = \mu \frac{\Delta V}{\Delta z} = \mu \frac{(\omega r - 0)}{(h - 0)} = \frac{\mu \omega r}{h}$$ $\tau_{z\theta}$

and

$$dT = r\,\tau_{z\theta}\,dA = r\,\mu \frac{\omega r}{h}\,2\pi r\,dr = \frac{2\pi \mu \omega r^3}{h}\,dr$$

Integrating

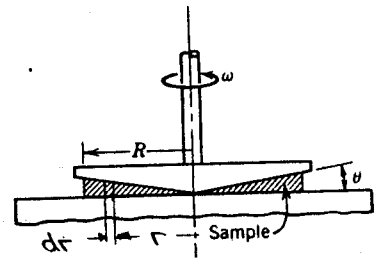
$$T = \int_A dT = \int_0^R \frac{2\pi \mu \omega r^3}{h}\,dr = \left.\frac{\pi \mu \omega r^4}{2h}\right]_0^R$$

$$T = \frac{\pi \mu \omega R^4}{2h}$$ T

The device could not be used to measure the viscosity of a non-Newtonian fluid because the applied shear stress is not uniform. It varies from zero at the center of the disks to $\mu\omega R/h$ at the edge

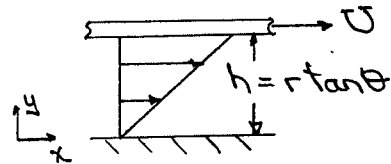
Problem 2.60

Given: Cone and plate viscometer shown.
Apex of cone just touches the plate,
θ is very small

Find: (a) Derive an expression for the shear rate in the liquid that fills the gap
(b) Evaluate the torque on the driven cone in terms of the shear stress and geometry of the system.

Solution:

Since the angle θ is very small, the average gap width is also very small.
It is reasonable to assume a linear velocity profile across the gap and to neglect end effects.

The shear (deformation) rate is given by

$$\dot{\gamma} = \frac{du}{dy} = \frac{\Delta u}{\Delta y}$$

At any radius, r,
the velocity $U = \omega r$ and
the gap width $h = r \tan\theta$

$$\therefore \dot{\gamma} = \frac{\omega r}{r \tan\theta} = \frac{\omega}{\tan\theta}$$

Since θ is very small, $\tan\theta \simeq \theta$ and

$$\dot{\gamma} = \frac{\omega}{\theta} \qquad \dot{\gamma}$$

Note: The shear rate is independent of r. The entire sample is subjected to the same shear rate.

The torque on the driven cone is given by

$$T = \int r \, dF \qquad \text{where } dF = \tau_{yx} \, dA$$

Since $\dot{\gamma}$ is a constant (for a given ω) then τ_{yx} = constant

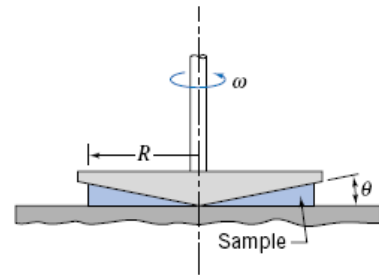
and

$$T = \int r \, dF = \int_A r \tau_{yx} \, dA = \tau_{yx} \int_0^R r \, 2\pi r \, dr$$

$$T = \frac{2\pi}{3} R^3 \tau_{yx} \qquad T$$

Problem 2.61

2.61 The viscometer of Problem 2.60 is used to measure the apparent viscosity of a fluid. The data below are obtained. What kind of non-Newtonian fluid is this? Find the values of k and n used in Eqs. 2.16 and 2.17 in defining the apparent viscosity of a fluid. (Assume θ is 0.5 degrees.) Predict the viscosity at 90 and 100 rpm.

Speed (rpm)	10	20	30	40	50	60	70	80
μ (N·s/m^2)	0.121	0.139	0.153	0.159	0.172	0.172	0.183	0.185

Given: Data on the viscometer

Find: The values of coefficients k and n; determine the kind of non-Newtonial fluid it is; estimate viscosity at 90 and 100 rpm

Solution:

The velocity gradient at any radius r is $$\frac{du}{dy} = \frac{r \cdot \omega}{r \cdot \tan(\theta)}$$

where ω (rad/s) is the angular velocity $$\omega = \frac{2 \cdot \pi \cdot N}{60}$$ where N is the speed in rpm

For small θ, $\tan(\theta)$ can be replace with θ, so $$\frac{du}{dy} = \frac{\omega}{\theta}$$

From Eq 2.11. $$k \cdot \left(\left| \frac{du}{dy} \right| \right)^{n-1} \cdot \frac{du}{dy} = \eta \cdot \frac{du}{dy}$$

where η is the apparent viscosity. Hence $$\eta = k \cdot \left(\frac{du}{dy} \right)^{n-1} = k \cdot \left(\frac{\omega}{\theta} \right)^{n-1}$$

The data is

N (rpm)	μ (N·s/m^2)
10	0.121
20	0.139
30	0.153
40	0.159
50	0.172
60	0.172
70	0.183
80	0.185

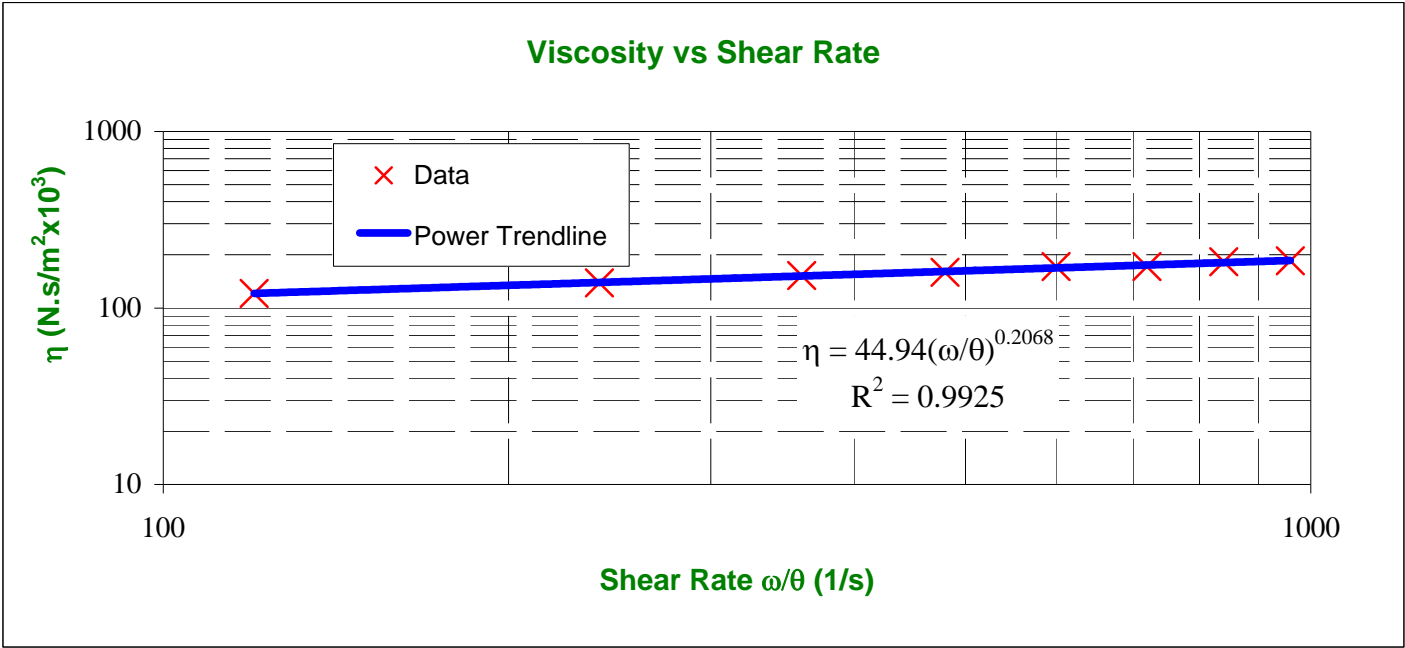
The computed data is

ω (rad/s)	ω/θ (1/s)	η (N·s/m²x10³)
1.047	120	121
2.094	240	139
3.142	360	153
4.189	480	159
5.236	600	172
6.283	720	172
7.330	840	183
8.378	960	185

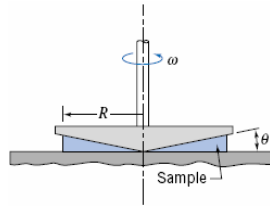
From the *Trendline* analysis

k = 0.0449
n - 1 = 0.2068
n = 1.21 The fluid is dilatant

The apparent viscosities at 90 and 100 rpm can now be computed

N (rpm)	ω (rad/s)	ω/θ (1/s)	η (N·s/m²x10³)
90	9.42	1080	191
100	10.47	1200	195

Problem 2.62 (In Excel)

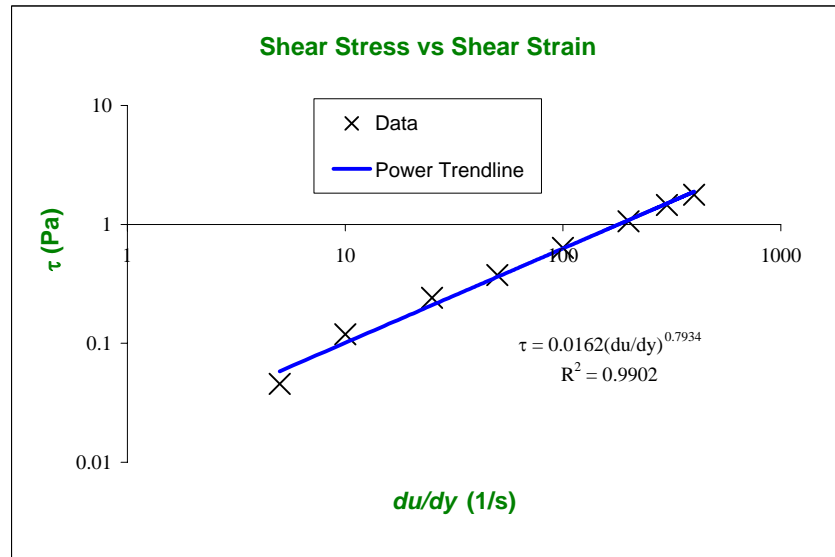
2.62 A viscometer is used to measure the viscosity of a patient's blood. The deformation rate (shear rate)–shear stress data is shown below. Plot the apparent viscosity versus deformation rate. Find the value of k and n in Eq. 2.17, and from this examine the aphorism "Blood is thicker than water."

du/dy (s^{-1})	5	10	25	50	100	200	300	400
τ (Pa)	0.0457	0.119	0.241	0.375	0.634	1.06	1.46	1.78

Given: Viscometer data

Find: Value of k and n in Eq. 2.17

Solution:

The data is

τ (Pa)	*du/dy* (s^{-1})
0.0457	5
0.119	10
0.241	25
0.375	50
0.634	100
1.06	200
1.46	300
1.78	400

Hence we have $k = 0.0162$

$n = 0.7934$ Blood is pseudoplastic (shear thinning)

We can compute the apparent viscosity fron $\eta = k\,(du/dy)^{n-1}$

du/dy (s^{-1})	η ($N{\cdot}s/m^2$)
5	0.0116
10	0.0101
25	0.0083
50	0.0072
100	0.0063
200	0.0054
300	0.0050
400	0.0047

$\mu_{water} = 0.001\ N{\cdot}s/m^2$ at 20^oC

Hence, blood is "thicker" than water!

Problem 2.63 (In Excel)

2.63 An insulation company is examining a new material for extruding into cavities. The experimental data is given below for the speed U of the upper plate, which is separated from a fixed lower plate by a 1-mm-thick sample of the material, when a given shear stress is applied. Determine the type of material. If a replacement material with a minimum yield stress of 250 Pa is needed, what viscosity will the material need to have the same behavior as the current material at a shear stress of 450 Pa?

τ (Pa)	50	100	150	163	171	170	202	246	349	444
U (m/s)	0	0	0	0.005	0.01	0.025	0.05	0.1	0.2	0.3

Given: Data on insulation material

Find: Type of material; replacement material

Solution:

The velocity gradient is

$du/dy = U/\delta$ where δ = 0.001 m

Data and computations

τ (Pa)	U (m/s)	du/dy (s^{-1})
50	0.000	0
100	0.000	0
150	0.000	0
163	0.005	5
171	0.01	10
170	0.03	25
202	0.05	50
246	0.1	100
349	0.2	200
444	0.3	300

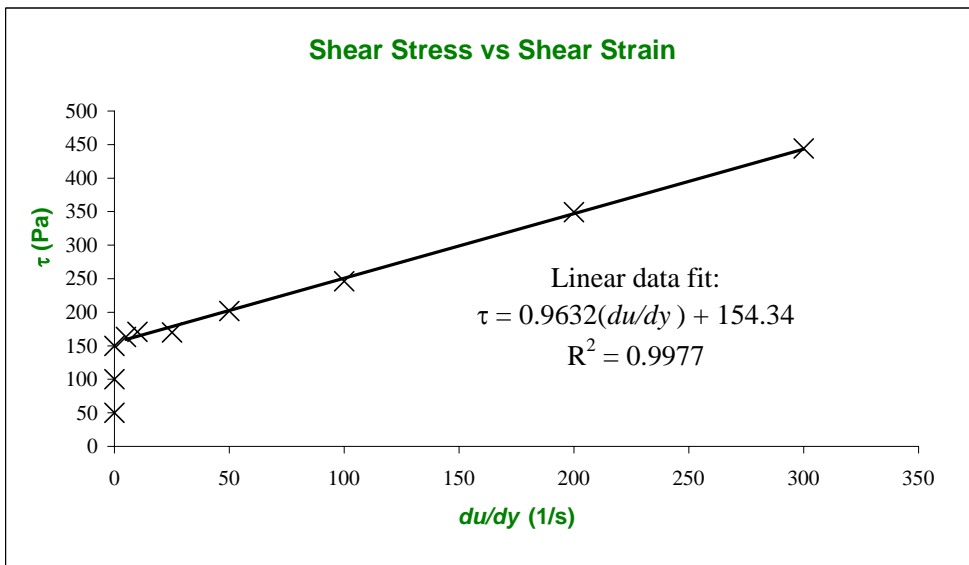
Hence we have a Bingham plastic, with τ_y = 154 Pa

μ_p = 0.963 $N{\cdot}s/m^2$

At τ = 450 Pa, based on the linear fit du/dy = 307 s^{-1}

For a fluid with τ_y = 250 Pa

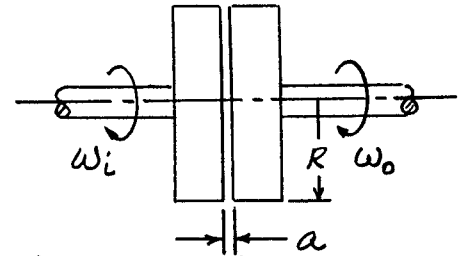
we can use the Bingham plastic formula to solve for μ_p given τ, τ_y and du/dy from above

μ_p = 0.652 $N{\cdot}s/m^2$

Problem 2.64

Given: Viscous clutch made from pair of closely spaced disks.

Input speed, ω_i

Output speed, ω_o

Viscous oil in gap, μ

Find algebraic expressions in terms of μ, R, a, ω_i, and ω_o for:
(a) Torque transmitted, T
(b) Power transmitted
(c) Slip ratio, $s = \Delta\omega/\omega_i$, in terms of T
(d) Efficiency, η, in terms of s, ω_i, and T

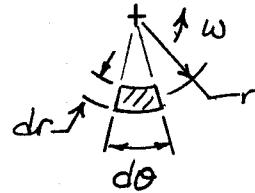
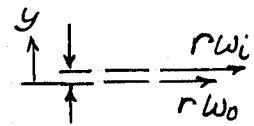
Solution: Apply Newton's law of viscosity

Basic equations: $\tau = \mu \frac{du}{dy}$ $\quad dF = \tau dA \quad dT = r dF$

Assumptions: (1) Newtonian liquid
(2) Narrow gap so velocity profile is linear

Consider a segment of plates:

$$\tau = \mu \frac{du}{dy} = \mu \frac{\Delta u}{\Delta y} = \mu \frac{r(\omega_i - \omega_o)}{a}$$

End View (labels: ω, r, dr, $d\theta$) — Bottom View (labels: y, $r\omega_i$, $r\omega_o$)

$$dA = r\,dr\,d\theta$$

$$dF = \tau dA = \frac{\mu r \Delta\omega}{a} r\,dr\,d\theta = \frac{\mu \Delta\omega}{a} r^2 dr\,d\theta \; ; \; dT = r dF = \frac{\mu\Delta\omega}{a} r^3 dr\,d\theta$$

Integrating

$$T = \int_0^{2\pi}\int_0^R dT = \frac{\mu\Delta\omega}{a}\int_0^{2\pi}\int_0^R r^3 dr\,d\theta = \frac{2\pi\mu\Delta\omega}{a}\int_0^R r^3 dr = \frac{\pi\mu\Delta\omega R^4}{2a}$$

T

$$P_o = T\omega_o = \frac{\pi\mu\omega_o\Delta\omega R^4}{2a} \quad \text{(power transmitted)}$$

P

$$s = \frac{\Delta\omega}{\omega_i} = \frac{2aT}{\pi\mu R^4\omega_i}$$

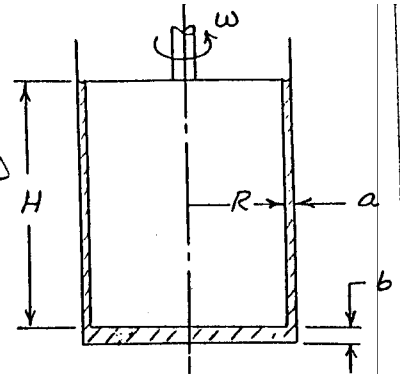
s

Efficiency is $\eta = \frac{\text{Power out}}{\text{Power in}} = \frac{T\omega_o}{T\omega_i} = \frac{\omega_o}{\omega_i}$. But $\omega_o = \omega_i - \Delta\omega$, so

$$\eta = \frac{\omega_i - \Delta\omega}{\omega_i} = 1 - \frac{\Delta\omega}{\omega_i} = 1 - s$$

η

Problem 2.65

Given: Concentric-cylinder viscometer shown. When inner cylinder rotates at angular speed ω viscous retarding torque arises around circumference of inner cylinder and on cylinder bottom.

Find: (a) expression for viscous torque due to gap of width, a

(b) expression for viscous torque on bottom due to gap of width b

(c) For $T_{bottom}/T_{annulus} \leq 0.01$, plot b/a vs geometric variables.

(d) What are design implications?

(e) What design modifications can you recommend?

Solution: Basic equation $\tau_{yx} = \mu \frac{du}{dy}$

Assumptions: (1) linear velocity profile (2) Newtonian liquid

(a) in annular gap

$$\tau = \mu \frac{du}{dr} = \mu \frac{\Delta u}{\Delta r} = \mu \frac{U}{a} = \mu \frac{\omega R}{a}$$

$$\text{Torque} = RF_t = R\tau A = R\mu \frac{\omega R}{a}(2\pi R H) = \frac{2\pi \mu \omega R^3 H}{a} \quad \text{(a)}$$

(b) in bottom gap

$$\tau = \mu \frac{du}{dz} = \mu \frac{\Delta u}{\Delta z} = \mu \frac{U}{a} = \mu \frac{\omega r}{b}$$

(varies with r)

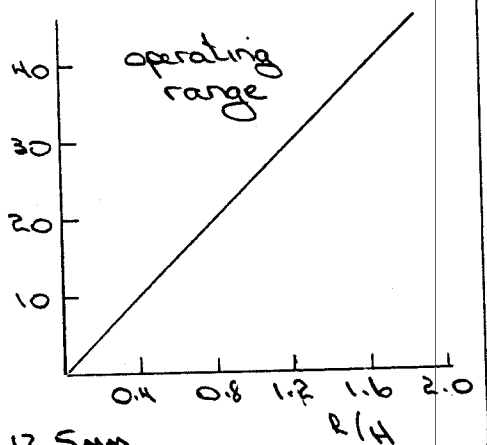
$$\text{Torque} = \int dT = \int r\, dF = \int r\tau\, dA = \int_0^R r \mu \frac{\omega r}{b} 2\pi r\, dr$$

$$\text{Torque} = \frac{2\pi \mu \omega}{b} \int_0^R r^3\, dr = \frac{2\pi \mu \omega}{b}\left[\frac{r^4}{4}\right]_0^R = \frac{\pi \mu \omega}{2b} R^4 \quad \text{(b)}$$

(c) For $T_{bottom}/T_{annulus} \leq \frac{1}{100}$, then

$$\frac{T_{bot}}{T_{an}} = \frac{\pi \mu \omega}{2b} R^4 \times \frac{a}{2\pi \mu \omega R^3 H} \leq \frac{1}{100}$$

$$\frac{aR}{4bH} \leq \frac{1}{100}$$

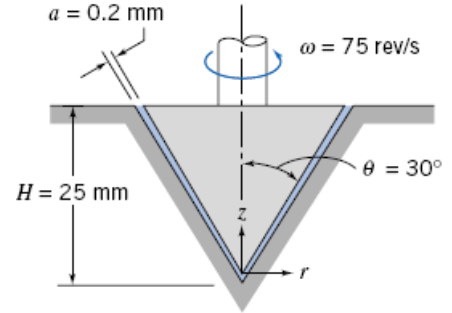
or $$\frac{b}{a} \geq 25 \frac{R}{H}$$

(d) The plot shows the operating range. Specific design would depend on other constraints.

For $a = 1\,mm$ with $R/H = 1/2$ gives $b = 12.5\,mm$

(e) For a given value of R/H, the dimension b could be effectively increased by "hollowing out" the inner cylinder as shown by the dashed lines in the diagram above.

Problem 2.66

[4]

2.66 A conical pointed shaft turns in a conical bearing. The gap between shaft and bearing is filled with heavy oil having the viscosity of SAE 30 at 30°C. Obtain an algebraic expression for the shear stress that acts on the surface of the conical shaft. Calculate the viscous torque that acts on the shaft.

Given: Conical bearing geometry

Find: Expression for shear stress; Viscous torque on shaft

Solution:

Basic equation $\tau = \mu \cdot \frac{du}{dy}$ $\quad dT = r \cdot \tau \cdot dA$ $\quad$ Infinitesimal shear torque

Assumptions: Newtonian fluid, linear velocity profile (in narrow clearance gap), no slip condition

$$\tan(\theta) = \frac{r}{z} \qquad \text{so} \qquad r = z \cdot \tan(\theta)$$

Then $$\tau = \mu \cdot \frac{du}{dy} = \mu \cdot \frac{\Delta u}{\Delta y} = \mu \cdot \frac{(\omega \cdot r - 0)}{(a - 0)} = \frac{\mu \cdot \omega \cdot z \cdot \tan(\theta)}{a}$$

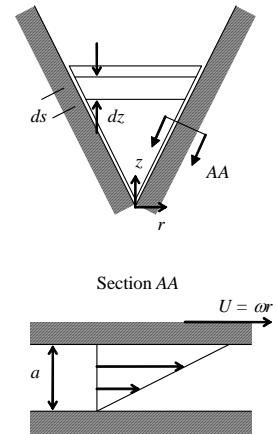
As we move up the device, shear stress increases linearly (because rate of shear strain does)

But from the sketch $\quad dz = ds \cdot \cos(\theta) \qquad dA = 2 \cdot \pi \cdot r \cdot ds = 2 \cdot \pi \cdot r \cdot \frac{dz}{\cos(\theta)}$

The viscous torque on the element of area is $\quad dT = r \cdot \tau \cdot dA = r \cdot \frac{\mu \cdot \omega \cdot z \cdot \tan(\theta)}{a} \cdot 2 \cdot \pi \cdot r \cdot \frac{dz}{\cos(\theta)} \qquad dT = \frac{2 \cdot \pi \cdot \mu \cdot \omega \cdot z^3 \cdot \tan(\theta)^3}{a \cdot \cos(\theta)} \cdot dz$

Integrating and using limits z = H and z = 0 $\quad T = \frac{\pi \cdot \mu \cdot \omega \cdot \tan(\theta)^3 \cdot H^4}{2 \cdot a \cdot \cos(\theta)}$

Using given data, and $\quad \mu = 0.2 \cdot \frac{N \cdot s}{m^2} \quad$ from Fig. A.2

$$T = \frac{\pi}{2} \times 0.2 \cdot \frac{N \cdot s}{m^2} \times 75 \cdot \frac{rev}{s} \times \tan(30 \cdot deg)^3 \times (0.025 \cdot m)^4 \times \frac{1}{0.2 \times 10^{-3} \cdot m} \times \frac{1}{\cos(30 \cdot deg)} \times \frac{2 \cdot \pi \cdot rad}{rev} \qquad T = 0.0643 \cdot N \cdot m$$

Problem 2.67

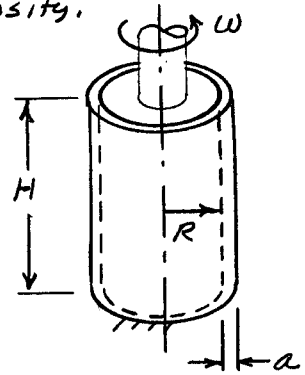
[5]

Given: Concentric-cylinder viscometer, liquid similar to water. Goal is to obtain ±1 percent accuracy in viscosity value.

Specify: Configuration and dimensions to achieve ±1% measurement. Parameter to be measured to compute viscosity.

Solution: Apply definition of Newtonian fluid

Computing equation: $\tau = \mu \frac{du}{dy}$

Assumptions: (1) Steady
(2) Newtonian liquid
(3) Narrow gap, so "unroll" it
(4) Linear velocity profile in gap
(5) Neglect end effects

Flow model:

$u = V\frac{y}{a} = \omega R \frac{y}{a}$; $\frac{du}{dy} = \frac{\omega R}{a}$

Thus $\tau = \mu \frac{du}{dy} = \mu \frac{\omega R}{a}$ and torque on rotor is $T = R\tau A$, where $A = 2\pi R H$

Consequently $T = R\mu \frac{\omega R}{a} 2\pi R H = \frac{2\pi \mu \omega R^3 H}{a}$, or

$$\mu = \frac{Ta}{2\pi \omega R^3 H}$$

μ

From this equation the uncertainty in μ is (see Appendix F),

$$u_\mu = \pm\left[u_T^2 + u_a^2 + u_\omega^2 + (3u_R)^2 + u_H^2\right]^{\frac{1}{2}} = \pm\left[13\,u^2\right]^{\frac{1}{2}} = \pm 3.61\,u$$

if the uncertainty of each parameter equals u. Thus

$$u = \pm\frac{u_\mu}{3.61} = \pm\frac{1 \text{ percent}}{3.61} = \pm 0.277 \text{ percent}$$

u

Typical dimensions for a bench-top unit might be

$H = 200$ mm, $R = 75$ mm, $a = 0.02$ mm, and $\omega = 10.5$ rad/s (100 rpm)

From Appendix A, Table A.8, water has $\mu = 1.00\times10^{-3}$ N·s/m² at $T = 20°$C.

The corresponding torque would be

$$T = 2\pi \times 1.00\times10^{-3}\frac{\text{N}\cdot\text{s}}{\text{m}^2}\times\frac{10.5}{\text{s}}\times(0.075)^3\text{m}^3\times 0.2\,\text{m}\times\frac{1}{0.00002\,\text{m}} = 0.278\ \text{N}\cdot\text{m}$$

T

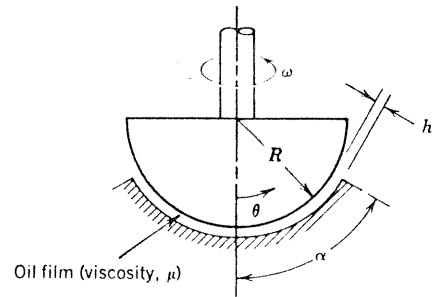
It should be possible to measure this torque quite accurately.

{Many details would need to be considered (e.g. bearings, temperature rise, etc.) to produce a workable device.}

Problem 2.68

[5]

Given: Spherical thrust bearing shown:

Find: Obtain and plot an algebraic expression for the torque on the spherical member, as a function of α.

Solution: Apply definitions

Computing equations: $\tau = \mu \frac{du}{dy}$ $\qquad T = \int_A r\tau \, dA$

Assumptions: (1) Newtonian fluid, (2) Narrow gap, (3) Laminar flow

From the figure, $r = R\sin\theta \qquad u = \omega r = \omega R \sin\theta$

$$\tau = \mu \frac{du}{dy} = \mu\left(\frac{u-0}{h}\right) = \mu\frac{u}{h} = \mu\frac{\omega R\sin\theta}{h}$$

$$dA = 2\pi r R\, d\theta = 2\pi R^2 \sin\theta \, d\theta$$

Thus

$$T = \int_0^\alpha R\sin\theta\left(\frac{\mu\omega R\sin\theta}{h}\right) 2\pi R^2 \sin\theta\, d\theta = \frac{2\pi\mu\omega R^4}{h}\int_0^\alpha \sin^3\theta\, d\theta$$

$$T = \frac{2\pi\mu\omega R^4}{h}\left[\frac{\cos^3\theta}{3} - \cos\theta\right]_0^\alpha = \frac{2\pi\mu\omega R^4}{h}\left[\frac{\cos^3\alpha}{3} - \cos\alpha + \frac{2}{3}\right]$$

T

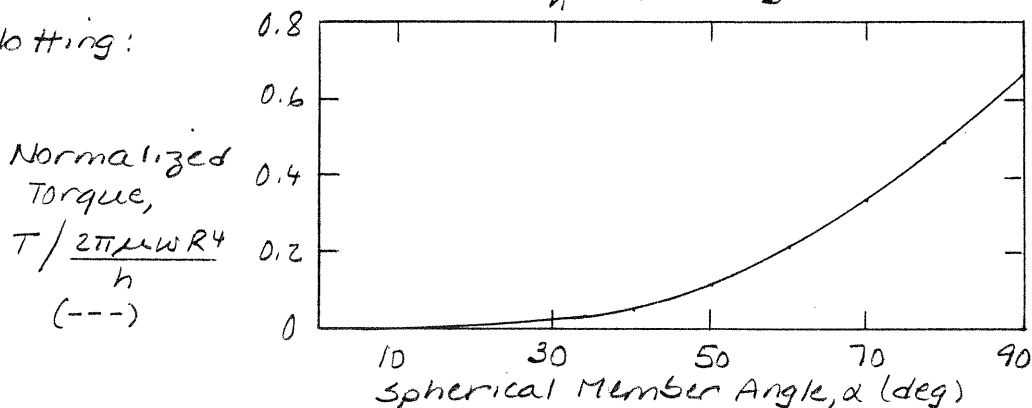
To plot, normalize to $\left[T \Big/ \frac{2\pi\mu\omega R^4}{h}\right] = \left[\frac{\cos^3\alpha}{3} - \cos\alpha + \frac{2}{3}\right]$

Plotting:

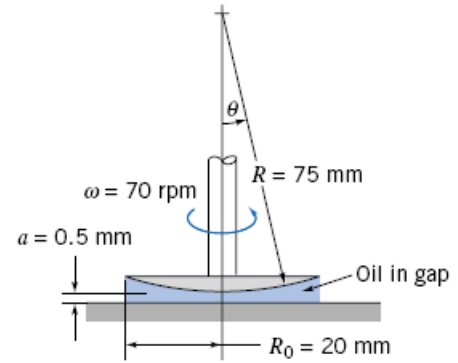
Normalized Torque, $T \Big/ \frac{2\pi\mu\omega R^4}{h}$ (---)

0.8
0.6
0.4
0.2
0

10 30 50 70 90

Spherical Member Angle, α (deg)

{Check dimensions: $\left[\frac{\mu\omega R^4}{h}\right] = \frac{Ft}{L^2} \times \frac{1}{t} \times L^4 \times \frac{1}{L} = FL$ ✓✓}

Problem 2.69

2.69 A cross section of a rotating bearing is shown. The spherical member rotates with angular speed ω, a small distance, a, above the plane surface. The narrow gap is filled with viscous oil, having $\mu = 1250$ cp. Obtain an algebraic expression for the shear stress acting on the spherical member. Evaluate the maximum shear stress that acts on the spherical member for the conditions shown. (Is the maximum necessarily located at the maximum radius?) Develop an algebraic expression (in the form of an integral) for the total viscous shear torque that acts on the spherical member. Calculate the torque using the dimensions shown.

Given: Geometry of rotating bearing

Find: Expression for shear stress; Maximum shear stress; Expression for total torque; Total torque

Solution:

Basic equation $\tau = \mu \cdot \frac{du}{dy}$ $\qquad dT = r \cdot \tau \cdot dA$

Assumptions: Newtonian fluid, narrow clearance gap, laminar motion

From the figure $\quad r = R \cdot \sin(\theta) \qquad u = \omega \cdot r = \omega \cdot R \cdot \sin(\theta) \qquad \frac{du}{dy} = \frac{u - 0}{h} = \frac{u}{h}$

$$h = a + R \cdot (1 - \cos(\theta)) \qquad dA = 2 \cdot \pi \cdot r \cdot dr = 2 \cdot \pi R \cdot \sin(\theta) \cdot R \cdot \cos(\theta) \cdot d\theta$$

Then $\quad \tau = \mu \cdot \frac{du}{dy} = \frac{\mu \cdot \omega \cdot R \cdot \sin(\theta)}{a + R \cdot (1 - \cos(\theta))}$

To find the maximum τ set $\quad \frac{d}{d\theta}\left[\frac{\mu \cdot \omega \cdot R \cdot \sin(\theta)}{a + R \cdot (1 - \cos(\theta))}\right] = 0 \qquad$ so $\qquad \frac{R \cdot \mu \cdot \omega \cdot (R \cdot \cos(\theta) - R + a \cdot \cos(\theta))}{(R + a - R \cdot \cos(\theta))^2} = 0$

$$R \cdot \cos(\theta) - R + a \cdot \cos(\theta) = 0 \qquad \theta = \mathrm{acos}\left(\frac{R}{R + a}\right) = \mathrm{acos}\left(\frac{75}{75 + 0.5}\right) \qquad \theta = 6.6 \cdot \mathrm{deg}$$

$$\tau = 12.5 \cdot \mathrm{poise} \times 0.1 \cdot \frac{\frac{kg}{m \cdot s}}{\mathrm{poise}} \times 2 \cdot \pi \cdot \frac{70}{60} \cdot \frac{rad}{s} \times 0.075 \cdot m \times \sin(6.6 \cdot \mathrm{deg}) \times \frac{1}{[0.0005 + 0.075 \cdot (1 - \cos(6.6 \cdot \mathrm{deg}))] \cdot m} \times \frac{N \cdot s^2}{m \cdot kg}$$

$$\tau = 79.2 \cdot \frac{N}{m^2}$$

The torque is $\quad T = \int r \cdot \tau \cdot A \, d\theta = \int_0^{\theta_{max}} \frac{\mu \cdot \omega \cdot R^4 \cdot \sin(\theta)^2 \cdot \cos(\theta)}{a + R \cdot (1 - \cos(\theta))} \, d\theta \qquad$ where $\quad \theta_{max} = \mathrm{asin}\left(\frac{R_0}{R}\right) \qquad \theta_{max} = 15.5 \cdot \mathrm{deg}$

This integral is best evaluated numerically using Excel, Mathcad, or a good calculator $\qquad T = 1.02 \times 10^{-3} \cdot N \cdot m$

Problem 2.70

[2]

Given: Small gas bubbles form in soda when opened; $D = 0.1$ mm.

Find: Estimate pressure difference from inside to outside such a bubble.

Solution: Consider a free-body diagram of half a bubble:

Two forces act:

Pressure: $F_p = \Delta p \frac{\pi D^2}{4}$

Surface tension: $F_\sigma = \sigma \pi D$

D

x

Summing forces for equilibrium

$$\Sigma F_x = F_p - F_\sigma = \Delta p \frac{\pi D^2}{4} - \sigma \pi D = 0$$

so $\frac{\Delta p D}{4} - \sigma = 0$ or $\Delta p = \frac{4\sigma}{D}$

Assuming soda-gas interface is similar to water-air, then $\sigma = 72.8$ mN/m, and

$$\Delta p = 4 \times 72.8 \times 10^{-3} \frac{N}{m} \times \frac{1}{0.1 \times 10^{-3} m} = 2.91 \times 10^3 \frac{N}{m^2} = 2.91 \text{ kPa}$$

Δp

Problem 2.71

[2]

Slowly fill a glass with water to the maximum possible level. Observe the water level closely. Explain how it can be higher than the rim of the glass.

Open-Ended Problem Statement: Slowly fill a glass with water to the maximum possible level before it overflows. Observe the water level closely. Explain how it can be higher than the rim of the glass.

Discussion: Surface tension can cause the maximum water level in a glass to be higher than the rim of the glass. The same phenomenon causes an isolated drop of water to "bead up" on a smooth surface.

Surface tension between the water/air interface and the glass acts as an invisible membrane that allows trapped water to rise above the level of the rim of the glass. The mechanism can be envisioned as forces that act in the surface of the liquid above the rim of the glass. Thus the water appears to defy gravity by attaining a level higher than the rim of the glass.

To experimentally demonstrate that this phenomenon is the result of surface tension, set the liquid level nearly as far above the glass rim as you can get it, using plain water. Add a drop of liquid detergent (the detergent contains additives that reduce the surface tension of water). Watch as the excess water runs over the side of the glass.

Problem 2.72

[2]

2.72 You intend to gently place several steel needles on the free surface of the water in a large tank. The needles come in two lengths: Some are 5 cm long, and some are 10 cm long. Needles of each length are available with diameters of 1 mm, 2.5 mm, and 5 mm. Make a prediction as to which needles, if any, will float.

Given: Data on size of various needles

Find: Which needles, if any, will float

Solution:

For a steel needle of length *L*, diameter *D*, density ρ_s, to float in water with surface tension σ and contact angle θ, the vertical force due to surface tension must equal or exceed the weight

$$2 \cdot L \cdot \sigma \cdot \cos(\theta) \geq W = m \cdot g = \frac{\pi \cdot D^2}{4} \cdot \rho_s \cdot L \cdot g \qquad \text{or} \qquad D \leq \sqrt{\frac{8 \cdot \sigma \cdot \cos(\theta)}{\pi \cdot \rho_s \cdot g}}$$

From Table A.4 $\quad \sigma = 72.8 \times 10^{-3} \cdot \frac{N}{m} \quad \theta = 0 \cdot deg \quad$ and for water $\quad \rho = 1000 \cdot \frac{kg}{m^3}$

From Table A.1, for steel $\quad SG = 7.83$

Hence $\quad \sqrt{\frac{8 \cdot \sigma \cdot \cos(\theta)}{\pi \cdot SG \cdot \rho \cdot g}} = \sqrt{\frac{8}{\pi \cdot 7.83} \times 72.8 \times 10^{-3} \cdot \frac{N}{m} \times \frac{m^3}{999 \cdot kg} \times \frac{s^2}{9.81 \cdot m} \times \frac{kg \cdot m}{N \cdot s^2}} = 1.55 \times 10^{-3} \cdot m = 1.55 \cdot mm$

Hence *D* < 1.55 mm. Only the 1 mm needles float (needle length is irrelevant)

Problem 2.73

[5]

Plan an experiment to measure the surface tension of a liquid similar to water. If necessary, review the NCFMF video *Surface Tension* for ideas. Which method would be most suitable for use in an undergraduate laboratory? What experimental precision could be expected?

Open-Ended Problem Statement: Plan an experiment to measure the surface tension of a liquid similar to water. If necessary, review the NCFMF video *Surface Tension* for ideas. Which method would be most suitable for use in an undergraduate laboratory? What experimental precision could be expected?

Discussion: Two basic kinds of experiment are possible for an undergraduate laboratory:

1. Using a clear small-diameter tube, compare the capillary rise of the unknown liquid with that of a known liquid (compare with water, because it is similar to the unknown liquid).

   This method would be simple to set up and should give fairly accurate results. A vertical traversing optical microscope could be used to increase the precision of measuring the liquid height in each tube.

   A drawback to this method is that the specific gravity and co ntact angle of the two liquids must be the same to allow the capillary rises to be compared.

   The capillary rise would be largest and therefore easiest to measure accurately in a tube with the smallest practical diameter. Tubes of several diameters could be used if desired.

2. Dip an object into a pool of test liquid and measure the vertical force required to pull the object from the liquid surface.

   The object might be made rectangular (e.g., a sheet of plastic material) or circular (e.g., a metal ring). The net force needed to pull the same object from each liquid should be proportional to the surface tension of each liquid.

   This method would be simple to set up. However, the force magnitudes to be measured would be quite small.

   A drawback to this method is that the contact angles of the two liquids must be the same.

The first method is probably best for undergraduate laboratory use. A quantitative estimate of experimental measurement uncertainty is impossible without knowing details of the test setup. It might be reasonable to expect results accurate to within ± 10% of the true surface tension.

---

*Net force is the total vertical force minus the weight of the object. A buoyancy correction would be necessary if part of the object were submerged in the test liquid.

Problem 2.74

Given: Water, with bulk modulus assumed constant.

Find: (a) Percent change in density at 100 atm
(b) Plot percent change vs. p/p_{atm} up to 50,000 psi.
(c) Comment on assumption of constant density.

Solution: By definition, $E_v = \frac{dp}{d\rho/\rho}$. Assume E_v = Constant. Then

$$\frac{d\rho}{\rho} = \frac{dp}{E_v}$$

Integrating, from ρ_0 to ρ gives $\ln\frac{\rho}{\rho_0} = \frac{p-p_0}{E_v} = \frac{\Delta p}{E_v}$, so $\frac{\rho}{\rho_0} = e^{\Delta p/E_v}$

The relative change in density is

$$\frac{\Delta\rho}{\rho_0} = \frac{\rho-\rho_0}{\rho_0} = \frac{\rho}{\rho_0} - 1 = e^{\Delta p/E_v} - 1$$

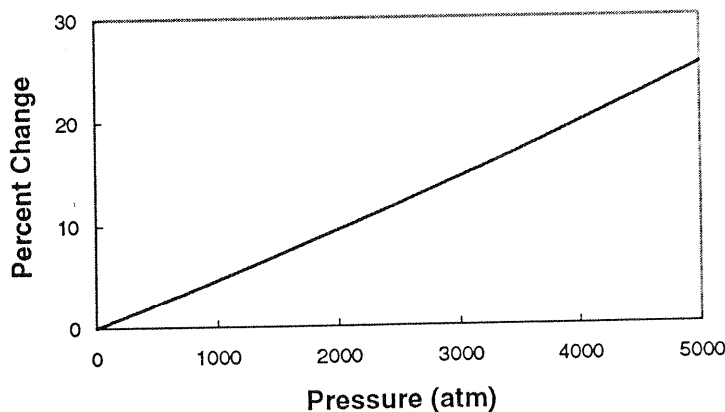
From Table A.2, E_v = 2.24 GPa for water at 20°C.

For p = 100 atm (gage), Δp = 100 atm, so

$$\frac{\Delta\rho}{\rho_0} = \exp\left(100\ \text{atm} \times \frac{1}{2.24\times10^9\ \text{Pa}} \times 101.325\times10^3\ \frac{\text{Pa}}{\text{atm}}\right) - 1 = 0.00453, \text{ or } 0.453\%$$

For Δp = 50,000 psi,

$$\frac{\Delta\rho}{\rho_0} = \exp\left(50{,}000\ \text{psi} \times \frac{1}{2.24\times10^9\ \text{Pa}} \times \frac{101.325\times10^3\ \text{Pa}}{14.696\ \text{psi}}\right) - 1 = 0.166 \text{ or } 16.6\%$$

Thus constant density is not a reasonable assumption for a cutting jet operating at 50,000 psi. Constant density (5% change) would be reasonable up to $\Delta p \approx$ 16,000 psi.

Problem 2.75

[2]

2.75 The viscous boundary layer velocity profile shown in Fig. 2.15 can be approximated by a parabolic equation,

$$u(y) = a + b\left(\frac{y}{\delta}\right) + c\left(\frac{y}{\delta}\right)^2$$

The boundary condition is $u = U$ (the free stream velocity) at the boundary edge δ (where the viscous friction becomes zero). Find the values of a, b, and c.

Given: Boundary layer velocity profile in terms of constants a, b and c

Find: Constants a, b and c

Solution:

Basic equation $$u = a + b\cdot\left(\frac{y}{\delta}\right) + c\cdot\left(\frac{y}{\delta}\right)^2$$

Assumptions: No slip, at outer edge u = U and τ = 0

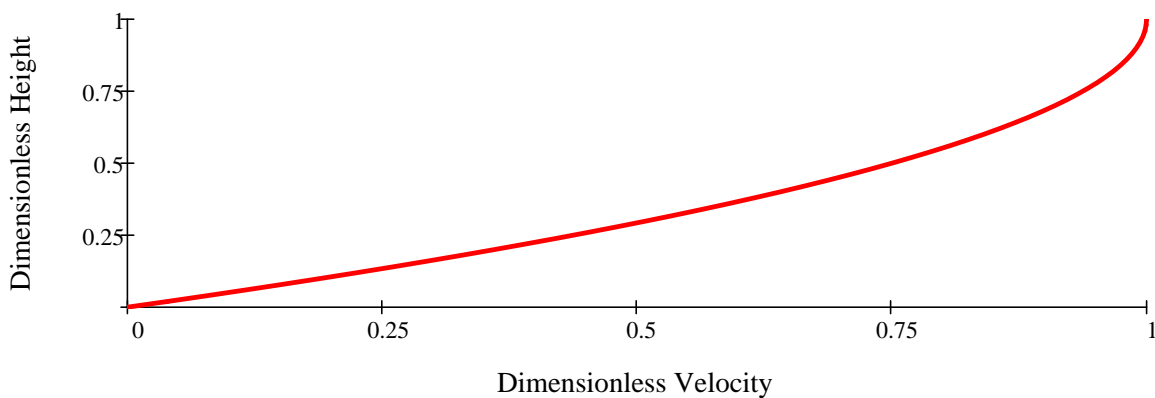
At y = 0 $\quad 0 = a \qquad a = 0$

At y = δ $\quad U = a + b + c \qquad b + c = U \qquad (1)$

At y = δ $$\tau = \mu\cdot\frac{du}{dy} = 0$$

$$0 = \frac{d}{dy}a + b\cdot\left(\frac{y}{\delta}\right) + c\cdot\left(\frac{y}{\delta}\right)^2 = \frac{b}{\delta} + 2\cdot c\cdot\frac{y}{\delta^2} = \frac{b}{\delta} + 2\cdot\frac{c}{\delta} \qquad b + 2\cdot c = 0 \qquad (2)$$

From 1 and 2 $\quad c = -U \qquad b = 2\cdot U$

Hence $$u = 2\cdot U\cdot\left(\frac{y}{\delta}\right) - U\cdot\left(\frac{y}{\delta}\right)^2 \qquad \frac{u}{U} = 2\cdot\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$$

Problem 2.76

2.76 The viscous boundary layer velocity profile shown in Fig. 2.15 can be approximated by a cubic equation,

$$u(y) = a + b\left(\frac{y}{\delta}\right) + c\left(\frac{y}{\delta}\right)^3$$

The boundary condition is $u = U$ (the free stream velocity) at the boundary edge δ (where the viscous friction becomes zero). Find the values of a, b, and c.

Given: Boundary layer velocity profile in terms of constants a, b and c

Find: Constants a, b and c

Solution:

Basic equation $\quad u = a + b\cdot\left(\frac{y}{\delta}\right) + c\cdot\left(\frac{y}{\delta}\right)^3$

Assumptions: No slip, at outer edge u = U and τ = 0

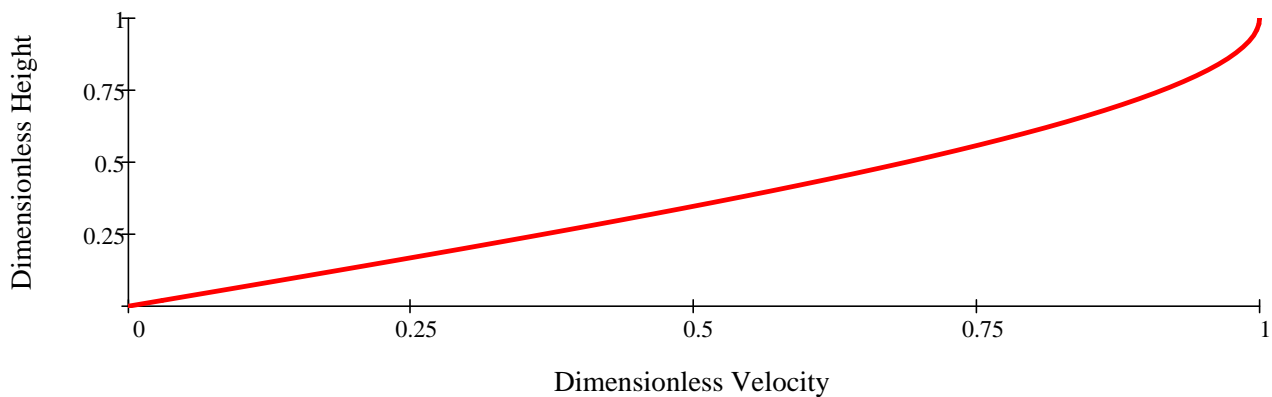
At y = 0 $\quad 0 = a \qquad a = 0$

At y = δ $\quad U = a + b + c \qquad b + c = U \qquad (1)$

At y = δ $\quad \tau = \mu\cdot\frac{du}{dy} = 0$

$$0 = \frac{d}{dy}a + b\cdot\left(\frac{y}{\delta}\right) + c\cdot\left(\frac{y}{\delta}\right)^3 = \frac{b}{\delta} + 3\cdot c\cdot\frac{y^2}{\delta^3} = \frac{b}{\delta} + 3\cdot\frac{c}{\delta} \qquad b + 3\cdot c = 0 \qquad (2)$$

From 1 and 2 $\quad c = -\frac{U}{2} \qquad b = \frac{3}{2}\cdot U$

Hence $\quad u = \frac{3\cdot U}{2}\cdot\left(\frac{y}{\delta}\right) - \frac{U}{2}\cdot\left(\frac{y}{\delta}\right)^3 \qquad \frac{u}{U} = \frac{3}{2}\cdot\left(\frac{y}{\delta}\right) - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^3$

Problem 2.77

2.77 At what minimum speed (in mph) would an automobile have to travel for compressibility effects to be important? Assume the local air temperature is 60°F.

Given: Local temperature

Find: Minimum speed for compressibility effects

Solution:

Basic equation $V = M \cdot c$ and $M = 0.3$ for compressibility effects

$c = \sqrt{k \cdot R \cdot T}$ For air at STP, k = 1.40 and R = 286.9J/kg.K (53.33 ft.lbf/lbmºR).

Hence $V = M \cdot c = M \cdot \sqrt{k \cdot R \cdot T}$

$$V = 0.3 \times \left[1.4 \times 53.33 \cdot \frac{ft \cdot lbf}{lbm \cdot R} \times \frac{32.2 \cdot lbm \cdot ft}{lbf \cdot s^2} \times (60 + 460) \cdot R\right]^{\frac{1}{2}} \cdot \frac{60 \cdot mph}{88 \cdot \frac{ft}{s}} \qquad V = 229 \cdot mph$$

Problem 2.78

[2]

2.78 Water flows through a 1-in. ID garden hose at a rate of 0.075 ft^3/min. A 5-in.-long, cone-shaped nozzle is attached to the hose to accelerate the flow. If the nozzle reduces the flow area by a factor of 4, at what distance from the inlet of the nozzle does the flow become turbulent? Assume the water temperature is 60°F.

NOTE: Flow rate should be $0.75\cdot\frac{ft^3}{min}$

Given: Geometry of and flow rate through garden hose

Find: At which point becomes turbulent

Solution:

Basic equation — For pipe flow (Section 2-6) $Re = \frac{\rho\cdot V\cdot D}{\mu} = 2300$ for transition to turbulence

Also flow rate Q is given by $Q = \frac{\pi\cdot D^2}{4}\cdot V$

We can combine these equations and eliminate V to obtain an expression for Re in terms of D

$$Re = \frac{\rho\cdot V\cdot D}{\mu} = \frac{\rho\cdot D}{\mu}\cdot\frac{4\cdot Q}{\pi\cdot D^2} = \frac{4\cdot Q\cdot\rho}{\pi\cdot\mu\cdot D} = 2300$$

Hence $D = \frac{4\cdot Q\cdot\rho}{2300\cdot\pi\cdot\mu}$ From Appendix A: $\rho = 1.94\cdot\frac{slug}{ft^3}$ (Approximately)

$$\mu = 1.25\times 10^{-3}\cdot\frac{N\cdot s}{m^2}\times\frac{0.209\cdot\frac{lbf\cdot s}{ft^2}}{1\cdot\frac{N\cdot s}{m^2}}$$ (Approximately, from Fig. A.2) $\mu = 2.61\times 10^{-4}\cdot\frac{lbf\cdot s}{ft^2}$

Hence $$D = \frac{4}{2300\cdot\pi}\times\frac{0.75\cdot ft^3}{min}\times\frac{1\cdot min}{60\cdot s}\times\frac{1.94\cdot slug}{ft^3}\times\frac{ft^2}{2.61\cdot 10^{-4}\cdot lbf\cdot s}\times\frac{lbf\cdot s^2}{slug\cdot ft}\times\frac{12\cdot in}{1\cdot ft}$$ $D = 0.617\cdot in$

NOTE: For wrong flow rate, will be 1/10th of this!

The nozzle is tapered: $D_{in} = 1\cdot in$ $D_{out} = \frac{D_{in}}{\sqrt{4}}$ $D_{out} = 0.5\cdot in$ $L = 5\cdot in$

Linear ratios leads to the distance from D_{in} at which D = 0.617 in $\frac{L_{turb}}{L} = \frac{D - D_{in}}{D_{out} - D_{in}}$

$$L_{turb} = L\cdot\frac{D - D_{in}}{D_{out} - D_{in}}$$ $L_{turb} = 3.83\cdot in$

NOTE: For wrong flow rate, this does not apply! Flow will not become turbulent.

Problem 2.79

2.79 A supersonic aircraft travels at 2700 km/hr at an altitude of 27 km. What is the Mach number of the aircraft? At what approximate distance measured from the leading edge of the aircraft's wing does the boundary layer change from laminar to turbulent?

Given: Data on supersonic aircraft

Find: Mach number; Point at which boundary layer becomes turbulent

Solution:

Basic equation $V = M\cdot c$ and $c = \sqrt{k\cdot R\cdot T}$ For air at STP, k = 1.40 and R = 286.9J/kg.K (53.33 ft.lbf/lbm°R).

Hence $M = \frac{V}{c} = \frac{V}{\sqrt{k\cdot R\cdot T}}$

At 27 km the temperature is approximately (from Table A.3) $T = 223.5\cdot K$

$$M = \left(2700\times 10^{3}\cdot\frac{m}{hr}\times\frac{1\cdot hr}{3600\cdot s}\right)\cdot\left(\frac{1}{1.4}\times\frac{1}{286.9}\cdot\frac{kg\cdot K}{N\cdot m}\times\frac{1\cdot N\cdot s^{2}}{kg\cdot m}\times\frac{1}{223.5}\cdot\frac{1}{K}\right)^{\frac{1}{2}} \qquad M = 2.5$$

For boundary layer transition, from Section 2-6 $Re_{trans} = 500000$

Then $Re_{trans} = \frac{\rho\cdot V\cdot x_{trans}}{\mu}$ so $x_{trans} = \frac{\mu\cdot Re_{trans}}{\rho\cdot V}$

We need to find the viscosity and density at this altitude and pressure. The viscosity depends on temperature only, but at 223.5 K = - 50°C, it is off scale of Fig. A.3. Instead we need to use formulas as in Appendix A

At this altitude the density is (Table A.3) $\rho = 0.02422\times 1.225\cdot\frac{kg}{m^{3}}$ $\rho = 0.0297\frac{kg}{m^{3}}$

For μ $\mu = \frac{b\cdot T^{\frac{1}{2}}}{1+\frac{S}{T}}$ where $b = 1.458\times 10^{-6}\cdot\frac{kg}{m\cdot s\cdot K^{\frac{1}{2}}}$ $S = 110.4\cdot K$

$\mu = 1.459\times 10^{-5}\frac{kg}{m\cdot s}$ $\mu = 1.459\times 10^{-5}\cdot\frac{N\cdot s}{m^{2}}$

Hence $x_{trans} = 1.459\times 10^{-5}\cdot\frac{kg}{m\cdot s}\times 500000\times\frac{1}{0.0297}\cdot\frac{m^{3}}{kg}\times\frac{1}{2700}\times\frac{1}{10^{3}}\cdot\frac{hr}{m}\times\frac{3600\cdot s}{1\cdot hr}$ $x_{trans} = 0.327\,m$

Problem 2.80

[2]

2.80 What is the Reynolds number of water at 20°C flowing at 0.25 m/s through a 5-mm-diameter tube? If the pipe is now heated, at what mean water temperature will the flow transition to turbulence? Assume the velocity of the flow remains constant.

Given: Data on water tube

Find: Reynolds number of flow; Temperature at which flow becomes turbulent

Solution:

Basic equation For pipe flow (Section 2-6) $Re = \frac{\rho \cdot V \cdot D}{\mu} = \frac{V \cdot D}{\nu}$

At 20ºC, from Fig. A.3 $\nu = 9 \times 10^{-7} \cdot \frac{m^2}{s}$ and so $Re = 0.25 \cdot \frac{m}{s} \times 0.005 \cdot m \times \frac{1}{9 \times 10^{-7}} \cdot \frac{s}{m^2}$ $Re = 1389$

For the heated pipe $Re = \frac{V \cdot D}{\nu} = 2300$ for transition to turbulence

Hence $\nu = \frac{V \cdot D}{2300} = \frac{1}{2300} \times 0.25 \cdot \frac{m}{s} \times 0.005 \cdot m$ $\nu = 5.435 \times 10^{-7} \frac{m^2}{s}$

From Fig. A.3, the temperature of water at this viscosity is approximately $T = 52 \cdot C$

Problem 2.81

[2]

2.81 SAE 30 oil at 100°C flows through a 12-mm-diameter stainless-steel tube. What is the specific gravity and specific weight of the oil? If the oil discharged from the tube fills a 100-mL graduated cylinder in 9 seconds, is the flow laminar or turbulent?

Given: Type of oil, flow rate, and tube geometry

Find: Whether flow is laminar or turbulent

Solution:

Data on SAE 30 oil SG or density is limited in the Appendix. We can Google it or use the following $\nu = \frac{\mu}{\rho}$ so $\rho = \frac{\mu}{\nu}$

At 100ºC, from Figs. A.2 and A.3 $\mu = 9\times 10^{-3}\cdot\frac{N\cdot s}{m^2}$ $\nu = 1\times 10^{-5}\cdot\frac{m^2}{s}$

$$\rho = 9\times 10^{-3}\cdot\frac{N\cdot s}{m^2}\times\frac{1}{1\times 10^{-5}}\cdot\frac{s}{m^2}\times\frac{kg\cdot m}{s^2\cdot N} \qquad \rho = 900\frac{kg}{m^3}$$

Hence $SG = \frac{\rho}{\rho_{water}}$ $\rho_{water} = 1000\cdot\frac{kg}{m^3}$ $SG = 0.9$

The specific weight is $\gamma = \rho\cdot g$ $\gamma = 900\cdot\frac{kg}{m^3}\times 9.81\cdot\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}$ $\gamma = 8.829\times 10^3\cdot\frac{N}{m^3}$

For pipe flow (Section 2-6) $Q = \frac{\pi\cdot D^2}{4}\cdot V$ so $V = \frac{4\cdot Q}{\pi\cdot D^2}$

$$Q = 100\cdot mL\times\frac{10^{-6}\cdot m^3}{1\cdot mL}\times\frac{1}{9}\cdot\frac{1}{s} \qquad Q = 1.111\times 10^{-5}\frac{m^3}{s}$$

Then $V = \frac{4}{\pi}\times 1.11\times 10^{-5}\cdot\frac{m^3}{s}\times\left(\frac{1}{12}\cdot\frac{1}{mm}\times\frac{1000\cdot mm}{1\cdot m}\right)^2$ $V = 0.0981\frac{m}{s}$

Hence $Re = \frac{\rho\cdot V\cdot D}{\mu}$

$$Re = 900\cdot\frac{kg}{m^3}\times 0.0981\cdot\frac{m}{s}\times 0.012\cdot m\times\frac{1}{9\times 10^{-3}}\cdot\frac{m^2}{N\cdot s}\times\frac{N\cdot s^2}{kg\cdot m} \qquad Re = 118$$

Flow is laminar

Problem 2.82

[2]

2.82 A seaplane is flying at 100 mph through air at 45°F. At what distance from the leading edge of the underside of the fuselage does the boundary layer transition to turbulence? How does this boundary layer transition change as the underside of the fuselage touches the water during landing? Assume the water temperature is also 45°F.

Given: Data on seaplane

Find: Transition point of boundary layer

Solution:

For boundary layer transition, from Section 2-6 $\quad Re_{trans} = 500000$

Then

$$Re_{trans} = \frac{\rho \cdot V \cdot x_{trans}}{\mu} = \frac{V \cdot x_{trans}}{\nu} \quad \text{so} \quad x_{trans} = \frac{\nu \cdot Re_{trans}}{V}$$

At 45ºF = 7.2ºC (Fig A.3)

$$\nu = 0.8 \times 10^{-5} \cdot \frac{m^2}{s} \times \frac{10.8 \cdot \frac{ft^2}{s}}{1 \cdot \frac{m^2}{s}} \qquad \nu = 8.64 \times 10^{-5} \cdot \frac{ft^2}{s}$$

$$x_{trans} = 8.64 \times 10^{-5} \cdot \frac{ft^2}{s} \cdot 500000 \times \frac{1}{100 \cdot mph} \times \frac{60 \cdot mph}{88 \cdot \frac{ft}{s}} \qquad x_{trans} = 0.295 \cdot ft$$

As the seaplane touches down:

At 45ºF = 7.2ºC (Fig A.3)

$$\nu = 1.5 \times 10^{-5} \cdot \frac{m^2}{s} \times \frac{10.8 \cdot \frac{ft^2}{s}}{1 \cdot \frac{m^2}{s}} \qquad \nu = 1.62 \times 10^{-4} \cdot \frac{ft^2}{s}$$

$$x_{trans} = 1.62 \times 10^{-4} \cdot \frac{ft^2}{s} \cdot 500000 \times \frac{1}{100 \cdot mph} \times \frac{60 \cdot mph}{88 \cdot \frac{ft}{s}} \qquad x_{trans} = 0.552 \cdot ft$$

Problem 2.83 (In Excel)

[3]

2.83 An airliner is cruising at an altitude of 5.5 km with a speed of 700 km/hr. As the airliner increases its altitude, it adjusts its speed so that the Mach number remains constant. Provide a sketch of speed vs. altitude. What is the speed of the airliner at an altitude of 8 km?

Given: Data on airliner
Find: Sketch of speed versus altitude (M = const)
Solution:

Data on temperature versus height can be obtained from Table A.3

At 5.5 km the temperature is approximate 252 K

The speed of sound is obtained from $c = \sqrt{k \cdot R \cdot T}$

where k = 1.4
R = 286.9 J/kg·K (Table A.6)

c = 318 m/s

We also have
V = 700 km/hr

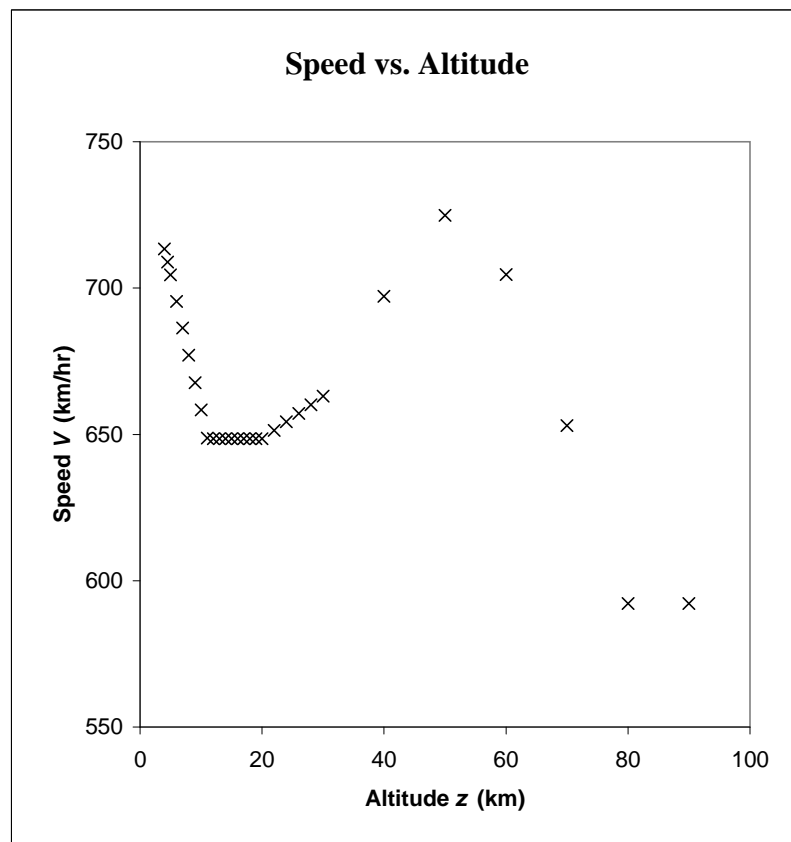
or V = 194 m/s

Hence $M = V/c$ or

M = 0.611

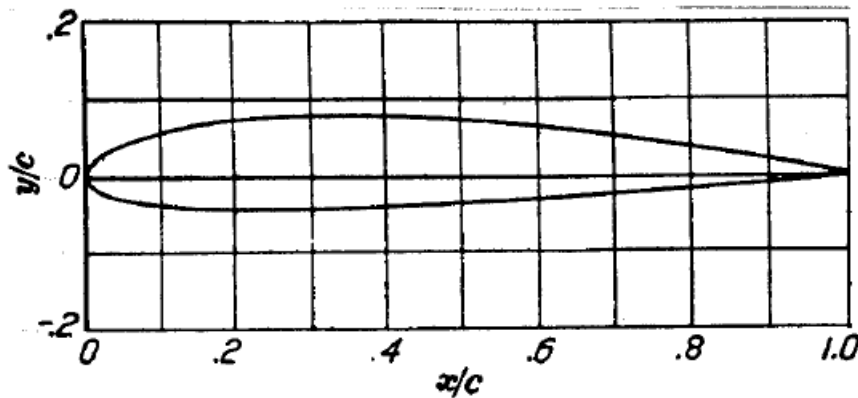
To compute V for constant M, we use $V = M \cdot c = 0.611 \cdot c$

At a height of 8 km V = 677 km/hr
NOTE: Realistically, the aiplane will fly to a maximum height of about 10 km!

z (km)	T (K)	c (m/s)	V (km/hr)
4	262	325	713
5	259	322	709
5	256	320	704
6	249	316	695
7	243	312	686
8	236	308	677
9	230	304	668
10	223	299	658
11	217	295	649
12	217	295	649
13	217	295	649
14	217	295	649
15	217	295	649
16	217	295	649
17	217	295	649
18	217	295	649
19	217	295	649
20	217	295	649
22	219	296	651
24	221	298	654
26	223	299	657
28	225	300	660
30	227	302	663
40	250	317	697
50	271	330	725
60	256	321	705
70	220	297	653
80	181	269	592
90	181	269	592

Problem 2.84

How does an airplane wing develop lift?

Open-Ended Problem Statement: How does an airplane wing develop lift?

Discussion: The sketch shows the cross-section of a typical airplane wing. The airfoil section is rounded at the front, curved across the top, reaches maximum thickness about a third of the way back, and then tapers slowly to a fine trailing edge. The bottom of the airfoil section is relatively flat. (The discussion below also applies to a symmetric airfoil at an angle of incidence that produces lift.)

NACA 2412 Wing Section

It is both a popular expectation and an experimental fact that air flows more rapidly over the curved top surface of the airfoil section than along the relatively flat bottom. In the NCFMF video *Flow Visualization*, timelines placed in front of the airfoil indicate that fluid flows more rapidly along the top of the section than along the bottom.

In the absence of viscous effects (this is a valid assumption outside the boundary layers on the airfoil) pressure falls when flow speed increases. Thus the pressures on the top surface of the airfoil where flow speed is higher are lower than the pressures on the bottom surface where flow speed does not increase. (Actual pressure profiles measured for a lifting section are shown in the NCFMF video *Boundary Layer Control*.) The unbalanced pressures on the top and bottom surfaces of the airfoil section create a net force that tends to develop lift on the profile.

Problem 3.1

[2]

3.1 Compressed nitrogen is stored in a spherical tank of diameter $D = 0.75$ m. The gas is at an absolute pressure of 25 MPa and a temperature of 25°C. What is the mass in the tank? If the maximum allowable wall stress in the tank is 210 MPa, find the minimum theoretical wall thickness of the tank.

Given: Data on nitrogen tank

Find: Mass of nitrogen; minimum required wall thickness

Solution:

Assuming ideal gas behavior: $p \cdot V = M \cdot R \cdot T$

where, from Table A.6, for nitrogen $R = 297 \cdot \frac{J}{kg \cdot K}$

Then the mass of nitrogen is $M = \frac{p \cdot V}{R \cdot T} = \frac{p}{R \cdot T} \cdot \left(\frac{\pi \cdot D^3}{6} \right)$

$$M = \frac{25 \cdot 10^6 \cdot N}{m^2} \times \frac{kg \cdot K}{297 \cdot J} \times \frac{1}{298 \cdot K} \times \frac{J}{N \cdot m} \times \frac{\pi \cdot (0.75 \cdot m)^3}{6}$$

$$M = 62.4\,kg$$

To determine wall thickness, consider a free body diagram for one hemisphere:

$$\Sigma F = 0 = p \cdot \frac{\pi \cdot D^2}{4} - \sigma_c \cdot \pi \cdot D \cdot t$$

where σ_c is the circumferential stress in the container

Then $t = \frac{p \cdot \pi \cdot D^2}{4 \cdot \pi \cdot D \cdot \sigma_c} = \frac{p \cdot D}{4 \cdot \sigma_c}$

$$t = 25 \cdot 10^6 \cdot \frac{N}{m^2} \times \frac{0.75 \cdot m}{4} \times \frac{1}{210 \cdot 10^6} \cdot \frac{m^2}{N}$$

$$t = 0.0223\,m \qquad t = 22.3\,mm$$

Problem 3.2

[2]

3.2 Ear "popping" is an unpleasant phenomenon sometimes experienced when a change in pressure occurs, for example in a fast-moving elevator or in an airplane. If you are in a two-seater airplane at 3000 m and a descent of 100 m causes your ears to "pop," what is the pressure change that your ears "pop" at, in millimeters of mercury? If the airplane now rises to 8000 m and again begins descending, how far will the airplane descend before your ears "pop" again? Assume a U.S. Standard Atmosphere.

Given: Data on flight of airplane

Find: Pressure change in mm Hg for ears to "pop"; descent distance from 8000 m to cause ears to "pop."

Solution:

Assume the air density is approximately constant constant from 3000 m to 2900 m.
From table A.3

$$\rho_{SL} = 1.225 \cdot \frac{kg}{m^3} \qquad \rho_{air} = 0.7423 \cdot \rho_{SL} \qquad \rho_{air} = 0.909 \frac{kg}{m^3}$$

We also have from the manometer equation, Eq. 3.7

$$\Delta p = -\rho_{air} \cdot g \cdot \Delta z \qquad \text{and also} \qquad \Delta p = -\rho_{Hg} \cdot g \cdot \Delta h_{Hg}$$

Combining $$\Delta h_{Hg} = \frac{\rho_{air}}{\rho_{Hg}} \cdot \Delta z = \frac{\rho_{air}}{SG_{Hg} \cdot \rho_{H2O}} \cdot \Delta z \qquad SG_{Hg} = 13.55 \text{ from Table A.2}$$

$$\Delta h_{Hg} = \frac{0.909}{13.55 \times 999} \times 100 \cdot m \qquad \Delta h_{Hg} = 6.72\,mm$$

For the ear popping descending from 8000 m, again assume the air density is approximately constant constant, this time at 8000 m.
From table A.3

$$\rho_{air} = 0.4292 \cdot \rho_{SL} \qquad \rho_{air} = 0.526 \frac{kg}{m^3}$$

We also have from the manometer equation

$$\rho_{air8000} \cdot g \cdot \Delta z_{8000} = \rho_{air3000} \cdot g \cdot \Delta z_{3000}$$

where the numerical subscripts refer to conditions at 3000m and 8000m.
Hence

$$\Delta z_{8000} = \frac{\rho_{air3000} \cdot g}{\rho_{air8000} \cdot g} \cdot \Delta z_{3000} = \frac{\rho_{air3000}}{\rho_{air8000}} \cdot \Delta z_{3000} \qquad \Delta z_{8000} = \frac{0.909}{0.526} \times 100 \cdot m \qquad \Delta z_{8000} = 173\,m$$

Problem 3.3 [3]

3.3 When you are on a mountain face and boil water, you notice that the water temperature is 195°F. What is your approximate altitude? The next day, you are at a location where it boils at 185°F. How high did you climb between the two days? Assume a U.S. Standard Atmosphere.

Given: Boiling points of water at different elevations

Find: Change in elevation

Solution:

From the steam tables, we have the following data for the boiling point (saturation temperature) of water

T_{sat} (°F)	p (psia)
195	10.39
185	8.39

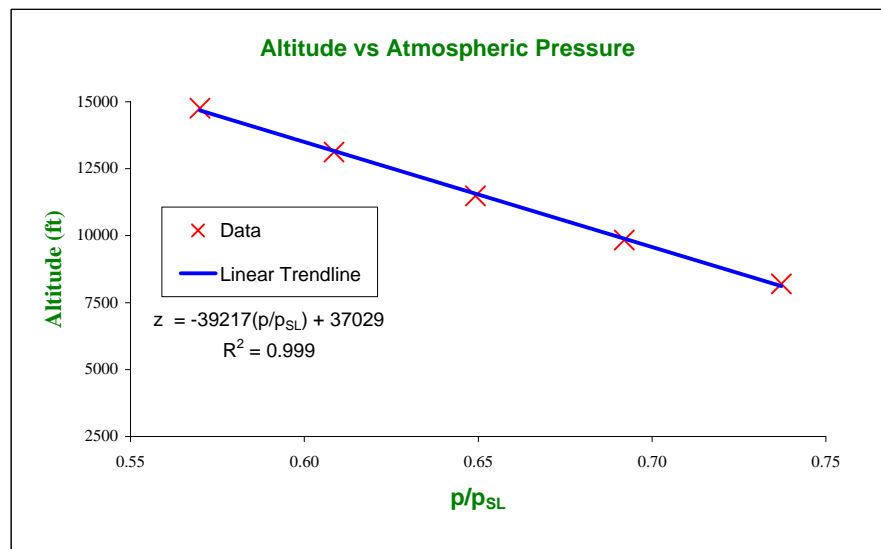
The sea level pressure, from Table A.3, is

p_{SL} = 14.696 psia

Hence

T_{sat} (°F)	p/p_{SL}
195	0.707
185	0.571

From Table A.3

p/p_{SL}	Altitude (m)	Altitude (ft)
0.7372	2500	8203
0.6920	3000	9843
0.6492	3500	11484
0.6085	4000	13124
0.5700	4500	14765

Then, any one of a number of *Excel* functions can be used to interpolate
(Here we use *Excel*'s *Trendline* analysis)

p/p_{SL}	Altitude (ft)
0.707	9303
0.571	14640

Current altitude is approximately 9303 ft

The change in altitude is then 5337 ft

Alternatively, we can interpolate for each altitude by using a linear regression between adjacent data points

For

p/p_{SL}	Altitude (m)	Altitude (ft)
0.7372	2500	8203
0.6920	3000	9843

p/p_{SL}	Altitude (m)	Altitude (ft)
0.6085	4000	13124
0.5700	4500	14765

Then

0.7070	2834	9299

0.5730	4461	14637

The change in altitude is then 5338 ft

Problem 3.4

Given: Pure water on a standard day

Find: Boiling temperature at (a) 1000 m, and (b) 2000 m. Compare with sea level value

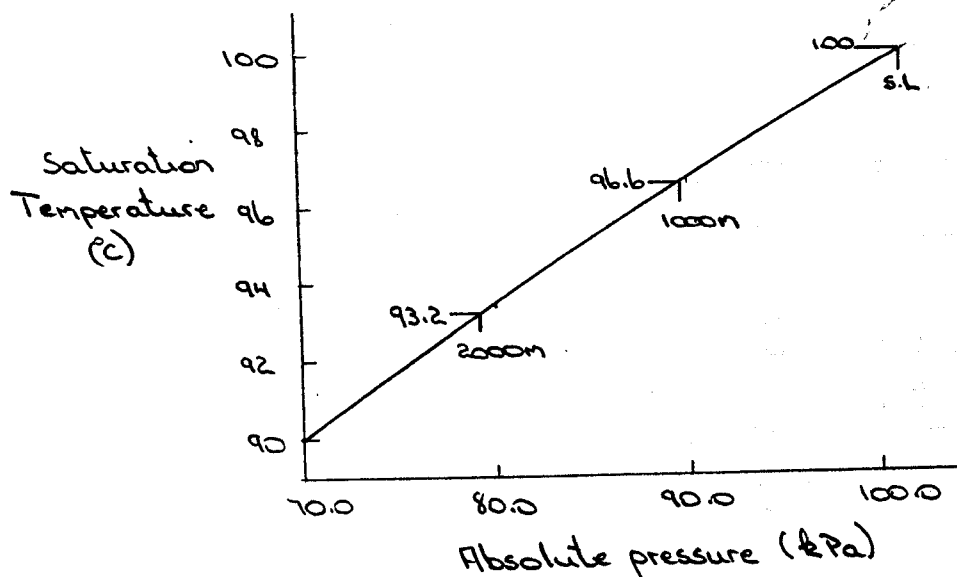
Solution

We can determine the atmospheric pressure at the given altitudes from table A.3, Appendix A

Elevation (m)	$\frac{P}{P_o}$	P (kPa)	T^*_{sat} (°C)
0	1.000	101	100
1000	0.887	89.6	96.6
2000	0.785	79.3	93.2

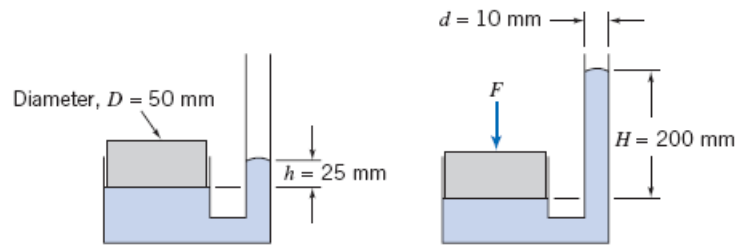
* T_{sat} obtained from plot of T_{sat} vs P given below

Data from Steam Tables gives T_{sat}

P (kPa)	T_{sat} (°C)
70	90.0
80	93.5
90	96.7
101.325	100.0

{ These data show that T_{sat} drops about 3.4°C / 1000m }

Problem 3.5

[2]

3.5 The tube shown is filled with mercury at 20°C. Calculate the force applied to the piston.

Given: Data on system before and after applied force

Find: Applied force

Solution:

Basic equation $\frac{dp}{dy} = -\rho \cdot g$ or, for constant ρ $p = p_{atm} - \rho \cdot g \cdot (y - y_0)$ with $p(y_0) = p_{atm}$

For initial state $p_1 = p_{atm} + \rho \cdot g \cdot h$ and $F_1 = p_1 \cdot A = \rho \cdot g \cdot h \cdot A$ (Gage; F_1 is hydrostatic upwards force)

For the initial FBD $\Sigma F_y = 0$ $F_1 - W = 0$ $W = F_1 = \rho \cdot g \cdot h \cdot A$

For final state $p_2 = p_{atm} + \rho \cdot g \cdot H$ and $F_2 = p_2 \cdot A = \rho \cdot g \cdot H \cdot A$ (Gage; F_2 is hydrostatic upwards force)

For the final FBD $\Sigma F_y = 0$ $F_2 - W - F = 0$ $F = F_2 - W = \rho \cdot g \cdot H \cdot A - \rho \cdot g \cdot h \cdot A = \rho \cdot g \cdot A \cdot (H - h)$

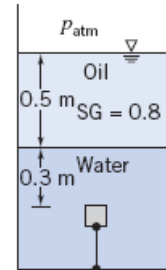
$$F = \rho_{H2O} \cdot SG \cdot g \cdot \frac{\pi \cdot D^2}{4} \cdot (H - h)$$

From Fig. A.1 $SG = 13.54$

$$F = 1000 \cdot \frac{kg}{m^3} \times 13.54 \times 9.81 \cdot \frac{m}{s^2} \times \frac{\pi}{4} \times (0.05 \cdot m)^2 \times (0.2 - 0.025) \cdot m \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F = 45.6\,N$$

Problem 3.6

3.6 A 125-mL cube of solid oak is held submerged by a tether as shown. Calculate the actual force of the water on the bottom surface of the cube and the tension in the tether.

Given: Data on system

Find: Force on bottom of cube; tension in tether

Solution:

Basic equation $\frac{dp}{dy} = -\rho \cdot g$ or, for constant ρ $\Delta p = \rho \cdot g \cdot h$ where h is measured downwards

The absolute pressure at the interface is $p_{interface} = p_{atm} + SG_{oil} \cdot \rho \cdot g \cdot h_{oil}$

Then the pressure on the lower surface is $p_L = p_{interface} + \rho \cdot g \cdot h_L = p_{atm} + \rho \cdot g \cdot \left(SG_{oil} \cdot h_{oil} + h_L\right)$

For the cube $V = 125 \cdot mL$ $V = 1.25 \times 10^{-4} \cdot m^3$

Then the size of the cube is $d = V^{\frac{1}{3}}$ $d = 0.05\,m$ and the depth in water to the upper surface is $h_U = 0.3 \cdot m$

Hence $h_L = h_U + d$ $h_L = 0.35\,m$ where h_L is the depth in water to the lower surface

The force on the lower surface is $F_L = p_L \cdot A$ where $A = d^2$ $A = 0.0025\,m^2$

$$F_L = \left[p_{atm} + \rho \cdot g \cdot \left(SG_{oil} \cdot h_{oil} + h_L\right)\right] \cdot A$$

$$F_L = \left[101 \times 10^3 \cdot \frac{N}{m^2} + 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times (0.8 \times 0.5 \cdot m + 0.35 \cdot m) \times \frac{N \cdot s^2}{kg \cdot m}\right] \times 0.0025 \cdot m^2$$

$F_L = 270.894\,N$ Note: Extra decimals needed for computing T later!

For the tension in the tether, an FBD gives $\Sigma F_y = 0$ $F_L - F_U - W - T = 0$ or $T = F_L - F_U - W$

where $F_U = \left[p_{atm} + \rho \cdot g \cdot \left(SG_{oil} \cdot h_{oil} + h_U\right)\right] \cdot A$

Note that we could instead compute $\Delta F = F_L - F_U = \rho \cdot g \cdot SG_{oil} \cdot \left(h_L - h_U\right) \cdot A$ and $T = \Delta F - W$

Using F_U

$$F_U = \left[101 \times 10^3 \cdot \frac{N}{m^2} + 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times (0.8 \times 0.5 \cdot m + 0.3 \cdot m) \times \frac{N \cdot s^2}{kg \cdot m}\right] \times 0.0025 \cdot m^2$$

$F_U = 269.668\,N$ Note: Extra decimals needed for computing T later!

For the oak block (Table A.1) $SG_{oak} = 0.77$ so $W = SG_{oak} \cdot \rho \cdot g \cdot V$

$$W = 0.77 \times 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 1.25 \times 10^{-4} \cdot m^3 \times \frac{N \cdot s^2}{kg \cdot m} \qquad W = 0.944\,N$$

$T = F_L - F_U - W$ $T = 0.282\,N$

Problem 3.7

3.7 The following pressure and temperature measurements were taken by a meteorological balloon rising through the lower atmosphere:

p (in 10^3 Pa)	T (in °C)
101.4	12.0
100.8	11.1
100.2	10.5
99.6	10.2
99.0	10.1
98.4	10.0
97.8	10.3
97.2	10.8
96.6	11.6
96.0	12.2
95.4	12.1

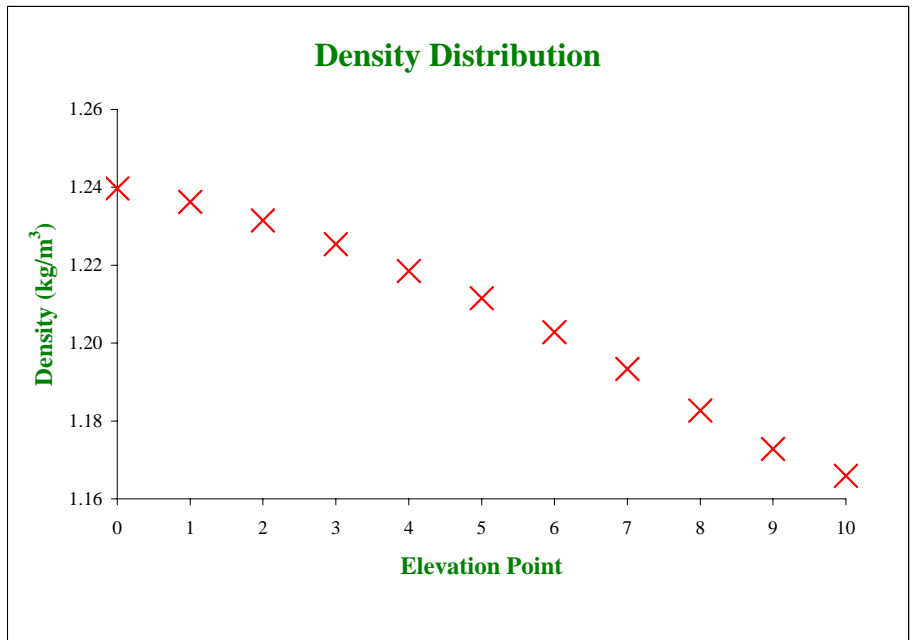
The initial values (top of table) correspond to ground level. Using the ideal gas law ($p = \rho RT$ with R = 287 $m^2/(s^2 \cdot K)$, compute and plot the variation of air density (in kg/m^3) with height.

Given: Pressure and temperature data from balloon

Find: Plot density change as a function of elevation

Solution:

Using the ideal gas equation, $\rho = p/RT$

p (kPa)	T (°C)	ρ (kg/m³)
101.4	12.0	1.240
100.8	11.1	1.236
100.2	10.5	1.231
99.6	10.2	1.225
99.0	10.1	1.218
98.4	10.0	1.212
97.8	10.3	1.203
97.2	10.8	1.193
96.6	11.6	1.183
96.0	12.2	1.173
95.4	12.1	1.166

Problem 3.8

3.8 Your pressure gage indicates that the pressure in your cold tires is 0.25 MPa (gage) on a mountain at an elevation of 3500 m. What is the absolute pressure? After you drive down to sea level, your tires have warmed to 25°C. What pressure does your gage now indicate? Assume a U.S. Standard Atmosphere.

Given: Data on tire at 3500 m and at sea level

Find: Absolute pressure at 3500 m; pressure at sea level

Solution:

At an elevation of 3500 m, from Table A.3:

$$p_{SL} = 101 \cdot kPa \qquad p_{atm} = 0.6492 \cdot p_{SL} \qquad p_{atm} = 65.6 \cdot kPa$$

and we have $p_g = 0.25 \cdot MPa \qquad p_g = 250 \cdot kPa \qquad p = p_g + p_{atm} \qquad p = 316 \cdot kPa$

At sea level $p_{atm} = 101 \cdot kPa$

Meanwhile, the tire has warmed up, from the ambient temperature at 3500 m, to 25°C.

At an elevation of 3500 m, from Table A.3 $T_{cold} = 265.4 \cdot K$ and $T_{hot} = (25 + 273) \cdot K \qquad T_{hot} = 298\,K$

Hence, assuming ideal gas behavior, $pV = mRT$, and that the tire is approximately a rigid container, the absolute pressure of the hot tire is

$$p_{hot} = \frac{T_{hot}}{T_{cold}} \cdot p \qquad p_{hot} = 354 \cdot kPa$$

Then the gage pressure is

$$p_g = p_{hot} - p_{atm} \qquad p_g = 253 \cdot kPa$$

Problem 3.9

[2]

3.9 A hollow metal cube with sides 100 mm floats at the interface between a layer of water and a layer of SAE 10W oil such that 10% of the cube is exposed to the oil. What is the pressure difference between the upper and lower horizontal surfaces? What is the average density of the cube?

Given: Properties of a cube floating at an interface

Find: The pressures difference between the upper and lower surfaces; average cube density

Solution:

The pressure difference is obtained from two applications of Eq. 3.7

$$p_U = p_0 + \rho_{SAE10} \cdot g \cdot (H - 0.1 \cdot d) \qquad p_L = p_0 + \rho_{SAE10} \cdot g \cdot H + \rho_{H2O} \cdot g \cdot 0.9 \cdot d$$

where p_U and p_L are the upper and lower pressures, p_0 is the oil free surface pressure, H is the depth of the interface, and d is the cube size

Hence the pressure difference is

$$\Delta p = p_L - p_U = \rho_{H2O} \cdot g \cdot 0.9 \cdot d + \rho_{SAE10} \cdot g \cdot 0.1 \cdot d \qquad \Delta p = \rho_{H2O} \cdot g \cdot d \cdot \left(0.9 + SG_{SAE10} \cdot 0.1\right)$$

From Table A.2 $\quad SG_{SAE10} = 0.92$

$$\Delta p = 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 0.1 \cdot m \times (0.9 + 0.92 \times 0.1) \times \frac{N \cdot s^2}{kg \cdot m} \qquad \Delta p = 972\, Pa$$

For the cube density, set up a free body force balance for the cube

$$\Sigma F = 0 = \Delta p \cdot A - W$$

Hence $\quad W = \Delta p \cdot A = \Delta p \cdot d^2$

$$\rho_{cube} = \frac{m}{d^3} = \frac{W}{d^3 \cdot g} = \frac{\Delta p \cdot d^2}{d^3 \cdot g} = \frac{\Delta p}{d \cdot g}$$

$$\rho_{cube} = 972 \cdot \frac{N}{m^2} \times \frac{1}{0.1 \cdot m} \times \frac{s^2}{9.81 \cdot m} \times \frac{kg \cdot m}{N \cdot s^2} \qquad \rho_{cube} = 991 \frac{kg}{m^3}$$

Problem 3.10

3.10 A cube with 6 in. sides is suspended in a fluid by a wire. The top of the cube is horizontal and 8 in. below the free surface. If the cube has a mass of 2 slugs and the tension in the wire is $T = 50.7$ lbf, compute the fluid specific gravity, and from this determine the fluid. What are the gage pressures on the upper and lower surfaces?

Given: Properties of a cube suspended by a wire in a fluid

Find: The fluid specific gravity; the gage pressures on the upper and lower surfaces

Solution:

From a free body analysis of the cube: $\Sigma F = 0 = T + (p_L - p_U) \cdot d^2 - M \cdot g$

where M and d are the cube mass and size and p_L and p_U are the pressures on the lower and upper surfaces

For each pressure we can use Eq. 3.7 $p = p_0 + \rho \cdot g \cdot h$

Hence $p_L - p_U = [p_0 + \rho \cdot g \cdot (H + d)] - (p_0 + \rho \cdot g \cdot H) = \rho \cdot g \cdot d = SG \cdot \rho_{H2O} \cdot d$

where H is the depth of the upper surface

Hence the force balance gives $SG = \dfrac{M \cdot g - T}{\rho_{H2O} \cdot g \cdot d^3}$

$$SG = \frac{2 \cdot slug \times 32.2 \cdot \frac{ft}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} - 50.7 \cdot lbf}{1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} \times (0.5 \cdot ft)^3} \qquad SG = 1.75$$

From Table A.1, the fluid is Meriam blue.

The individual pressures are computed from Eq 3.7

$$p = p_0 + \rho \cdot g \cdot h \quad \text{or} \quad p_g = \rho \cdot g \cdot h = SG \cdot \rho_{H2O} \cdot h$$

For the upper surface

$$p_g = 1.754 \times 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times \frac{2}{3} \cdot ft \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \left(\frac{1 \cdot ft}{12 \cdot in}\right)^2 \qquad p_g = 0.507\, psi$$

For the lower surface

$$p_g = 1.754 \times 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times \left(\frac{2}{3} + \frac{1}{2}\right) \cdot ft \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \left(\frac{1 \cdot ft}{12 \cdot in}\right)^2 \qquad p_g = 0.888\, psi$$

Note that the SG calculation can also be performed using a buoyancy approach (discussed later in the chapter):

Consider a free body diagram of the cube: $\Sigma F = 0 = T + F_B - M \cdot g$

where M is the cube mass and F_B is the buoyancy force $F_B = SG \cdot \rho_{H2O} \cdot L^3 \cdot g$

Hence $T + SG \cdot \rho_{H2O} \cdot L^3 \cdot g - M \cdot g = 0$ or $SG = \dfrac{M \cdot g - T}{\rho_{H2O} \cdot g \cdot L^3}$ as before $SG = 1.75$

Problem 3.11

[2]

3.11 An air bubble, 0.3 in. in diameter, is released from the regulator of a scuba diver swimming 100 ft below the sea surface. (The water temperature is 86°F.) Estimate the diameter of the bubble just before it reaches the water surface.

Given: Data on air bubble

Find: Bubble diameter as it reaches surface

Solution:

Basic equation $\frac{dp}{dy} = -\rho_{sea} \cdot g$ and the ideal gas equation $p = \rho \cdot R \cdot T = \frac{M}{V} \cdot R \cdot T$

We assume the temperature is constant, and the density of sea water is constant

For constant sea water density $p = p_{atm} + SG_{sea} \cdot \rho \cdot g \cdot h$ where p is the pressure at any depth h

Then the pressure at the initial depth is $p_1 = p_{atm} + SG_{sea} \cdot \rho \cdot g \cdot h_1$

The pressure as it reaches the surface is $p_2 = p_{atm}$

For the bubble $p = \frac{M \cdot R \cdot T}{V}$ but M and T are constant $M \cdot R \cdot T = const = p \cdot V$

Hence $p_1 \cdot V_1 = p_2 \cdot V_2$ or $V_2 = V_1 \cdot \frac{p_1}{p_2}$ or $D_2^{\ 3} = D_1^{\ 3} \cdot \frac{p_1}{p_2}$

Then the size of the bubble at the surface i $D_2 = D_1 \cdot \left(\frac{p_1}{p_2}\right)^{\frac{1}{3}} = D_1 \cdot \left[\frac{\left(p_{atm} + \rho_{sea} \cdot g \cdot h_1\right)}{p_{atm}}\right]^{\frac{1}{3}} = D_1 \cdot \left(1 + \frac{\rho_{sea} \cdot g \cdot h_1}{p_{atm}}\right)^{\frac{1}{3}}$

From Table A.2 $SG_{sea} = 1.025$ (This is at 68ºF)

$$D_2 = 0.3 \cdot in \times \left[1 + 1.025 \times 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \times \frac{ft}{s^2} \times 100 \cdot ft \times \frac{in^2}{14.7 \cdot lbf} \times \left(\frac{1 \cdot ft}{12 \cdot in}\right)^2 \times \frac{lbf \cdot s^2}{slug\,ft}\right]^{\frac{1}{3}}$$

$$D_2 = 0.477 \cdot in$$

Problem 3.12

Given: Model behavior of seawater by assuming constant bulk modulus

Find: (a) expression density as a function of depth, h.
(b) Show that result may be written as
$\rho \approx \rho_0 + bh$
(c) evaluate the constant b
(d) use results of (b) to obtain equation for $p(h)$
(e) determine percent error in predicted pressure at h=1000m

Solution: From Table A.2, App. A, $SG)_0 = 1.025$, $E_v = 2.42\ GN/m^2$

Basic equation: $\frac{dp}{dh} = \rho g$ Definition: $E_v = \frac{dp}{d\rho/\rho}$

Then, $dp = \rho g\, dh = E_v \frac{d\rho}{\rho}$ and $\frac{d\rho}{\rho^2} = \frac{g}{E_v} dh$

Integrating, $\int_{\rho_0}^{\rho} \frac{d\rho}{\rho^2} = \int_0^h \frac{g}{E_v} dh$ and $-\frac{1}{\rho}\Big]_{\rho_0}^{\rho} = \frac{gh}{E_v}$

then, $\frac{gh}{E_v} = -\frac{1}{\rho} + \frac{1}{\rho_0} = \frac{-\rho_0 + \rho}{\rho\rho_0}$ or $\rho - \rho_0 = \rho\rho_0 \frac{gh}{E_v}$

$\therefore \rho\left(1 - \rho_0 \frac{gh}{E_v}\right) = \rho_0$ and $\frac{\rho}{\rho_0} = \frac{1}{\left\{1 - \frac{\rho_0 gh}{E_v}\right\}}$ ← $\rho(h)$

For $\frac{\rho_0 g}{E_v} \ll 1$, $\frac{\rho}{\rho_0} \approx 1 + \frac{\rho_0 g}{E_v} h$

Thus,
$$\rho \approx \rho_0 + \frac{\rho_0^2 g}{E_v} h = \rho_0 + bh \quad \text{where } b = \frac{\rho_0^2 g}{E_v}$$ ← Q.E.D.

Since $dp = \rho g\, dh$, then an approximate expression for $p(h)$ is

$$p - p_{atm} = \int_{p_{atm}}^{p} dp = \int_0^h (\rho_0 + bh) g\, dh = \left(\rho_0 h + \frac{bh^2}{2}\right) g$$

$$p_{approx} = p_{atm} + \left(\rho_0 h + \frac{\rho_0^2 g h^2}{E_v}\right) g = p_{atm} + \rho_0 h g\left[1 + \frac{\rho_0 gh}{E_v}\right]$$ ← p_{approx}

The exact solution for $p(h)$ is obtained by utilizing the exact equation for $\rho(h)$. Thus.

$$p - p_{atm} = \int_{p_{atm}}^{p} dp = \int_{\rho_0}^{\rho} E_v \frac{d\rho}{\rho} = E_v \ln\frac{\rho}{\rho_0}$$

$$p = p_{atm} + E_v \ln\left\{1 - \frac{\rho_0 gh}{E_v}\right\}^{-1}$$ ← p_{exact}

$$\frac{\rho_0 gh}{E_v} = (1.025)1000\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 10^3 m \times \frac{m^2}{2.42\times10^9 N} \times \frac{N \cdot s^2}{kg \cdot m} = 4.16 \times 10^{-3}$$

Substituting numerical values, $p_{approx} = p_{atm} + 9.851$ MPa
$p_{exact} = p_{atm} + 10.076$ MPa

$$\text{error} = \frac{p_{exact} - p_{app}}{p_{exact}} = \frac{10.076 - 9.851}{10.076} = 0.0224 = 2.24\%$$ ← error

Problem 3.13

Given: Behavior of seawater to be modeled by assuming constant bulk modulus

Find: The percent deviations in (a) density, and (b) pressure, at depth $h = 10$ km, as compared to values obtained assuming constant density.

Plot: the results over range of $0 \le h \le 10$ km

Solution

Basic equation: $\frac{dp}{dh} = \rho g$ Definition: $E_v = \frac{dp}{d\rho/\rho}$

Then, $dp = \rho g\, dh = \frac{d\rho}{\rho} E_v$ and $\int_{\rho_0}^{\rho} \frac{d\rho}{\rho^2} = \int_0^h \frac{g\, dh}{E_v}$

We obtain

$$-\frac{1}{\rho}\Big]_{\rho_0}^{\rho} = -\frac{1}{\rho} + \frac{1}{\rho_0} = \frac{-\rho_0 + \rho}{\rho\rho_0} = \frac{gh}{E_v} \quad \text{or} \quad \rho - \rho_0 = \rho\rho_0 \frac{gh}{E_v}$$

Then

$$\rho\left(1 - \frac{\rho_0 gh}{E_v}\right) = \rho_0 \quad \text{and} \quad \frac{\rho}{\rho_0} = \frac{1}{\left(1 - \frac{\rho_0 gh}{E_v}\right)}$$

Finally,

$$\frac{\Delta\rho}{\rho_0} = \frac{\rho - \rho_0}{\rho_0} = \frac{\rho}{\rho_0} - 1 = \frac{\rho_0 gh/E_v}{(1 - \rho_0 gh/E_v)} \quad \text{---- (1)}$$

To determine an expression for the percent deviation in pressure we write $\int_{p_{atm}}^{p} dp = E_v \int_{\rho_0}^{\rho} \frac{d\rho}{\rho}$

Then $p - p_{atm} = E_v \ln \rho/\rho_0$

For ρ = constant, $\int_{p_{atm}}^{p} dp = \rho_0 g \int_0^h dh$ and $p_{\rho=c} - p_{atm} = \rho_0 gh$

Then

$$\frac{p - p_{\rho=c}}{p_{\rho=c}} = \frac{\Delta p}{p_{\rho=c}} = \frac{E_v \ln \rho/\rho_0 - \rho_0 gh}{\rho_0 gh} = \frac{E_v}{\rho_0 gh} \ln\frac{\rho}{\rho_0} - 1 \quad \text{--- (2)}$$

From Table A.2 for seawater SG = 1.025, $E_v = 2.42$ GN/m². Then

$$\frac{E_v}{\rho_0 g} = 2.42 \times 10^9 \frac{N}{m^2} \times \frac{1}{(1000)(1.025)} \frac{m^3}{kg} \times \frac{s^2}{9.81\, m} \times \frac{kg \cdot m}{N \cdot s^2} \times \frac{km}{10^3 m} = 240.7 \text{ km}$$

Substituting into eqs (1) and (2)

$$\frac{\Delta\rho}{\rho_0} = \frac{4.155 \times 10^{-3} h}{1 - 4.155 \times 10^{-3} h} \quad \text{.....(1a)}$$

$$\frac{\Delta p}{p_0} = \frac{240.7}{h} \ln\left[\frac{1}{1 - 4.155 \times 10^{-3} h}\right] - 1 \quad \text{.... (2a)}$$

At $h = 10$ km, $\frac{\Delta\rho}{\rho_0} = 0.0434$ or 4.34% $\underline{\quad\quad\quad} \frac{\Delta\rho}{\rho_0}\Big)_{h=10 km}$

$\frac{\Delta p}{p_o} = 0.0215$ or 2.15% ← $\left.\frac{\Delta p}{p_o}\right)_{h=10\,km}$

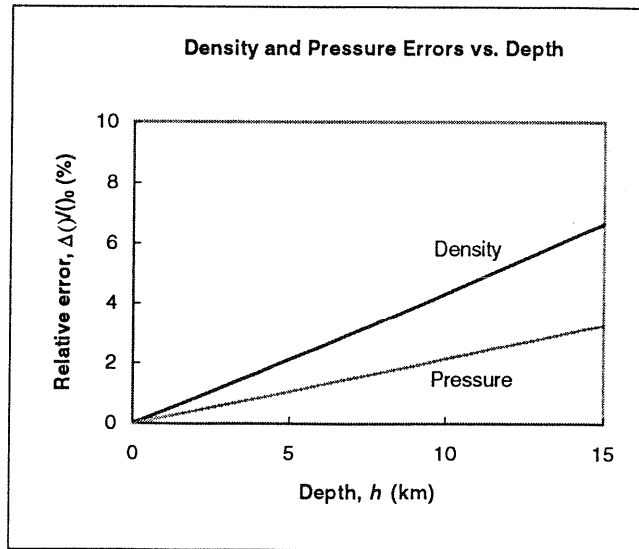
Both $\Delta\rho/\rho_o$ and $\Delta p/p_o$ are plotted as a function of depth h (in km) below.
The computing equations are

$$\Delta\rho/\rho_o = \frac{\rho_o g h/E_v}{(1-\rho_o g h/E_v)}$$

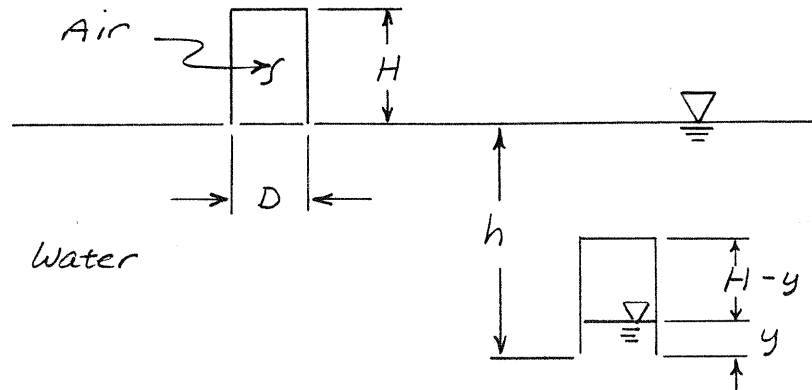
$$\Delta p/p_o = \frac{E_v}{\rho_o g h} \ln\frac{\rho}{\rho_o} - 1$$

Density and pressure variation of seawater:

E_v = 2.42 GN/m^2 Bulk modulus of seawater

Depth, h (km)	Density Error, $\Delta\rho/\rho_o$ (---)	Pressure Error, $\Delta p/p_o$ (---)
0	0	0
1	0.417	0.219
2	0.838	0.429
3	1.26	0.639
4	1.69	0.851
5	2.12	1.06
6	2.56	1.28
7	3.00	1.49
8	3.44	1.71
9	3.88	1.93
10	4.34	2.15
11	4.79	2.37
12	5.25	2.59
13	5.71	2.81
14	6.18	3.04
15	6.65	3.26

Problem 3.14

[3]

Given: Cylindrical cup lowered slowly beneath pool surface.

Find: Expression for y in terms of h and H. Plot: y/H vs. h/H.

Solution: Apply ideal gas and hydrostatic equations.

Basic equations: $p\forall = mRT \qquad \frac{dp}{dh} = \rho g$

Assumptions: (1) T = constant
(2) Static liquid
(3) Incompressible liquid

Using (1), $p\forall = p_a \frac{\pi D^2}{4} H = p \frac{\pi D^2}{4}(H-y)$; or $\quad p_a H = p(H-y)$

Integrating $\frac{dp}{dh} = \rho g$ gives $p - p_a = \rho g(h-y)$ in container.

Thus

$$p_a H = [p_a + \rho g(h-y)](H-y) = p_a H - p_a y + \rho g(h-y)(H-y)$$

Expanding,

$$0 = \rho g h H - \rho g h y - \rho g y H + \rho g y^2 - p_a y$$

or

$$0 = hH - \left[(h+H) + \frac{p_a}{\rho g}\right] y + y^2$$

Using the quadratic equation

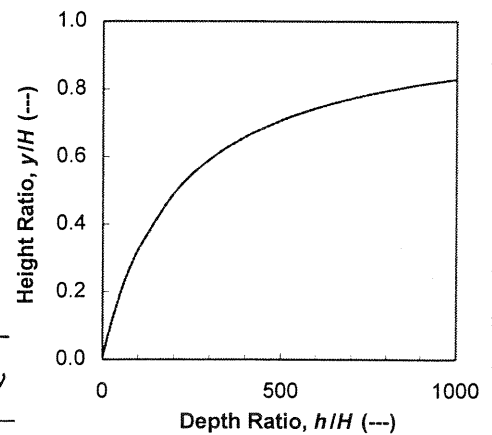
$$y = \frac{h + H + p_a/\rho g - \sqrt{\left[h + H + \frac{p_a}{\rho g}\right]^2 - 4hH}}{2}$$

Height Ratio, y/H (---)

1.0
0.8
0.6
0.4
0.2
0.0

0 500 1000

Depth Ratio, h/H (---)

y

(Note $y \le H$, so the minus sign must be used.) In terms of y/H, this becomes

$$\frac{y}{H} = \frac{\frac{h}{H} + 1 + \frac{p_a}{\rho g H} - \sqrt{\left[\frac{h}{H} + 1 + \frac{p_a}{\rho g H}\right]^2 - 4\frac{h}{H}}}{2}$$

y/H

(see plot above.)

Problem 3.15

[1]

3.15 You close the top of your straw using your thumb and lift it out of your glass containing Coke. Holding it vertically, the total length of the straw is 17 in., but the Coke held in the straw is in the bottom 6 in. What is the pressure in the straw just below your thumb? Ignore any surface tension effects.

Given: Geometry of straw

Find: Pressure just below the thumb

Solution:

Basic equation $\frac{dp}{dy} = -\rho \cdot g$ or, for constant ρ $\Delta p = \rho \cdot g \cdot h$ where h is measured downwards

This equation only applies in the 6 in of coke in the straw - in the other 11 inches of air the pressure is essentially constant.

The gage pressure at the coke surface is $p_{coke} = \rho \cdot g \cdot h_{coke}$ assuming coke is about as dense as water (it's actually a bit dens

Hence, with $h_{coke} = -6 \cdot in$ because h is measured downwards

$$p_{coke} = -1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times 6 \cdot in \times \frac{1 \cdot ft}{12 \cdot in} \times \frac{lbf \cdot s^2}{slug\, ft}$$

$$p_{coke} = -31.2 \cdot \frac{lbf}{ft^2} \qquad p_{coke} = -0.217 \cdot psi \qquad \text{gage}$$

$$p_{coke} = 14.5 \cdot psi$$

Problem 3.16

[2]

3.16 A water tank filled with water to a depth of 5 m has an inspection cover (2.5 cm × 2.5 cm square) at its base, held in place by a plastic bracket. The bracket can hold a load of 40 N. Is the bracket strong enough? If it is, what would the water depth have to be to cause the bracket to break?

Given: Data on water tank and inspection cover

Find: If the support bracket is strong enough; at what water depth would it fail

Solution:

Basic equation $\frac{dp}{dy} = -\rho \cdot g$ or, for constant ρ $\Delta p = \rho \cdot g \cdot h$ where h is measured downwards

The absolute pressure at the base is $p_{base} = p_{atm} + \rho \cdot g \cdot h$ where $h = 5 \cdot m$

The gage pressure at the base is $p_{base} = \rho \cdot g \cdot h$ This is the pressure to use as we have p_{atm} on the outside of the cover.

The force on the inspection cover is $F = p_{base} \cdot A$ where $A = 2.5 \cdot cm \times 2.5 \cdot cm$ $A = 6.25 \times 10^{-4} m^2$

$$F = \rho \cdot g \cdot h \cdot A$$

$$F = 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 5 \cdot m \times 6.25 \times 10^{-4} \cdot m^2 \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F = 30.7 N$$

The bracket is strong enough (it can take 40 N). To find the maximum depth we start wi $F = 40 \cdot N$

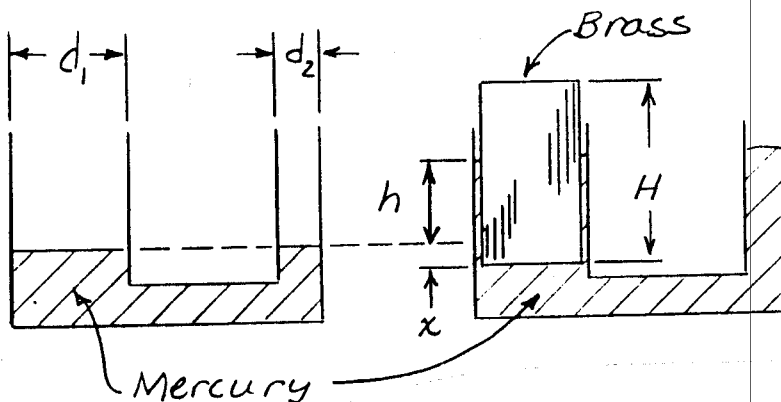
$$h = \frac{F}{\rho \cdot g \cdot A}$$

$$h = 40 \cdot N \times \frac{1}{1000} \cdot \frac{m^3}{kg} \times \frac{1}{9.81} \cdot \frac{s^2}{m} \times \frac{1}{6.25 \times 10^{-4}} \cdot \frac{1}{m^2} \times \frac{kg \cdot m}{N \cdot s^2}$$

$$h = 6.52 m$$

Problem 3.17

Given: Container of mercury with vertical tubes $d_1 = 39.5$ mm and $d_2 = 12.7$ mm.

Brass cylinder with $D = 37.5$ mm and $H = 76.2$ mm is introduced into larger tube, where it floats.

Find: (a) Pressure on bottom of cylinder.
(b) New equilibrium level, h, of mercury.

<u>Solution:</u> Analyze free-body diagram of cylinder, apply hydrostatics.

Computing equations: $\Sigma F_z = 0$; $\frac{dp}{dz} = -\rho g$; $\rho = SG\,\rho_{H_2O}$

Assumptions: (1) Static liquid
(2) Incompressible liquid

For the cylinder $\Sigma F_z = p\frac{\pi D^2}{4} - \rho_{brass}\, g \frac{\pi D^2}{4} H = 0$

Thus $\quad p = \rho_{brass}\, g H = SG_{brass}\, \rho_{H_2O}\, g H$

From Table A.1, $SG_{brass} = 8.55$ at $20°C$, so

$$p = 8.55 \times 1000 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 0.0762\, m \times \frac{N \cdot s^2}{kg \cdot m} = 6.39 \text{ kPa (gage)}$$

This pressure must be produced by a column of mercury $h+x$ in height. Thus, using SG_{Hg} from Table A.1,

$$p = \rho_{Hg}\, g(h+x) = SG_{Hg}\, \rho_{H_2O}\, g(h+x) = SG_{brass}\, \rho_{H_2O}\, g H$$

Thus $\quad h + x = \frac{SG_{brass}}{SG_{Hg}} H = \frac{8.55}{13.55} H = 0.631\, H \qquad (1)$

But the volume of mercury must remain constant. Therefore

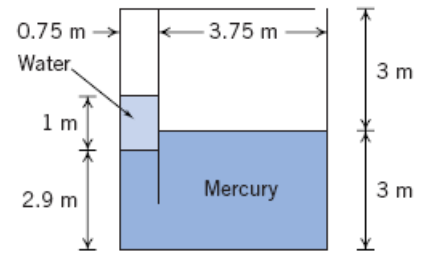
$$\frac{\pi D^2}{4} x = \frac{\pi (d_1^2 - D^2)}{4} h + \frac{\pi d_2^2}{4} h \quad \text{or} \quad x\left[\left(\frac{d_1}{D}\right)^2 - 1 + \left(\frac{d_2}{D}\right)^2\right] = 0.224\, h$$

Substituting into Eq. 1,

$$h + x = h + 0.224\, h = 1.224\, h = 0.631\, H \quad \text{or} \quad h = \frac{0.631}{1.224} H = 0.516\, H$$

h = 39.3 mm

Problem 3.18

3.18 A partitioned tank as shown contains water and mercury. What is the gage pressure in the air trapped in the left chamber? What pressure would the air on the left need to be pumped to in order to bring the water and mercury free surfaces level?

Given: Data on partitioned tank

Find: Gage pressure of trapped air; pressure to make water and mercury levels equal

Solution:

The pressure difference is obtained from repeated application of Eq. 3.7, or in other words, from Eq. 3.8. Starting from the right air chamber

$$p_{gage} = SG_{Hg} \times \rho_{H2O} \times g \times (3 \cdot m - 2.9 \cdot m) - \rho_{H2O} \times g \times 1 \cdot m$$

$$p_{gage} = \rho_{H2O} \times g \times \left(SG_{Hg} \times 0.1 \cdot m - 1.0 \cdot m\right)$$

$$p_{gage} = 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times (13.55 \times 0.1 \cdot m - 1.0 \cdot m) \times \frac{N \cdot s^2}{kg \cdot m} \qquad p_{gage} = 3.48 \cdot kPa$$

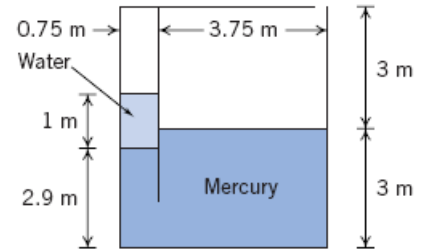
If the left air pressure is now increased until the water and mercury levels are now equal, Eq. 3.8 leads to

$$p_{gage} = SG_{Hg} \times \rho_{H2O} \times g \times 1.0 \cdot m - \rho_{H2O} \times g \times 1.0 \cdot m$$

$$p_{gage} = \rho_{H2O} \times g \times \left(SG_{Hg} \times 1 \cdot m - 1.0 \cdot m\right)$$

$$p_{gage} = 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times (13.55 \times 1 \cdot m - 1.0 \cdot m) \times \frac{N \cdot s^2}{kg \cdot m} \qquad p_{gage} = 123 \cdot kPa$$

Problem 3.19

3.19 In the tank of Problem 3.18, if the opening to atmosphere on the right chamber is first sealed, what pressure would the air on the left now need to be pumped to in order to bring the water and mercury free surfaces level? (Assume the air trapped in the right chamber behaves isothermally.)

Given: Data on partitioned tank

Find: Pressure of trapped air required to bring water and mercury levels equal if right air opening is sealed

Solution:

First we need to determine how far each free surface moves.

In the tank of Problem 3.15, the ratio of cross section areas of the partitions is 0.75/3.75 or 1:5. Suppose the water surface (and therefore the mercury on the left) must move down distance x to bring the water and mercury levels equal. Then by mercury volume conservation, the mercury free surface (on the right) moves up $(0.75/3.75)x = x/5$. These two changes in level must cancel the original discrepancy in free surface levels, of (1m + 2.9m) - 3 m = 0.9 m. Hence $x + x/5 =$ 0.9 m, or $x = 0.75$ m. The mercury level thus moves up $x/5 = 0.15$ m.

Assuming the air (an ideal gas, $pV=RT$) in the right behaves isothermally, the new pressure there will be

$$p_{right} = \frac{V_{rightold}}{V_{rightnew}} \cdot p_{atm} = \frac{A_{right} \cdot L_{rightold}}{A_{right} \cdot L_{rightnew}} \cdot p_{atm} = \frac{L_{rightold}}{L_{rightnew}} \cdot p_{atm}$$

where V, A and L represent volume, cross-section area, and vertical length

Hence

$$p_{right} = \frac{3}{3 - 0.15} \times 101 \cdot kPa \qquad p_{right} = 106\,kPa$$

When the water and mercury levels are equal application of Eq. 3.8 gives:

$$p_{left} = p_{right} + SG_{Hg} \times \rho_{H2O} \times g \times 1.0 \cdot m - \rho_{H2O} \times g \times 1.0 \cdot m$$

$$p_{left} = p_{right} + \rho_{H2O} \times g \times \left(SG_{Hg} \times 1.0 \cdot m - 1.0 \cdot m\right)$$

$$p_{left} = 106 \cdot kPa + 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times (13.55 \cdot 1.0 \cdot m - 1.0 \cdot m) \times \frac{N \cdot s^2}{kg \cdot m} \qquad p_{left} = 229\,kPa$$

$$p_{gage} = p_{left} - p_{atm} \qquad p_{gage} = 229 \cdot kPa - 101 \cdot kPa \qquad p_{gage} = 128\,kPa$$

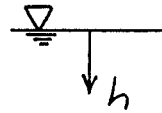
Problem 3.20

[2]

Given: U-tube manometer, partially filled with water, then $\forall_{oil} = 3.25\ cm^3$ of Meriam red oil is added to the left side.

Find: Equilibrium height, H, when both legs are open to atmosphere.

Solution: Apply basic pressure-height relation.

Basic equation: $\frac{dp}{dh} = +\rho g$

Assumptions: (1) Incompressible liquid
(2) h measured down

Integration gives

$$p_2 - p_1 = \rho g (h_2 - h_1)$$

Thus

$$p_B = p_A + \rho_{oil}\, g\, L$$

$$p_D = p_C + \rho_{water}\, g (L - H)$$

Since $p_A = p_C = p_{atm}$, then

$$\rho_{oil}\, g\, L = \rho_{water}\, g (L - H)$$

or

$$SG_{oil}\, L = L - H$$

Thus

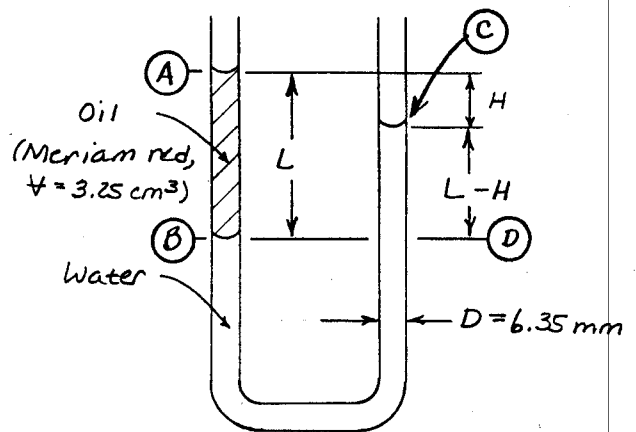
$$H = L (1 - SG_{oil})$$

From the volume of oil, $\forall = \frac{\pi D^2}{4} L$, so

$$L = \frac{4\forall}{\pi D^2} = \frac{4}{\pi} \times 3.25\ cm^3 \times \frac{1}{(6.35)^2 mm^2} \times (10)^3 \frac{mm^3}{cm^3} = 103\ mm$$

Finally, since $SG = 0.827$ (Table A.1, Appendix A), then

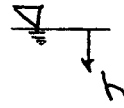
$$H = 103\ mm\ (1 - 0.827) = 17.8\ mm$$

H

Problem 3.21

Given: Two-fluid manometer shown.

Find: Pressure difference, $p_1 - p_2$

Solution:

Basic equation: $\frac{dp}{dh} = \rho g$

Assumptions: (1) static liquid
(2) incompressible
(3) g = constant

Then, $dp = \rho g \, dh$ and $\Delta p = \rho g \Delta h$

Starting at point ① and progressing to point ② we have

$$p_1 + \rho_{H_2O} g (d + l) - \rho_{c.t.} g l - \rho_{H_2O} g d = p_2$$

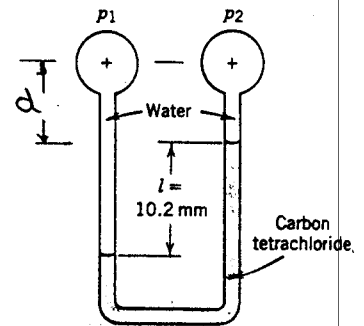
$$\therefore p_1 - p_2 = \rho_{ct} g l - \rho_{H_2O} g l = SG_{ct} \rho_{H_2O} g l - \rho_{H_2O} g l$$

$$p_1 - p_2 = \rho_{H_2O} g l (SG_{ct} - 1)$$

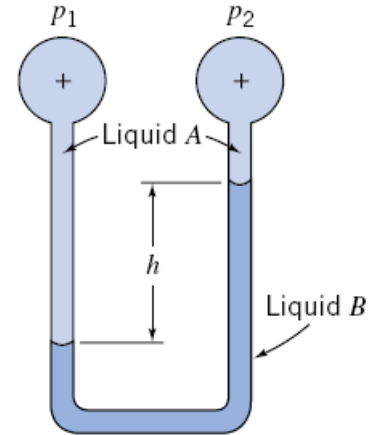
From Table A.2, Appendix A, $SG_{ct} = 1.595$

$$\therefore p_1 - p_2 = 1000 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 10.2 \, mm \times \frac{m}{10^3 \, mm} (1.595 - 1) \frac{N \cdot s^2}{kg \cdot m}$$

$$p_1 - p_2 = 59.5 \, N/m^2$$

$p_1 - p_2$

Problem 3.22

[2]

3.22 The manometer shown contains two liquids. Liquid A has SG = 0.88 and liquid B has SG = 2.95. Calculate the deflection, h, when the applied pressure difference is $p_1 - p_2 = 18 \text{ lbf/ft}^2$.

Given: Data on manometer

Find: Deflection due to pressure difference

Solution:

Basic equation $\quad \dfrac{dp}{dy} = -\rho \cdot g \quad$ or, for constant ρ $\quad \Delta p = \rho \cdot g \cdot \Delta h \quad$ where h is measured downwards

Starting at p_1 $\quad p_A = p_1 + SG_A \cdot \rho \cdot g \cdot (h + l) \quad$ where l is the (unknown) distance from the level of the right interface

Next, from A to B $\quad p_B = p_A - SG_B \cdot \rho \cdot g \cdot h$

Finally, from A to the location of p_2 $\quad p_2 = p_B - SG_A \cdot \rho \cdot g \cdot l$

Combining the three equations $\quad p_2 = \left(p_A - SG_B \cdot \rho \cdot g \cdot h\right) - SG_A \cdot \rho \cdot g \cdot l = \left[p_1 + SG_A \cdot \rho \cdot g \cdot (h + l) - SG_B \cdot \rho \cdot g \cdot h\right] - SG_A \cdot \rho \cdot g \cdot l$

$$p_2 - p_1 = \left(SG_A - SG_B\right) \cdot \rho \cdot g \cdot h$$

$$h = \frac{p_1 - p_2}{\left(SG_B - SG_A\right) \cdot \rho \cdot g}$$

$$h = 18 \cdot \frac{\text{lbf}}{\text{ft}^2} \times \frac{1}{(2.95 - 0.88)} \times \frac{1}{1.94} \cdot \frac{\text{ft}^3}{\text{slug}} \times \frac{1}{32.2} \cdot \frac{\text{s}^2}{\text{ft}} \times \frac{\text{slug} \cdot \text{ft}}{\text{s}^2 \cdot \text{lbf}}$$

$h = 0.139 \cdot \text{ft} \qquad h = 1.67 \cdot \text{in}$

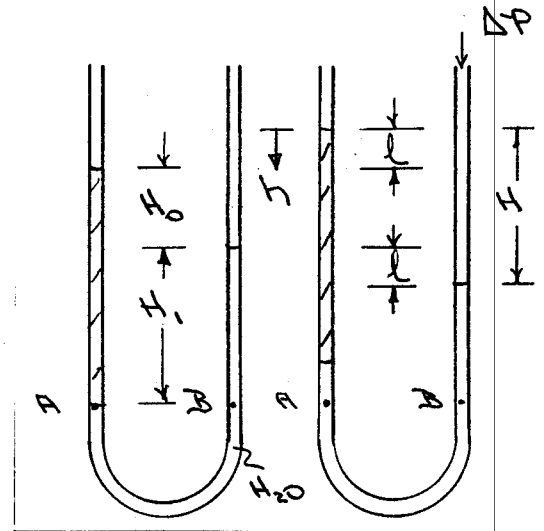
Problem 3.23

Given: Two fluid manometer contains water and kerosene. With both tubes open to atmosphere, the free surface elevations differ by $H_o = 20.0$ mm

Find: Elevation difference, H, between free-surface of fluids when a gage pressure of 98.0 Pa is applied to the right tube.

Solution:

Basic equation: $\frac{dp}{dh} = \rho g$; $\Delta p = \rho g \Delta h$

Assumptions: (1) static fluid
(2) gravity is the only body force

When the gage pressure $\Delta p = 98.0$ Pa is applied to the right tube, the water in the right tube is displaced downward a distance, l; the kerosene in the left tube is displaced upward the same distance, l.

Under the applied gage pressure, Δp, the elevation difference, H, is

$$H = H_o + 2l$$

Since points A & B are at the same elevation in the same fluid $p_A = p_B$.

Initially (left diagram), $p_A = \rho_k g(H_o + H_1)$, $p_B = \rho g H_1$ and hence

$$\rho_k g(H_o + H_1) = \rho g H_1$$

or

$$H_1 = \frac{\rho_k H_o}{\rho - \rho_k} = \frac{SG_k H_o}{(1 - SG_k)}$$. From table A.2, $SG_k = 0.82$

$$\therefore H_1 = \frac{0.82}{(1-0.82)} 20\text{mm} = 91.1 \text{ mm}$$

Under the applied pressure Δp (right diagram),

$$p_A = \rho_k g(H_o + H_1) + \rho g l \quad , \quad p_B = \Delta p + \rho g(H_1 - l).$$

$$\therefore SG_k(H_o + H_1) + l = \frac{\Delta p}{\rho g} + (H_1 - l)$$

Solving for l.

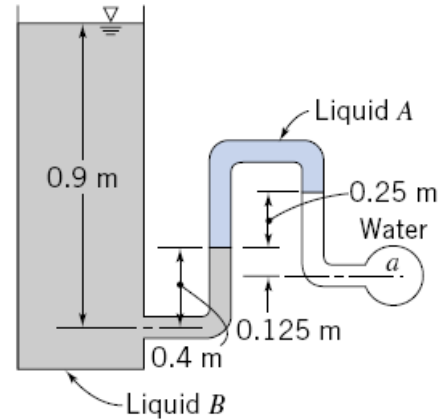
$$l = \frac{1}{2}\left[H_1 + \frac{\Delta p}{\rho g} - SG_k(H_o + H_1)\right]$$

$$= \frac{1}{2}\left[91.1\text{ mm} + \frac{98 N}{m^2} \times \frac{m^3}{999 kg} \times \frac{s^2}{9.81 m} \times \frac{kg \cdot m}{N \cdot s^2} \times \frac{10^3 mm}{m} - 0.82(20 + 91.1)\text{mm}\right]$$

$$l = 5 \text{ mm}$$

$$H = H_o + 2l = 30 \text{ mm}$$

Problem 3.24

3.24 Determine the gage pressure in psig at point *a*, if liquid *A* has SG = 0.75 and liquid *B* has SG = 1.20. The liquid surrounding point *a* is water and the tank on the left is open to the atmosphere.

Given: Data on manometer

Find: Gage pressure at point a

Solution:

Basic equation $\frac{dp}{dy} = -\rho \cdot g$ or, for constant ρ $\Delta p = \rho \cdot g \cdot \Delta h$ where Δh is height difference

Starting at point a $p_1 = p_a - \rho \cdot g \cdot h_1$ where $h_1 = 0.125 \cdot m + 0.25 \cdot m$ $h_1 = 0.375\,m$

Next, in liquid A $p_2 = p_1 + SG_A \cdot \rho \cdot g \cdot h_2$ where $h_2 = 0.25 \cdot m$

Finally, in liquid B $p_{atm} = p_2 - SG_B \cdot \rho \cdot g \cdot h_3$ where $h_3 = 0.9 \cdot m - 0.4 \cdot m$ $h_3 = 0.5\,m$

Combining the three equations $p_{atm} = \left(p_1 + SG_A \cdot \rho \cdot g \cdot h_2\right) - SG_B \cdot \rho \cdot g \cdot h_3 = p_a - \rho \cdot g \cdot h_1 + SG_A \cdot \rho \cdot g \cdot h_2 - SG_B \cdot \rho \cdot g \cdot h_3$

$$p_a = p_{atm} + \rho \cdot g \cdot \left(h_1 - SG_A \cdot h_2 + SG_B \cdot h_3\right)$$

or in gage pressures $p_a = \rho \cdot g \cdot \left(h_1 - SG_A \cdot h_2 + SG_B \cdot h_3\right)$

$$p_a = 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times [0.375 - (0.75 \times 0.25) + (1.20 \times 0.5)] \cdot m \times \frac{N \cdot s^2}{kg \cdot m}$$

$p_a = 7.73 \times 10^3\,Pa$ $p_a = 7.73 \cdot kPa$ (gage)

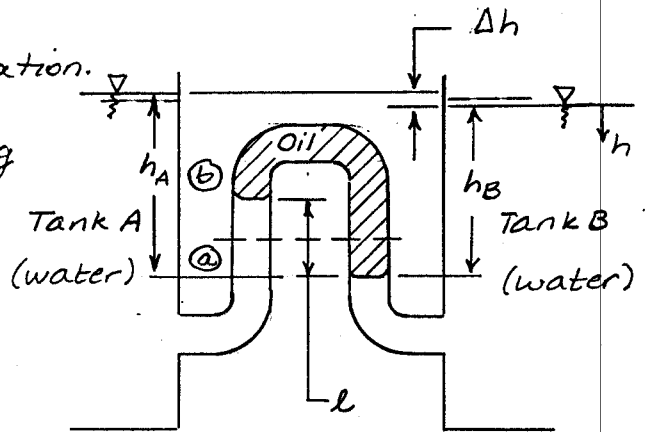
Problem 3.25

Given: Two-fluid manometer; oil is second fluid.

Find: SG needed for 10 to 1 amplification.

Solution: Basic equation $\frac{dp}{dz} = -\rho g$

Assumptions: (1) Static liquid
(2) Incompressible

Tank A (water) — Tank B (water); Δh, h_A, h_B, h, ℓ, (a), (b), Oil

Then $dp = \rho g dh$

$$p = p_0 + \rho g h$$

For left leg, $p_a = p_{atm} + \rho_{H_2O} g h_A$

$$p_b = p_a - \rho_{H_2O} g \ell = p_{atm} + \rho_{H_2O} g (h_A - \ell) \quad (1)$$

For right leg, $p_a = p_{atm} + \rho_{H_2O} g h_B$

$$p_b = p_a - SG_{oil} \rho_{H_2O} g \ell = p_{atm} + \rho_{H_2O} g (h_B - SG_{oil} \ell) \quad (2)$$

Combining,

$$\cancel{p_{atm}} + \rho_{H_2O} g (h_A - \ell) = \cancel{p_{atm}} + \rho_{H_2O} g (h_B - SG_{oil} \ell)$$

or

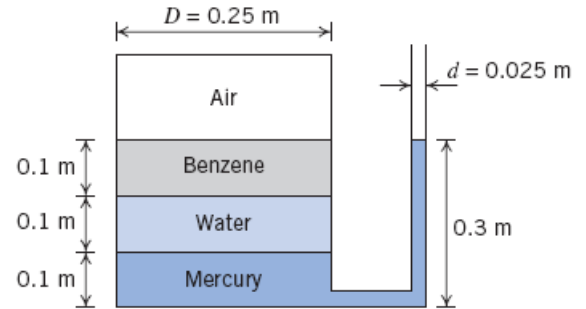
$$h_A - \ell = h_B - SG_{oil} \ell \; ; \; h_A - h_B = \Delta h = \ell (1 - SG_{oil})$$

Finally

$$SG_{oil} = 1 - \frac{\Delta h}{\ell} = 1 - \frac{1}{10} = 0.900$$

SG_{oil}

Problem 3.26

3.26 Consider a tank containing mercury, water, benzene, and air as shown. Find the air pressure (gage). If an opening is made in the top of the tank, find the equilibrium level of the mercury in the manometer.

Given:

Data on fluid levels in a tank

Find:

Air pressure; new equilibrium level if opening appears

Solution:

Using Eq. 3.8, starting from the open side and working in gage pressure

$$p_{air} = \rho_{H2O} \times g \times \left[SG_{Hg} \times (0.3 - 0.1) \cdot m - 0.1 \cdot m - SG_{Benzene} \times 0.1 \cdot m\right]$$

Using data from Table A.2

$$p_{air} = 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times (13.55 \times 0.2 \cdot m - 0.1 \cdot m - 0.879 \times 0.1 \cdot m) \times \frac{N \cdot s^2}{kg \cdot m} \qquad p_{air} = 24.7 \cdot kPa$$

To compute the new level of mercury in the manometer, assume the change in level from 0.3 m is an increase of x. Then, because the volume of mercury is constant, the tank mercury level will fall by distance $(0.025/0.25)^2 x$. Hence, the gage pressure at the bottom of the tank can be computed from the left and the right, providing a formula for x

$$SG_{Hg} \times \rho_{H2O} \times g \times (0.3 \cdot m + x) = SG_{Hg} \times \rho_{H2O} \times g \times \left[0.1 \cdot m - x \cdot \left(\frac{0.025}{0.25}\right)^2\right] \cdot m \ldots$$
$$+ \rho_{H2O} \times g \times 0.1 \cdot m + SG_{Benzene} \times \rho_{H2O} \times g \times 0.1 \cdot m$$

Hence

$$x = \frac{[0.1 \cdot m + 0.879 \times 0.1 \cdot m + 13.55 \times (0.1 - 0.3) \cdot m]}{\left[1 + \left(\frac{0.025}{0.25}\right)^2\right] \times 13.55} \qquad x = -0.184 \, m$$

(The negative sign indicates the manometer level actually fell)

The new manometer height is $h = 0.3 \cdot m + x$ $\qquad h = 0.116 \, m$

Problem 3.27

Given: Water flow in an inclined pipe as shown. Pressure difference, $p_A - p_B$, measured with two-fluid manometer $L = 5\,ft$, $h = 6\,in.$

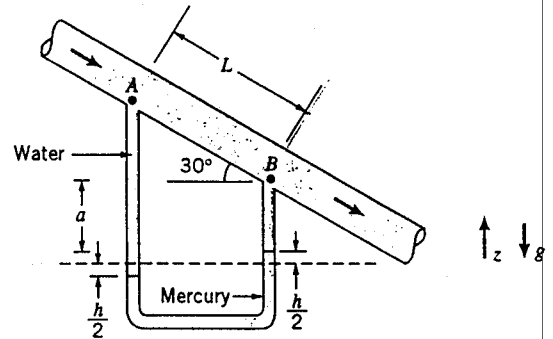
Water

30°

Mercury

L, A, B, a, $\frac{h}{2}$, $\frac{h}{2}$, z, g

Find: Pressure difference, $p_A - p_B$.

Solution:

Basic equation: $\frac{dp}{dh} = \rho g$ where h is measured positive down

Assumptions: (1) static liquid
(2) incompressible
(3) g = constant

Then,

$$dp = \rho g\, dh \quad \text{and} \quad \Delta p = \rho g \Delta h$$

Start at p_A and progress through manometer to p_B

$$p_A + \rho_{H_2O} g L \sin 30° + \rho_{H_2O} g a + \rho_{H_2O} g h - \rho_{Hg} g h - \rho_{H_2O} g a = p_B$$

$$p_A - p_B = \rho_{Hg} g h - \rho_{H_2O} g h - \rho_{H_2O} g L \sin 30°$$

$$= SG_{Hg} \rho_{H_2O} g h - \rho_{H_2O} g h - \rho_{H_2O} g L \sin 30°$$

$$p_A - p_B = \rho_{H_2O} g \left[h (SG_{Hg} - 1) - L \sin 30° \right]$$

From Table A.2, $SG_{Hg} = 13.55$

Then,

$$p_A - p_B = 1.94 \frac{slug}{ft^3} \times 32.2 \frac{ft}{s^2} \left[0.5\,ft (13.55 - 1) - 5\,ft \sin 30° \right] \cdot \frac{lbf \cdot s^2}{ft \cdot slug}$$

$$p_A - p_B = 236\ lbf/ft^2 \quad (1.64\ psi)$$

$p_A - p_B$

Problem 3.28

Given: A U-tube manometer is connected to the open tank filled with water as shown (manometer fluid is merian blue)

$D_1 = 2.5\,m$, $D_2 = 0.7\,m$, $d = 0.2\,m$

Find: The manometer deflection, l.

Solution:

Basic equation: $\frac{dP}{dh} = \rho g$

For γ = constant $\Delta P = \rho g \Delta h$

Then, beginning at the free surface and accounting for the changes in pressure with elevation,

$$P_{atm} + (P_1 - P_{atm}) + (P_2 - P_1) = P_2 = P_{atm}$$

$$\rho_{H_2O}\, g\left[(D_1 - D_2) + d + \frac{l}{2}\right] - \rho_{mb}\, g\, l = 0$$

$$(D_1 - D_2) + d + \frac{l}{2} = \frac{\rho_{mb}}{\rho_{H_2O}}\, l = (S.G)_{mb}\, l$$

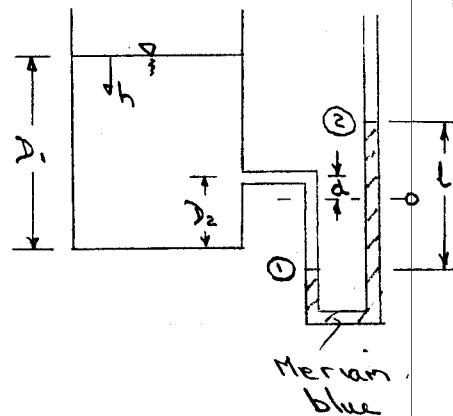
and

$$l = \frac{(D_1 - D_2) + d}{\left[(S.G)_{mb} - \frac{1}{2}\right]}$$

(From Table A.1, Appendix A, SG = 1.75.)

$$l = \frac{(2.5 - 0.7)\,m + 0.2\,m}{(1.75 - 0.5)}$$

$$l = 1.6\,m$$

l

Problem 3.29

[2]

Given: Reservoir manometer with vertical tubes $D = 18$ mm and $d = 6$ mm diameter. Gage liquid is Meriam red oil.

Find: (a) Algebraic expression for deflection L in small tube when gage pressure Δp is applied to the reservoir.
(b) Evaluate L when Δp is equivalent to 25 mm H_2O (gage).

Solution: Use the diagram of Example Problem 3.2, apply hydrostatics.

Computing equations: $\frac{dp}{dh} = +\rho g$; $\Delta p = \rho g \Delta h$; $\rho = SG\,\rho_{H_2O}$

Assumptions: (1) Static liquid
(2) Incompressible liquid

Δp, d, ②, D, L, x, Equilibrium liquid level, ①, z, Meriam red oil

Then $\Delta p = \rho_{oil}\, g\, (x + L)$

From conservation of volume,

$$\frac{\pi D^2}{4} x = \frac{\pi d^2}{4} L \;;\; x = \left(\frac{d}{D}\right)^2 L$$

so

$$\Delta p = \rho_{water}\, g\, \Delta h = \rho_{oil}\, g \left[\left(\frac{d}{D}\right)^2 L + L\right] = \rho_{oil}\, g L \left[1 + \left(\frac{d}{D}\right)^2\right]$$

Solving for L,

$$L = \frac{\Delta p}{\rho_{oil}\, g\left[1 + (d/D)^2\right]}$$

L

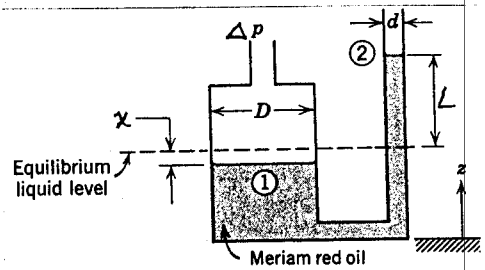
Substituting $\Delta p = \rho_{water}\, g\, \Delta h$,

$$L = \frac{\rho_{water}\, g\, \Delta h}{SG_{oil}\,\rho_{water}\, g\left[1 + (d/D)^2\right]} = \frac{\Delta h}{SG_{oil}\left[1 + (d/D)^2\right]}$$

Evaluating, with $SG_{oil} = 0.827$ (Table A.1),

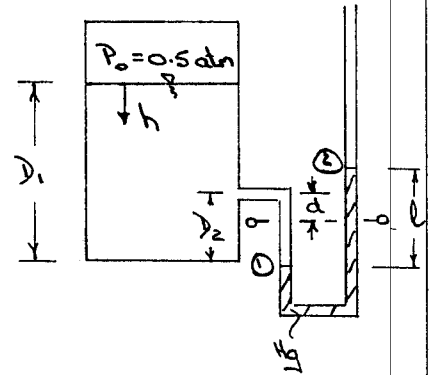
$$L = \frac{25.0 \text{ mm}}{0.827\left[1 + (6/18)^2\right]} = 27.2 \text{ mm}$$

L

{Note: $s \equiv \frac{L}{\Delta h_e} = \frac{27.2 \text{ mm}}{25.0 \text{ mm}} = 1.09$ for this manometer.}

Problem 3.30

Given: A U-tube manometer is connected to a closed tank filled with water as shown. The manometer fluid is Hg.

$D_1 = 2.5\,m$, $D_2 = 0.7\,m$, $d = 0.2\,m$

At the water surface $P_0 = 0.5$ atm (gage)

$P_0 = 0.5$ atm, h, D_1, D_2, d, l, (1), (2), Hg

Find: The manometer deflection l.

Solution

Basic equation $\frac{dP}{dh} = \rho g$

For γ = constant $\Delta P = \rho g \Delta h$

Then, beginning at the free surface and accounting for pressure changes with elevation,

$$P_0 + (P_1 - P_0) + (P_2 - P_1) = P_2 = P_{atm}$$

$$P_0 + \rho_{H_2O} g\left[(D_1 - D_2) + d + \frac{l}{2}\right] - \rho_{Hg} g l = P_{atm}$$

$$\frac{P_0 - P_{atm}}{\rho_{H_2O} g} + (D_1 - D_2) + d + \frac{l}{2} = \frac{\rho_{Hg} g}{\rho_{H_2O} g} l = (S.G)_{Hg} l$$

and

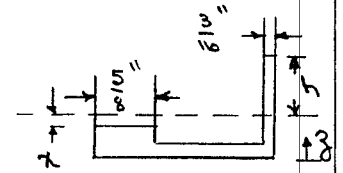
$$l = \frac{(P_0 - P_{atm})/\rho_{H_2O} g + (D_1 - D_2) + d}{(S.G)_{Hg} - 0.5}$$

$$= \frac{0.5\,atm \times 1.01 \times 10^5 \frac{N}{m^2 \cdot atm} \times \frac{m^3}{999\,kg} \times \frac{s^2}{9.81\,m} \times \frac{kg \cdot m}{N \cdot s^2} + (2.5 - 0.7)\,m + 0.2\,m}{13.6 - 0.5}$$

$$l = 0.546\,m \longleftarrow \quad l$$

Problem 3.31

Given: Reservoir manometer with dimensions shown
Manometer fluid SG = 0.827

Find: required distance between marks on vertical scale for 1 in of water ΔP

Solution:

Basic equation: $\frac{dP}{dz} = -\gamma$

Assumptions: (1) static fluid
(2) gravity is only body force
(3) z axis directed vertically

$dP = -\gamma\, dz$

For constant γ, $\Delta P = P_1 - P_2 = -\gamma(z_1 - z_2)$

Under applied pressure $\Delta P = \gamma_{oil}(x+h)$

But conditions of problem require $\Delta P = \gamma_{H_2O}\, l$ where $l = 1$ in

$\therefore\ \gamma_{oil}(x+h) = \gamma_{H_2O}\, l$

Since the volume of the oil must remain constant

$x\, A_{res} = h\, A_{tube}$

$\therefore\ x = h \frac{A_{tube}}{A_{res}}$

and $\gamma_{oil}\left(h \frac{A_t}{A_r} + h\right) = \gamma_{H_2O}\, l$

$$\therefore\ \frac{h}{l} = \frac{\gamma_{H_2O}}{\gamma_{oil}}\left(\frac{1}{\frac{A_t}{A_r}+1}\right) = \frac{1}{SG_{oil}\left[\left(\frac{D_t}{D_r}\right)^2+1\right]}$$

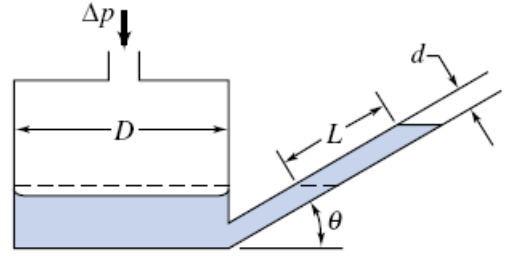
$$\frac{h}{l} = \frac{1}{0.827\left[\left(\frac{3}{16}\times\frac{8}{5}\right)^2+1\right]} = \frac{1}{0.827\left[(0.3)^2+1\right]}$$

$$\frac{h}{l} = 1.11$$

For $l = 1.0$ in as given, then $h = 1.11$ in.

Problem 3.32

[3]

3.32 The inclined-tube manometer shown has $D = 76$ mm and $d = 8$ mm, and is filled with Meriam red oil. Compute the angle, θ, that will give a 15-cm oil deflection along the inclined tube for an applied pressure of 25 mm of water (gage). Determine the sensitivity of this manometer.

Given:
Data on inclined manometer

Find:
Angle θ for given data; find sensitivity

Solution:

Basic equation $\frac{dp}{dy} = -\rho \cdot g$ or, for constant ρ $\Delta p = \rho \cdot g \cdot \Delta h$ where Δh is height difference

Under applied pressure $\Delta p = SG_{Mer} \cdot \rho \cdot g \cdot (L \cdot \sin(\theta) + x)$ (1)

From Table A.1 $SG_{Mer} = 0.827$

and Δp = 1 in. of water, or $\Delta p = \rho \cdot g \cdot h$ where $h = 25 \cdot mm$ $h = 0.025\,m$

$$\Delta p = 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 0.025 \cdot m \times \frac{N \cdot s^2}{kg \cdot m} \qquad \Delta p = 245\,Pa$$

The volume of liquid must remain constant, so $x \cdot A_{res} = L \cdot A_{tube}$ $x = L \cdot \frac{A_{tube}}{A_{res}} = L \cdot \left(\frac{d}{D}\right)^2$ (2)

Combining Eqs 1 and 2 $\Delta p = SG_{Mer} \cdot \rho \cdot g \cdot \left[L \cdot \sin(\theta) + L \cdot \left(\frac{d}{D}\right)^2\right]$

Solving for θ $\sin(\theta) = \frac{\Delta p}{SG_{Mer} \cdot \rho \cdot g \cdot L} - \left(\frac{d}{D}\right)^2$

$$\sin(\theta) = 245 \cdot \frac{N}{m^2} \times \frac{1}{0.827} \times \frac{1}{1000} \cdot \frac{m^3}{kg} \times \frac{1}{9.81} \cdot \frac{s^2}{m} \times \frac{1}{0.15} \cdot \frac{1}{m} \times \frac{kg \cdot m}{s^2 \cdot N} - \left(\frac{8}{76}\right)^2 = 0.186$$

$\theta = 11 \cdot deg$

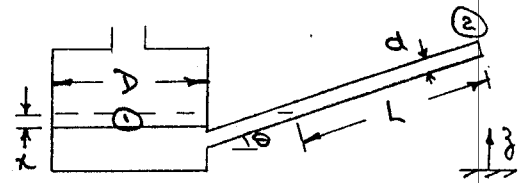
The sensitivity is the ratio of manometer deflection to a vertical water manometer

$$s = \frac{L}{h} = \frac{0.15 \cdot m}{0.025 \cdot m} \qquad s = 6$$

Problem 3.33

Given: Inclined manometer as shown

$D = 96\ mm$, $d = 8\ mm$

Angle θ is such that liquid deflection is five times that of U-tube manometer under same applied pressure difference

Find: angle, θ and manometer sensitivity

Solution:

Basic equation $\frac{dP}{dz} = -\rho g$

Then $dP = -\rho g\, dz$ and for constant ρ

$$\Delta P = P_1 - P_2 = -\rho g (z_1 - z_2)$$

For the inclined manometer,

$$P_1 - P_{atm} = \rho g (L \sin\theta + x)$$

Since the volume of the oil must remain constant,

$$x A_{res} = L A_{tube}$$

$$x = L \frac{A_{tube}}{A_{res}} = L\left(\frac{d}{D}\right)^2$$

Then

$$P_1 - P_{atm} = \rho g (L\sin\theta + x) = \rho g \left(L\sin\theta + L\left(\frac{d}{D}\right)^2\right) = \rho g L \left(\sin\theta + \left(\frac{d}{D}\right)^2\right)$$

For a U-tube manometer

$$P_1 - P_{atm} = -\rho g (z_1 - z_2) = \rho g h$$

Hence,

$$\frac{(P_1 - P_{atm})_{incl}}{(P_1 - P_{atm})_{U\text{-}tube}} = \frac{\rho g L\left[\sin\theta + \left(\frac{d}{D}\right)^2\right]}{\rho g h}$$

For same applied pressure and $L/h = 5$

$$1 = 5\left[\sin\theta + \left(\frac{d}{D}\right)^2\right]$$

$$\theta = \sin^{-1}\left[0.2 - \left(\frac{d}{D}\right)^2\right] = \sin^{-1}\left[0.2 - \left(\frac{8}{96}\right)^2\right] = 11.1° \quad \theta$$

$$s = L/\Delta h_e = L/(SG\ h) = 5/SG$$

Problem 3.34

Given: U-tube manometer with tubes of different diameter and two liquids, as shown.

Find: (a) the deflection, h, for $\Delta p = 250\ N/m^2$.
(b) the sensitivity of the manometer.

Plot: the manometer sensitivity as a function of d_2/d_1

Solution:

Basic equation: $\frac{dp}{dz} = -\rho g$

Assumptions: (1) static liquid (2) incompressible

Integrating the basic equation from reference state at z_0 to general state at z gives

$$p - p_0 = -\rho g(z - z_0) = \rho g(z_0 - z)$$

From the left diagram: $p_A - p_{atm} = \rho_w g l_1 = \rho_o g l_2$ ......(1)

From the right diagram $p_B - (p_{atm} + \Delta p) = \rho_w g l_3$ ......(2)

$p_B - p_{atm} = \rho_w g l_4 + \rho_o g l_2$ ......(3)

Subtracting Eq. 2 from Eq. 3 and then employing Eq. 1 gives

$$\Delta p = \rho_w g(l_4 - l_3) + \rho_o g l_2 = \rho_w g(l_4 + l_1 - l_3)$$

Define $l_w = l_1 - l_3$. Note $l_4 = h$. Then $\Delta p = \rho_w g(h + l_w)$ ......(4)

We can relate l_w to h by recognizing the volume of water must be conserved

$$\therefore \pi \frac{d_1^2}{4} l_w = \pi \frac{d_2^2}{4} h \quad \text{and} \quad l_w = h\left(\frac{d_2}{d_1}\right)^2$$

Substituting into Eq. 4 gives

$$\Delta p = \rho_w g\left[h + h\left(\frac{d_2}{d_1}\right)^2\right] = \rho_w g h\left[1 + \left(\frac{d_2}{d_1}\right)^2\right]$$

Solving for h,

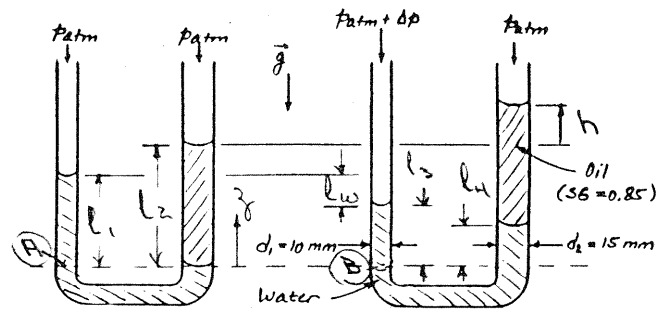
$$h = \frac{\Delta p}{\rho_w g[1 + (d_2/d_1)^2]} = 250\frac{N}{m^2} \times \frac{m^3}{999\,kg} \times \frac{s^2}{9.81\,m} \times \frac{1}{[1 + (15/10)^2]} \times \frac{kg \cdot m}{N \cdot s^2} \times 10^3 \frac{mm}{m}$$

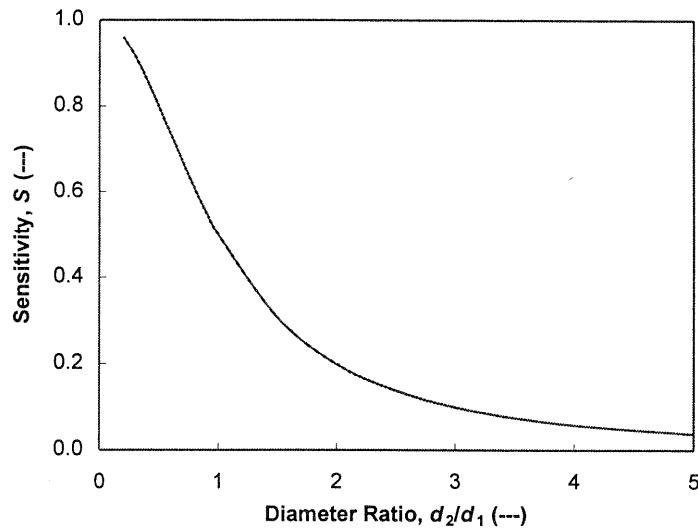
$h = 7.85$ mm $\longleftarrow$ h

(b) The sensitivity of the manometer is defined as

$$s = \frac{h}{\Delta h_e} = \frac{\text{actual deflection}}{\text{equivalent } \Delta h_{H_2O}} \quad \text{where } \Delta p = \rho_{H_2O} g \Delta h_e$$

$$\therefore s = \frac{h}{\Delta h_e} = \frac{1}{[1 + (d_2/d_1)^2]} = \frac{1}{[1 + (1.5)^2]} = 0.308$$ $\longleftarrow$ s

The design is a poor one. The sensitivity could be improved by interchanging d_2 and d_1, i.e. having $d_2/d_1 < 1.0$ as shown in the plot below.

$$S = \frac{1}{[1 + (d_2/d_1)^2]}$$

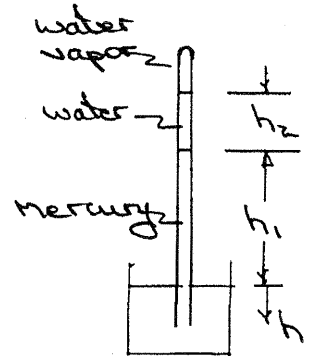
The manometer sensitivity, as a function of diameter ratio d_2/d_1 is shown below.

Problem 3.35

Given: Barometer with 6.5 in. of water on top of the mercury column of height 28.35 in.; Temperature $T = 70°F$.

Find: (a) Barometric pressure in psia.
(b) Effect of increase in ambient temperature (to $T_a = 85°F$) on length of mercury column for same barometric pressure.

Solution:

Basic equation: $\dfrac{dp}{dh} = \rho g$

Assumptions: (1) static liquid
(2) incompressible
(3) g = constant

Then, $dp = \rho g\, dh$ and $\Delta p = \rho g \Delta h$

Start at the free surface of the mercury ($p = p_{atm}$) and progress through the barometer to p_v (vapor pressure of the water)

$$p_{atm} - \rho_{Hg} g h_1 - \rho_{H_2O} g h_2 = p_v$$

$$p_{atm} = \rho_{Hg} g h_1 + \rho_{H_2O} g h_2 + p_v = \rho_{H_2O} SG_{Hg} g h_1 + \rho_{H_2O} g h_2 + p_v$$

$$p_{atm} = \rho_{H_2O}\, g\,[SG_{Hg} h_1 + h_2] + p_v$$

From Table A.2, $SG_{Hg} = 13.55$

Table A.7 $\rho_{H_2O} = 1.93\ \text{slug/ft}^3$, $p_v = 0.363$ psia.

Evaluating,

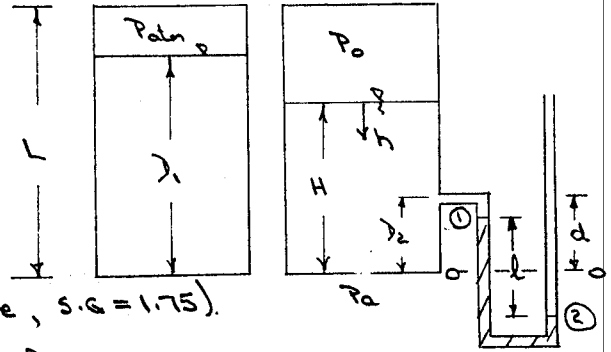
$$p_{atm} = 1.93\,\frac{\text{slug}}{\text{ft}^3} \times 32.2\,\frac{\text{ft}}{\text{s}^2}\left[13.55 \times 28.35\,\text{in} + 6.5\,\text{in}\right]\frac{\text{ft}}{12\,\text{in}} \times \frac{\text{ft}^2}{144\,\text{in}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}} + 0.363\ \text{psia}$$

$$p_{atm} = 14.4\ \text{psia}$$

At $T = 85°F$, the vapor pressure of water is estimated (from Table A.7) to be ≈ 0.60 psia. For the same barometric pressure the length of the mercury column would be shorter at the higher ambient temperature.

Problem 3.36

Given: Sealed tank of cross-section A and height, L = 3.0m, is filled with water to a depth $D_1 = 2.5m$.
Water drains slowly from the tank until system attains equilibrium.
U-tube manometer is connected to tank as shown. (manometer fluid is merian blue, s.G = 1.75).

$D_1 = 2.5\,m$, $D_2 = 0.7\,m$, $d = 0.2\,m$

Find: The manometer deflection, l, under equilibrium conditions

Solution:

Basic equations: $\frac{dP}{dh} = \rho g$ $\qquad P\forall = MRT$

For γ = constant $\Delta P = \rho g \Delta h$

To determine the surface pressure P_0 under equilibrium conditions treat air above water as an ideal gas

$$\frac{P_a \forall_a}{P_0 \forall_0} = \frac{MRT_a}{MRT_0} \qquad \text{Assuming } T_a = T_0 \text{, then}$$

$$P_0 = \frac{\forall_a}{\forall_0} P_a = \frac{A(L-D_1)}{A(L-H)} P_a = \frac{(L-D_1)}{(L-H)} P_a$$

Under equilibrium conditions, $P_0 + \rho_{H_2O} g H = P_a$

Hence, $\frac{(L-D_1)}{(L-H)} P_a + \rho_{H_2O} g H = P_a$ or $\rho_{H_2O} g H^2 - H(P_a + \rho_{H_2O} g L) + D_1 P_a = 0$

and

$$H = \frac{(P_a + \rho_{H_2O} g L) \pm \sqrt{(P_a + \rho_{H_2O} g L)^2 - 4\rho_{H_2O} g D_1 P_a}}{2\rho_{H_2O} g}$$

$$H = \frac{\left[1.01 \times 10^5 \frac{N}{m^2} + 999\frac{kg}{m^3} \times 9.81\frac{m}{sec^2} \times 3m \times \frac{N \cdot sec^2}{kg \cdot m}\right] \pm \sqrt{[\;]^2 - 4 \times 999\frac{kg}{m^3} \times 9.81\frac{m}{sec^2} \times 2.5m \times 1.01 \times 10^5 \frac{N}{m^2} \times \frac{N \cdot sec^2}{kg \cdot m}}}{2 \times 999\frac{kg}{m^3} \times 9.81\frac{m}{sec^2} \times \frac{N \cdot sec^2}{kg \cdot m}}$$

$H = 10.9m$ or $2.36m$. From physical considerations $H = 2.36m$

$$P_0 = \frac{(L-D_1)}{(L-H)} P_a = \frac{(3.0-2.5)}{(3.0-2.36)} \times 1.01 \times 10^5 \, N/m^2 = 7.89 \times 10^4 \, N/m^2$$

For the manometer, $P_0 + (P_1 - P_0) + (P_2 - P_1) = P_2 = P_{atm}$

$$P_0 + \rho_{H_2O} g \left(H - D_2 + d - \frac{l}{2}\right) + \rho_{mb} g l = P_{atm}$$

$$\frac{P_{atm} - P_0}{\rho_{H_2O} g} - H + D_2 - d = (S.G)_{mb} l - \frac{l}{2} = l\left[(S.G)_{mb} - 0.5\right]$$

$$l = \frac{(P_{atm} - P_0)/\rho_{H_2O} g - H + D_2 - d}{(S.G)_{mb} - 0.5} = \frac{(10.1 - 7.89) \times 10^4 \frac{N}{m^2} \times \frac{m^3}{999\,kg} \times \frac{s^2}{9.81m} \times \frac{kg\,m}{N \cdot s^2} - 2.36m + 0.7m - 0.2}{1.75 - 0.5}$$

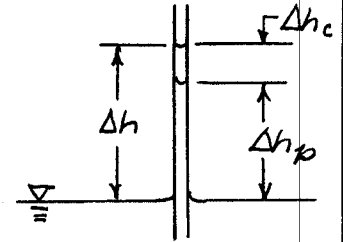
$l = 0.316m$ ← l

Problem 3.37

Given: Water column standing at $\Delta h = 50$ mm in $D = 2.5$ mm glass tube.

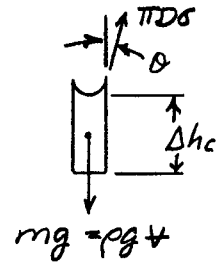
Find: (a) Column height if surface tension were zero.
(b) Column height in $D = 1$ mm tube.

Solution: Assume column height is sum of capillary rise and rise caused by pressure difference,

$$\Delta h = \Delta h_c + \Delta h_p$$

Choose a free-body diagram of Δh_c for analysis:

$$\sum F_z = \pi D \sigma \cos\theta - \frac{\pi D^2}{4} \rho g \, \Delta h_c = 0$$

Assumptions: (1) Neglect volume under meniscus
(2) Δh_p remains constant

Then $\Delta h_c = \dfrac{4\sigma}{\rho g D} \cos\theta$

For water (Table A.4), $\sigma = 72.8$ mN/m and $\theta \approx 0$, so $\cos\theta = 1$, and

$$\Delta h_c = \frac{4\sigma}{\rho g D}$$

For the $D = 2.5$ mm tube,

$$\Delta h_c = 4 \times 72.8 \times 10^{-3} \frac{N}{m} \times \frac{m^3}{999\,kg} \times \frac{s^2}{9.81\,m} \times \frac{1}{0.0025\,m} \times \frac{kg \cdot m}{N \cdot s^2} = 0.0119 \text{ m or } 11.9 \text{ mm}$$

Then

$$\Delta h_p = \Delta h - \Delta h_c = (50.0 - 11.9)\,mm = 38.1\,mm \quad (\sigma = 0)$$

Δh_1

For the $D = 1.0$ mm tube,

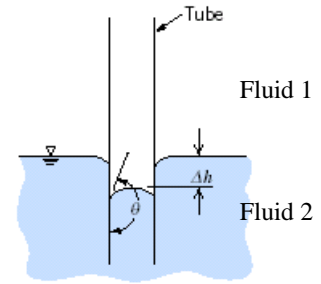
$$\Delta h_c = 4 \times 72.8 \times 10^{-3} \frac{N}{m} \times \frac{m^3}{999\,kg} \times \frac{s^2}{9.81\,m} \times \frac{1}{0.001\,m} \times \frac{kg \cdot m}{N \cdot s^2} = 0.0297 \text{ m or } 29.7 \text{ mm}$$

So

$$\Delta h = \Delta h_c + \Delta h_p = (29.7 + 38.1)\,mm = 67.8\,mm \quad (D = 1.0 \text{ mm tube})$$

Δh

Problem 3.38

3.38 Consider a small diameter open-ended tube inserted at the interface between two immiscible fluids of different densities. Derive an expression for the height difference Δh between the interface level inside and outside the tube in terms of tube diameter D, the two fluid densities, ρ_1 and ρ_2, and the surface tension σ and angle θ for the two fluids' interface. If the two fluids are water and mercury, find the tube diameter such that $\Delta h < 10$ mm.

Given: Two fluids inside and outside a tube

Find: An expression for height h; find diameter for h < 10 mm for water/mercury

Solution:

A free-body vertical force analysis for the section of fluid 1 height Δh in the tube below the "free surface" of fluid 2 leads to

$$\sum F = 0 = \Delta p \cdot \frac{\pi \cdot D^2}{4} - \rho_1 \cdot g \cdot \Delta h \cdot \frac{\pi \cdot D^2}{4} + \pi \cdot D \cdot \sigma \cdot \cos(\theta)$$

where Δp is the pressure difference generated by fluid 2 over height Δh, $\quad \Delta p = \rho_2 \cdot g \cdot \Delta h$

Assumption: Neglect meniscus curvature for column height and volume calculations

Hence $$\Delta p \cdot \frac{\pi \cdot D^2}{4} - \rho_1 \cdot g \cdot \Delta h \cdot \frac{\pi \cdot D^2}{4} = \rho_2 \cdot g \cdot \Delta h \cdot \frac{\pi \cdot D^2}{4} - \rho_1 \cdot g \cdot \Delta h \cdot \frac{\pi \cdot D^2}{4} = -\pi \cdot D \cdot \sigma \cdot \cos(\theta)$$

Solving for Δh $$\Delta h = -\frac{4 \cdot \sigma \cdot \cos(\theta)}{g \cdot D \cdot \left(\rho_2 - \rho_1\right)}$$

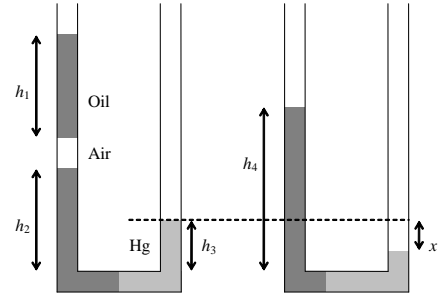
For fluids 1 and 2 being water and mercury (for mercury σ = 375 mN/m and θ = 140º, from Table A.4), solving for D to make Δh = 10 mm

$$D = -\frac{4 \cdot \sigma \cdot \cos(\theta)}{g \cdot \Delta h \cdot \left(\rho_2 - \rho_1\right)} = -\frac{4 \cdot \sigma \cdot \cos(\theta)}{g \cdot \Delta h \cdot \rho_{H2O} \cdot \left(SG_{Hg} - 1\right)}$$

$$D = -\frac{4 \times 0.375 \cdot \frac{N}{m} \times \cos(140 \cdot deg)}{9.81 \cdot \frac{m}{s^2} \times 0.01 \cdot m \times 1000 \cdot \frac{kg}{m^3} \times (13.6 - 1)} \times \frac{kg \cdot m}{N \cdot s^2} \qquad D = 0.93\,mm \qquad D \geq 1 \cdot mm$$

Problem 3.39

[2]

3.39 You have a manometer consisting of a tube that is 1.1-cm ID. On one side the manometer leg contains mercury, 10 cc of an oil (SG = 1.67), and 3 cc of air as a bubble in the oil. The other leg just contains mercury. Both legs are open to the atmosphere and are in a static condition. An accident occurs in which 3 cc of the oil and the air bubble are removed from the one leg. How much do the mercury height levels change?

Given: Data on manometer before and after an "accident"

Find: Change in mercury level

Solution:

Basic equation $\frac{dp}{dy} = -\rho \cdot g$ or, for constant ρ $\Delta p = \rho \cdot g \cdot \Delta h$ where Δh is height difference

For the initial state, working from right to left $p_{atm} = p_{atm} + SG_{Hg} \cdot \rho \cdot g \cdot h_3 - SG_{oil} \cdot \rho \cdot g \cdot (h_1 + h_2)$

$$SG_{Hg} \cdot \rho \cdot g \cdot h_3 = SG_{oil} \cdot \rho \cdot g \cdot (h_1 + h_2) \qquad (1)$$

Note that the air pocket has no effect!

For the final state, working from right to left $p_{atm} = p_{atm} + SG_{Hg} \cdot \rho \cdot g \cdot (h_3 - x) - SG_{oil} \cdot \rho \cdot g \cdot h_4$

$$SG_{Hg} \cdot \rho \cdot g \cdot (h_3 - x) = SG_{oil} \cdot \rho \cdot g \cdot h_4 \qquad (2)$$

The two unknowns here are the mercury levels before and after (i.e., h_3 and x)

Combining Eqs. 1 and 2 $SG_{Hg} \cdot \rho \cdot g \cdot x = SG_{oil} \cdot \rho \cdot g \cdot (h_1 + h_2 - h_4)$ $\quad x = \frac{SG_{oil}}{SG_{Hg}} \cdot (h_1 + h_2 - h_4) \qquad (3)$

From Table A.1 $SG_{Hg} = 13.55$

The term $h_1 + h_2 - h_4$ is the difference between the total height of oil before and after the accident

$$h_1 + h_2 - h_4 = \frac{\Delta V}{\left(\frac{\pi \cdot d^2}{4}\right)} = \frac{4}{\pi} \times \left(\frac{1}{0.011} \cdot \frac{1}{m}\right)^2 \times 3 \cdot cc \times \left(\frac{1 \cdot m}{100 \cdot cm}\right)^3 = 0.0316 \cdot m$$

Then from Eq. 3 $x = \frac{1.67}{13.55} \times 0.0316 \cdot m$ $\quad x = 3.895 \times 10^{-3} m$ $\quad x = 0.389 \cdot cm$

Problem 3.40

[3]

3.40 Based on the atmospheric temperature data of the U.S. Standard Atmosphere of Fig. 3.3, compute and plot the pressure variation with altitude, and compare with the pressure data of Table A.3.

Given: Atmospheric temperature data

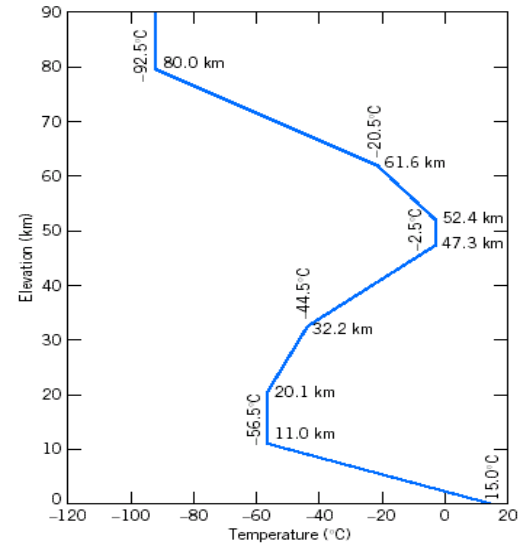
Find: Pressure variation; compare to Table A.3

Solution:

From Section 3-3: $\frac{dp}{dz} = -\rho \cdot z$

For a linear temperature variation $m = -\frac{dT}{dz} = \text{const}$ $\quad p = p_0 \cdot \left(\frac{T}{T_0}\right)^{\frac{g}{m \cdot R}}$

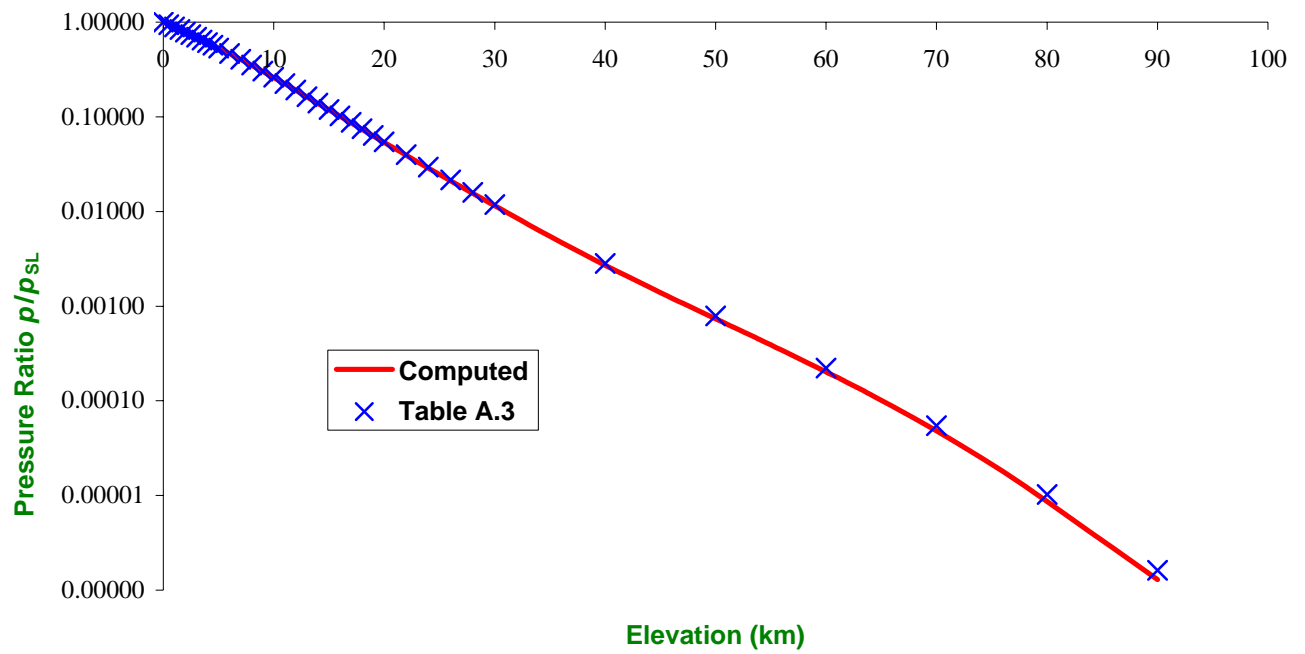
For isothermal conditions (Example 3.4) $\quad p = p_0 \cdot e^{-\frac{g \cdot (z - z_0)}{R \cdot T}}$

In these equations p_0, T_0, and z_0 are reference conditions

p_{SL} =	101	kPa
R =	286.9	J/kg.K
ρ =	999	kg/m^3

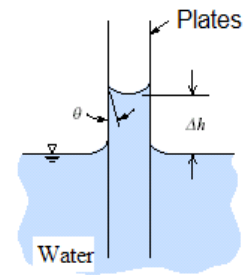
The temperature can be computed from the data in the figure
The pressures are then computed from the appropriate equation

z (km)	T (°C)	T (K)		p/p_{SL}
0.0	15.0	288.0	m =	1.000
2.0	2.0	275.00	0.0065	0.784
4.0	-11.0	262.0	(K/m)	0.608
6.0	-24.0	249.0		0.465
8.0	-37.0	236.0		0.351
11.0	-56.5	216.5		0.223
12.0	-56.5	216.5	T = const	0.190
14.0	-56.5	216.5		0.139
16.0	-56.5	216.5		0.101
18.0	-56.5	216.5		0.0738
20.1	-56.5	216.5		0.0530
22.0	-54.6	218.4	m =	0.0393
24.0	-52.6	220.4	-0.000991736	0.0288
26.0	-50.6	222.4	(K/m)	0.0211
28.0	-48.7	224.3		0.0155
30.0	-46.7	226.3		0.0115
32.2	-44.5	228.5		0.00824
34.0	-39.5	233.5	m =	0.00632
36.0	-33.9	239.1	-0.002781457	0.00473
38.0	-28.4	244.6	(K/m)	0.00356
40.0	-22.8	250.2		0.00270
42.0	-17.2	255.8		0.00206
44.0	-11.7	261.3		0.00158
46.0	-6.1	266.9		0.00122
47.3	-2.5	270.5		0.00104
50.0	-2.5	270.5	T = const	0.000736
52.4	-2.5	270.5		0.000544
54.0	-5.6	267.4	m =	0.000444
56.0	-9.5	263.5	0.001956522	0.000343
58.0	-13.5	259.5	(K/m)	0.000264
60.0	-17.4	255.6		0.000202
61.6	-20.5	252.5		0.000163
64.0	-29.9	243.1	m =	0.000117
66.0	-37.7	235.3	0.003913043	0.0000880
68.0	-45.5	227.5	(K/m)	0.0000655
70.0	-53.4	219.6		0.0000482
72.0	-61.2	211.8		0.0000351
74.0	-69.0	204.0		0.0000253
76.0	-76.8	196.2		0.0000180
78.0	-84.7	188.3		0.0000126
80.0	-92.5	180.5	T = const	0.00000861
82.0	-92.5	180.5		0.00000590
84.0	-92.5	180.5		0.00000404
86.0	-92.5	180.5		0.00000276
88.0	-92.5	180.5		0.00000189
90.0	-92.5	180.5		0.00000130

From Table A.3

z (km)	p/p_{SL}
0.0	1.000
0.5	0.942
1.0	0.887
1.5	0.835
2.0	0.785
2.5	0.737
3.0	0.692
3.5	0.649
4.0	0.609
4.5	0.570
5.0	0.533
6.0	0.466
7.0	0.406
8.0	0.352
9.0	0.304
10.0	0.262
11.0	0.224
12.0	0.192
13.0	0.164
14.0	0.140
15.0	0.120
16.0	0.102
17.0	0.0873
18.0	0.0747
19.0	0.0638
20.0	0.0546
22.0	0.0400
24.0	0.0293
26.0	0.0216
28.0	0.0160
30.0	0.0118
40.0	0.00283
50.0	0.000787
60.0	0.000222
70.0	0.0000545
80.0	0.0000102
90.0	0.00000162

Agreement between calculated and tabulated data is very good (as it should be, considering the table data is also computed!)

Problem 3.41

3.41 Two vertical glass plates 300 mm × 300 mm are placed in an open tank containing water. At one end the gap between the plates is 0.1 mm, and at the other it is 2 mm. Plot the curve of water height between the plates from one end of the pair to the other.

Given: Geometry of vertical plates

Find: Curve of water height due to capillary action

Solution:

Parallel Plates: A free-body vertical force analysis for the section of water height Δh above the "free surface" between plates arbitrary width w (similar to the figure above), leads to

$$\sum F = 0 = 2 \cdot w \cdot \sigma \cdot \cos(\theta) - \rho \cdot g \cdot \Delta h \cdot w \cdot a$$

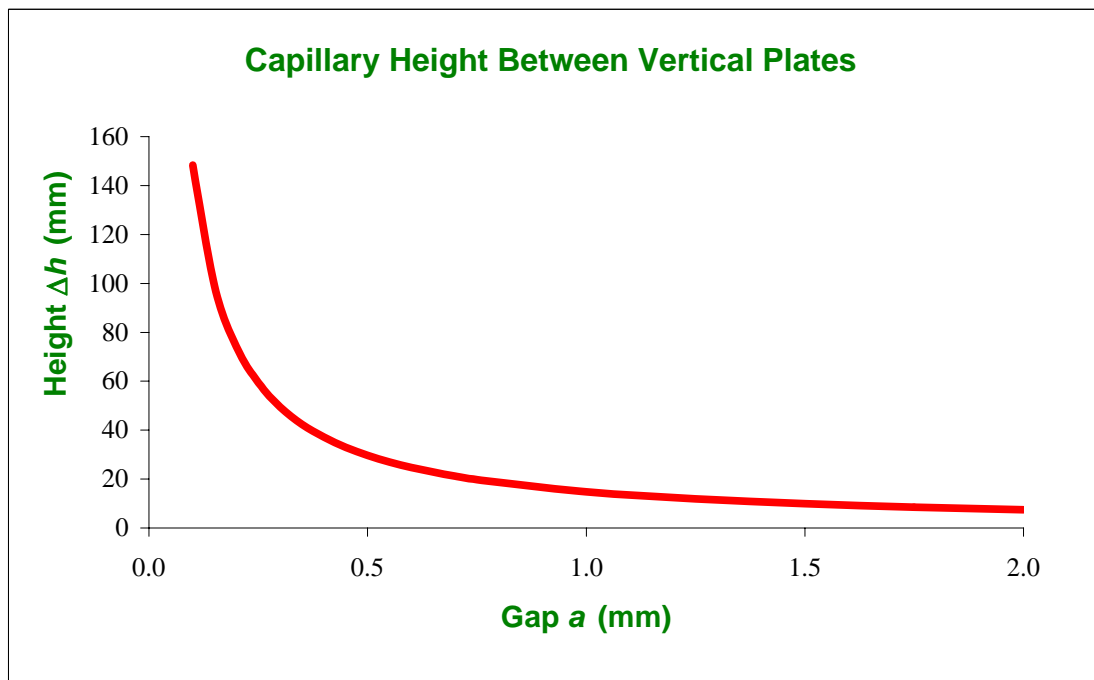
Solving for Δh

$$\Delta h = \frac{2 \cdot \sigma \cdot \cos(\theta)}{\rho \cdot g \cdot a}$$

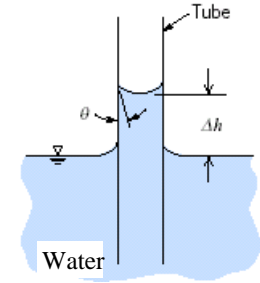
For water σ = 72.8 mN/m and $\theta = 0^{o}$ (Table A.4), so

σ = 72.8 mN/m
ρ = 1000 kg/m^3

Using the formula above

a (mm)	Δ*h* (mm)
0.10	148
0.15	98.9
0.20	74.2
0.25	59.4
0.30	49.5
0.35	42.4
0.40	37.1
0.45	33.0
0.50	29.7
0.55	27.0
0.60	24.7
0.65	22.8
0.70	21.2
0.75	19.8
1.00	14.8
1.25	11.9
1.50	9.89
1.75	8.48
2.00	7.42

Problem 3.42

[2]

3.42 Compare the height due to capillary action of water exposed to air in a circular tube of diameter $D = 0.5$ mm, and between two infinite vertical parallel plates of gap $a = 0.5$ mm.

Given: Water in a tube or between parallel plates

Find: Height Δ*h* for each system

Solution:

a) Tube: A free-body vertical force analysis for the section of water height Δ*h* above the "free surface" in the tube, as shown in the figure, leads to

$$\sum F = 0 = \pi \cdot D \cdot \sigma \cdot \cos(\theta) - \rho \cdot g \cdot \Delta h \cdot \frac{\pi \cdot D^2}{4}$$

Assumption: Neglect meniscus curvature for column height and volume calculations

Solving for Δ*h* $\quad \Delta h = \frac{4 \cdot \sigma \cdot \cos(\theta)}{\rho \cdot g \cdot D}$

b) Parallel Plates: A free-body vertical force analysis for the section of water height Δ*h* above the "free surface" between plates arbitrary width *w* (similar to the figure above), leads to

$$\sum F = 0 = 2 \cdot w \cdot \sigma \cdot \cos(\theta) - \rho \cdot g \cdot \Delta h \cdot w \cdot a$$

Solving for Δ*h* $\quad \Delta h = \frac{2 \cdot \sigma \cdot \cos(\theta)}{\rho \cdot g \cdot a}$

For water σ = 72.8 mN/m and θ = 0º (Table A.4), so

a) Tube $\quad \Delta h = \frac{4 \times 0.0728 \cdot \frac{N}{m}}{999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 0.005 \cdot m} \times \frac{kg \cdot m}{N \cdot s^2} \quad \Delta h = 5.94 \times 10^{-3} m \quad \Delta h = 5.94 mm$

b) Parallel Plates $\quad \Delta h = \frac{2 \times 0.0728 \cdot \frac{N}{m}}{999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 0.005 \cdot m} \times \frac{kg \cdot m}{N \cdot s^2} \quad \Delta h = 2.97 \times 10^{-3} m \quad \Delta h = 2.97 mm$

Problem 3.43

3.43 On a certain calm day, a mild inversion causes the atmospheric temperature to remain constant at 85°F between sea level and 16,000 ft altitude. Under these conditions, (a) calculate the elevation change for which a 2 percent reduction in air pressure occurs, (b) determine the change of elevation necessary to effect a 10 percent reduction in density, and (c) plot p_2/p_1 and ρ_2/ρ_1 as a function of Δz.

Given:
Data on isothermal atmosphere

Find:
Elevation changes for 2% and 10% density changes; plot of pressure and density versus elevation

Solution:

Basic equation $\quad \dfrac{dp}{dz} = -\rho \cdot g \quad$ and $\quad p = \rho \cdot R \cdot T$

Assumptions: static, isothermal fluid,; g = constant; ideal gas behavior

Then $\quad \dfrac{dp}{dz} = -\rho \cdot g = -\dfrac{p \cdot g}{R_{air} \cdot T} \quad$ and $\quad \dfrac{dp}{p} = -\dfrac{g}{R_{air} \cdot T} \cdot dz$

Integrating $\quad \Delta z = -\dfrac{R_{air} \cdot T_0}{g} \cdot \ln\left(\dfrac{p_2}{p_1}\right) \quad$ where $\quad T = T_0$

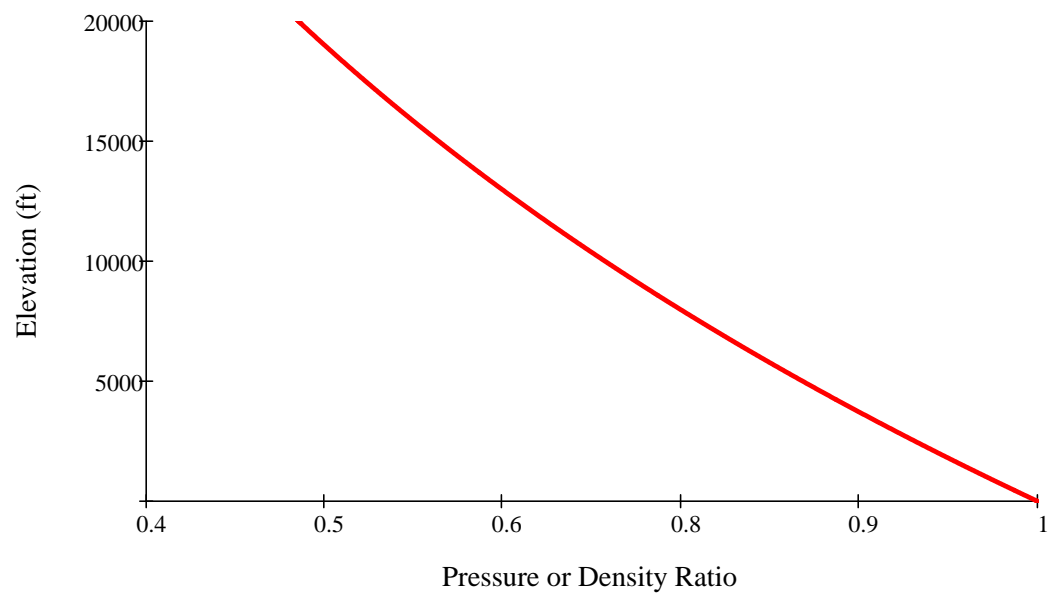
For an ideal with T constant $\quad \dfrac{p_2}{p_1} = \dfrac{\rho_2 \cdot R_{air} \cdot T}{\rho_1 \cdot R_{air} \cdot T} = \dfrac{\rho_2}{\rho_1} \quad$ so $\quad \Delta z = -\dfrac{R_{air} \cdot T_0}{g} \cdot \ln\left(\dfrac{\rho_2}{\rho_1}\right) = -C \cdot \ln\left(\dfrac{\rho_2}{\rho_1}\right) \quad (1)$

From Table A.6 $\quad R_{air} = 53.33 \cdot \dfrac{ft \cdot lbf}{lbm \cdot R}$

Evaluating $\quad C = \dfrac{R_{air} \cdot T_0}{g} = 53.33 \cdot \dfrac{ft \cdot lbf}{lbm \cdot R} \times (85 + 460) \cdot R \times \dfrac{1}{32.2} \cdot \dfrac{s^2}{ft} \times \dfrac{32.2 \cdot lbm \cdot ft}{s^2 \cdot lbf} \qquad C = 29065 \cdot ft$

For a 2% reduction in density $\quad \dfrac{\rho_2}{\rho_1} = 0.98 \quad$ so from Eq. 1 $\quad \Delta z = -29065 \cdot ft \cdot \ln(0.98) \qquad \Delta z = 587 \cdot ft$

For a 10% reduction in density $\quad \dfrac{\rho_2}{\rho_1} = 0.9 \quad$ so from Eq. 1 $\quad \Delta z = -29065 \cdot ft \cdot \ln(0.9) \qquad \Delta z = 3062 \cdot ft$

To plot $\dfrac{p_2}{p_1}$ and $\dfrac{\rho_2}{\rho_1}$ we rearrange Eq. 1 $\quad \dfrac{\rho_2}{\rho_1} = \dfrac{p_2}{p_1} = e^{-\frac{\Delta z}{C}}$

This plot can be plotted in Excel

Given: Martian atmosphere behaves as an ideal gas, T = constant
$M_m = 32.0$, $T = 200\ K$, $g = 3.92\ m/s^2$, $\rho_0 = 0.015\ kg/m^3$

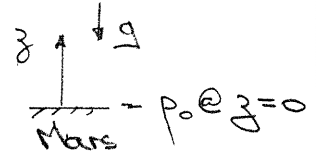
Find: Density at $z = 20\ km$.

Plot: the ratio ρ/ρ_0 (ratio of density to surface density) vs z; compare with earth's atmosphere

Solution:

Basic equations: $\frac{dp}{dz} = -\rho g$; $p = \rho RT$; $R = R_u/M_m$

Assumptions: (1) static fluid
(2) g constant
(3) ideal gas

$z \uparrow \downarrow g$ — Mars — ρ_0 @ $z=0$

Since T = constant, $dp = d(\rho RT) = RT\,d\rho$

$$\frac{dp}{dz} = RT\frac{d\rho}{dz} = -\rho g \quad \text{and} \quad \int_{\rho_0}^{\rho} \frac{d\rho}{\rho} = -\int_0^z \frac{g}{RT}\,dz$$

$$\ln\frac{\rho_z}{\rho_0} = -gz/RT \quad \text{and} \quad \frac{\rho}{\rho_0} = e^{-gz/RT} \qquad \text{---- (1)}$$

Evaluating

$$R = \frac{R_u}{M_m} = 8314.3\frac{N\cdot m}{kgmole\cdot K} \times \frac{kgmole}{32.0\,kg} = 260\frac{N\cdot m}{kg\cdot K}$$

$$\rho = 0.015\frac{kg}{m^3} \times \exp\left[-3.92\frac{m}{s^2} \times 20\times10^3 m \times \frac{kg\cdot K}{260\,N\cdot m} \times \frac{1}{200\,K} \times \frac{N\cdot s^2}{kg\cdot m}\right]$$

$$\rho = 0.00332\ kg/m^3 \qquad \rho_{z=20\,km}$$

For the Martian atmosphere, Eq. 1 gives $\rho/\rho_0 = e^{-0.0754\,z\,(km)}$

For the earth's atmosphere, ρ/ρ_0 is given in Table A.3

Both ρ/ρ_0 variations are plotted below

Note from the plot:
- on Mars $\rho/\rho_0 = 0.221$ at $z = 20\ km$, whereas
- on Earth, $\rho/\rho_0 = 0.073$ at $z = 20\ km$.

The difference is caused by (a) the larger gravity on Earth, and (b) temperature decrease with altitude in our atmosphere.

Problem 3.44

Density vs. elevation in Martian and Earth atmospheres:

Elevation Change Δz (m)	Density Ratio (Earth) ρ/ρ_{SL} (---)	Density Ratio (Mars) ρ/ρ_{SL} (---)
0	1.000	1.00
2000	0.8217	0.860
4000	0.6689	0.740
6000	0.5389	0.636
8000	0.4292	0.547
10000	0.3376	0.470
12000	0.2546	0.405
14000	0.1860	0.348
16000	0.1359	0.299
18000	0.09930	0.257
20000	0.07258	0.221

Problem 3.45

Given: Atmospheric conditions at ground level ($z=0$) in Denver, Colorado are $p_0 = 83.2$ kPa, $T_0 = 25°C$. Pike's peak is at elevation $z = 2690$ m

Find: Pressure on Pike's peak assuming (a) an incompressible, and (b) an adiabatic atmosphere.

Plot: p/p_0 vs z for both cases.

Solution:

Basic equations: $dp/dz = -\rho g$; $p = \rho R T$

Assumptions: (1) static fluid (2) g = constant (3) ideal gas behavior

(a) For an incompressible atmosphere $\int_{p_0}^{p} dp = -\int_0^z \rho g\, dz$

$$p - p_0 = \rho g z = \rho_0 g z = \frac{p_0}{RT_0} g z \quad \text{and} \quad p = p_0\left[1 - \frac{gz}{RT_0}\right] \quad \text{------ (1)}$$

At $z = 2690$ m

$$p = 83.2\,\text{kPa}\left[1 - 9.81\frac{\text{m}}{\text{s}^2} \times 2690\,\text{m} \times \frac{\text{kg}\cdot\text{K}}{287\,\text{N}\cdot\text{m}} \times \frac{1}{298\,\text{K}} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}\right] = 57.5\,\text{kPa} \quad p_{\rho=c}$$

(b) For an adiabatic atmosphere p/ρ^k = constant, $\rho = \rho_0 (p/p_0)^{1/k}$

$$\frac{dp}{dz} = -\rho g = -g\rho_0\left(\frac{p}{p_0}\right)^{1/k} dz \quad \text{or} \quad \int_{p_0}^{p} \frac{dp}{p^{1/k}} = -\int_0^z \rho_0 p_0^{-1/k} g\, dz$$

Then

$$\frac{k}{(k-1)} p^{-\frac{1}{k}+1}\Big]_{p_0}^{p} = -\rho_0 p_0^{-1/k} g z \quad \text{or} \quad \frac{k}{(k-1)}\left[p^{(k-1)/k} - p_0^{(k-1)/k}\right] = -\rho_0 p_0^{-1/k} g z$$

and

$$\left(\frac{k}{k-1}\right) p_0^{(k-1)/k}\left[\left(\frac{p}{p_0}\right)^{(k-1)/k} - 1\right] = -\rho_0 p_0^{-1/k} g z$$

$$\left(\frac{p}{p_0}\right)^{(k-1)/k} = 1 - \frac{(k-1)}{k} p_0^{-1/k-(k-1)/k} \rho_0 g z = 1 - \frac{(k-1)}{k} p_0^{-1} \rho_0 g z$$

and

$$\frac{p}{p_0} = \left[1 - \frac{(k-1)}{k}\frac{\rho_0}{p_0} g z\right]^{k/k-1} = \left[1 - \frac{(k-1)}{k}\frac{gz}{RT_0}\right]^{k/k-1} \quad \text{------ (2)}$$

Evaluating at $z = 2690$ m

$$p = 83.2\,\text{kPa}\left[1 - \frac{0.4}{1.4} \times 9.81\frac{\text{m}}{\text{s}^2} \times 2690\,\text{m} \times \frac{\text{kg}\cdot\text{K}}{287\,\text{N}\cdot\text{m}} \times \frac{1}{298\,\text{K}} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}\right]^{\frac{1.4}{0.4}}$$

$$p = 60.2\,\text{kPa} \quad p_{adiab}$$

The pressure ratio p/p_0 vs z is plotted for an incompressible atmosphere (Eq. 1) and an adiabatic atmosphere (Eq. 2) below

Incompressible case $p/p_0 = [1 - 0.115z]$ (z in km)

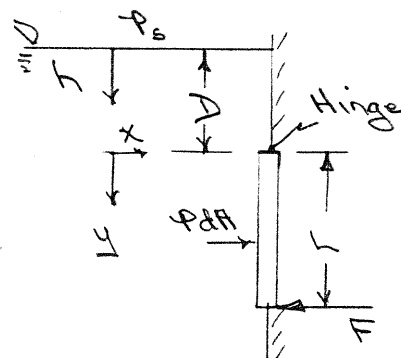
Adiabatic case $p/p_0 = [1 - 0.0328z]^{3.5}$ (z in km)

Problem 3.45

Pressure ratio vs. elevation above Denver:

Elevation z (m)	Elevation above Denver z (m)	Pressure Ratio (T = C) p/p_0 (---)	Pressure Ratio (adiabatic) p/p_0 (---)
0	-1610	1.185	1.20
500	-1110	1.127	1.13
1000	-610	1.070	1.07
1500	-110	1.013	1.01
2000	390	0.955	0.956
2500	890	0.898	0.902
3000	1390	0.841	0.849
3500	1890	0.783	0.800
4000	2390	0.726	0.752
4300	2690	0.691	0.724
4500	2890	0.669	0.706
5000	3390	0.611	0.662

Problem 3.46

Given: Door, of width $b = 1\,m$, located in plane vertical wall of water tank is hinged along upper edge.
$D = 1\,m$, $L = 1.5\,m$
Atmospheric pressure acts on outer surface of door; force F is applied at lower applied at lower edge to keep door closed

Find: (a) Force F, if $p_s = p_{atm}$.
(b) Force F, if $p_s = 0.5$ atm.

Plot: F/F_o over range of p_s/p_{atm}. (F_o is force required when $p_s = p_{atm}$)

Solution:

Basic equations: $\frac{dp}{dh} = \rho g$; $F_R = \int p\,dA$; $\sum M_z = 0$

Assumptions: (1) static fluid (2) ρ = constant
(3) door is in equilibrium

Since $\sum M_z = 0$ for equilibrium, taking moments about the hinge

$$\sum M_z = 0 = FL - \int y p\,dA = F \cdot L - \int_0^L y p b\,dy$$

and

$$F = \frac{1}{L}\int_0^L y p b\,dy \quad \text{------} (1)$$

Note: We will obtain a general expression for F (needed for the plot) and then simplify for cases (a) and (b)

Since $dp = \rho g\,dh$, then $p = p_s + \rho g h$
$h = D + y$ and hence $p = p_s + \rho g (D + y)$.

Because p_{atm} acts on the outside of the door, p_s is the surface gage pressure.

From Eq (1),

$$F = \frac{1}{L}\int_0^L y\left[p_{sg} + \rho g (D+y)\right] b\,dy$$

$$F = \frac{b}{L}\left[p_{sg}\frac{y^2}{2} + \rho g\left(\frac{Dy^2}{2} + \frac{y^3}{3}\right)\right]_0^L$$

$$F = \frac{b}{L}\left[p_{sg}\frac{L^2}{2} + \rho g\left(\frac{DL^2}{2} + \frac{L^3}{3}\right)\right] = b\left[p_{sg}\frac{L}{2} + \rho g L\left(\frac{D}{2} + \frac{L}{3}\right)\right] \quad \text{----} (2)$$

(a) For $p_s = p_{atm}$, $p_{sg} = 0$

$$F_o = \rho g b L\left(\frac{D}{2} + \frac{L}{3}\right) \quad \text{------} (3)$$

$$F_o = 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 1m \times 1.5m\left(\frac{1m}{2} + \frac{1.5m}{3}\right) \times \frac{N \cdot s^2}{kg \cdot m} \times \frac{kN}{10^3 N} = 14.7\,kN$$

(b) For $p_{sg} = 0.5$ atm (50.6 kPa), from Eq (2)

$$F = p_{sg}\frac{bL}{2} + \rho g L b\left(\frac{D}{2} + \frac{L}{3}\right)$$

$$F = 50.6\,\frac{\text{kN}}{\text{m}^2} \times 1\,\text{m} \times \frac{1.5\,\text{m}}{2} + 14.7\,\text{kN} = 52.7\,\text{kN}$$ ← $F)_{p_s = 0.5\,\text{atm}}$

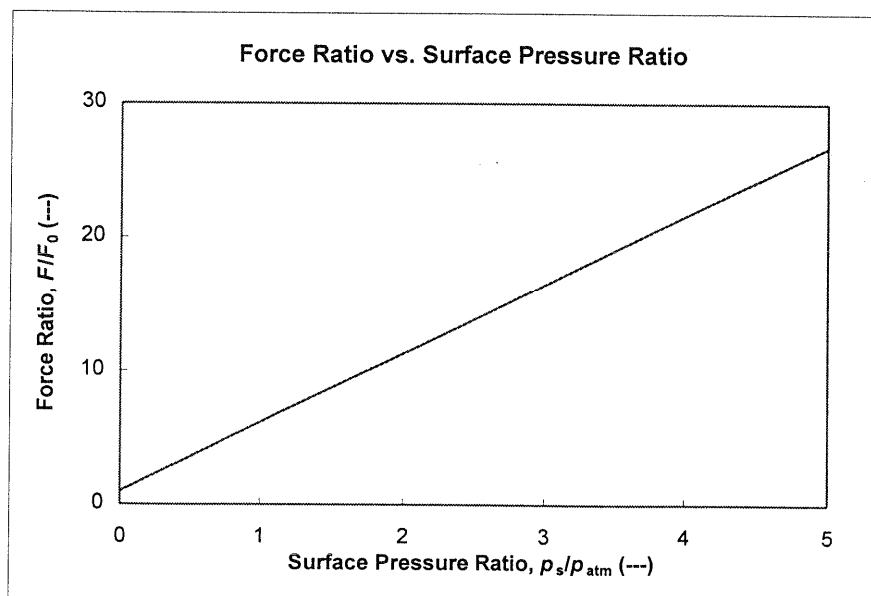
From Eqs (2) and (3) we can write

$$\frac{F}{F_0} = \frac{b\left[p_{sg}\frac{L}{2} + \rho g L\left(\frac{D}{2} + \frac{L}{3}\right)\right]}{\rho g b L\left(\frac{D}{2} + \frac{L}{3}\right)} = 1 + \frac{p_{sg}}{2\rho g\left(\frac{D}{2} + \frac{L}{3}\right)}$$

Substituting values

$$\frac{F}{F_0} = 1 + \frac{p_{sg}}{0.194} \quad \text{(with } p_{sg} \text{ in atmospheres)}. \qquad (4)$$

F/F_0 is plotted as a function of $p_{s\,gage}/p_{atm}$

Problem 3.47

Open-Ended Problem Statement: A hydropneumatic elevator consists of a piston-cylinder assembly to lift the elevator cab. Hydraulic oil, stored in an accumulator tank pressurized by air, is valved to the piston as needed to lift the elevator. When the elevator descends, oil is returned to the accumulator. Design the least expensive accumulator that can satisfy the system requirements. Assume the lift is 3 floors, the maximum load is 10 passengers, the maximum system pressure is 800 kPa (gage). For column bending strength, the piston diameter must be at least 150 mm. The elevator cab and piston have a combined mass of 3,000 kg, and are to be purchased. Perform the analysis needed to define, as a function of system operating pressure, the piston diameter, the accumulator volume and diameter, and the wall thickness. Discuss safety features that your company should specify for the complete elevator system. Would it be preferable to use a completely pneumatic design or a completely hydraulic design? Why?

Discussion: The design requirements are specified, except that a typical floor height is about 12 ft, making the total required lift about 36 ft.)

A spreadsheet was used to calculate the system properties for various pressures. Results are presented on the next page, followed by a sample calculation.

Total cost dropped quickly as system pressure was increased. A shallow minimum was reached in the 100-110 psig range.

The lowest-cost solution was obtained at a system pressure of about 100 psig. At this pressure, the reservoir of 140 gal required a 3.30 ft diameter pressure sphere with a 0.250 in. wall thickness. The welding cost was $311 and the material cost $433, for a total cost of $744.

Accumulator wall thickness was constrained at 0.250 in. for pressures below 100 psi; it increased for higher pressures (this caused the discontinuity in slope of the curve at 100 psig). The mass of steel became constant above 110 psig.

No allowance was made for the extra volume needed to pressurize the accumulator.

Fail-safe design is essential for an elevator to be used by the public. The control circuitry should be redundant. Failures must be easy to spot. For this reason, hydraulic actuation is good: leaks will be readily apparent. The final design must be reviewed, approved, and stamped by a professional engineer since the design involves public safety.

The terminology used in the solution is defined in Table 1.

Table 1. Symbols, definitions, and units

Symbol	Definition	Units
p	system pressure	psig
A_p	area of lift piston	in.^{2}
$\mathcal{V}_{oil}$	volume of oil	gal
D_s	diameter of (spherical) accumulator	ft
t	wall thickness of spherical accumulator	in.
A_w	area of weld	in.^{2}
C_w	cost of weld	$
M_s	mass of (steel) accumulator	lbm
C_s	cost of steel	$
C_t	total cost	$

Results of the system simulation and sample calculations are presented on the next page.

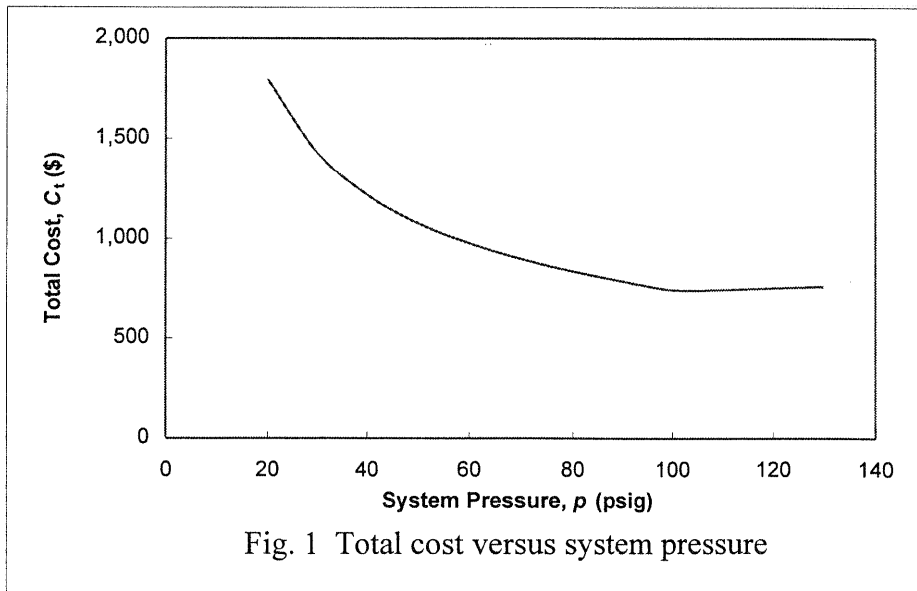
Problem 3.47

Table 2. Results of system simulation

Input Data:				
	Cab and piston weight:	W_{cab} =	6,000	lbf
	Passenger weight:	W_{pax} =	1,500	lbf
	Total weight:	W_{tot} =	7,500	lbf
	Allowable stress:	σ =	4,000	psi
	Minimum wall thickness:	t =	0.250	in.
	Welding cost factor:	cf_w =	5.00	\$/in.^{2}
	Steel cost factor:	cf_s =	1.25	\$/pound

Results:

p (psig)	A_p (in.^{2})	V_{oil} (gal)	D_s (ft)	t (in.)	A_w (in.^{2})	C_w (\$)	M_s (lbm)	C_s (\$)	C_t (\$)
20	375	701	5.64	0.250	106.2	$531	1012	$1,265	$1,796
30	250	468	4.92	0.250	92.8	$464	772	$965	$1,429
40	188	351	4.47	0.250	84.3	$422	638	$797	$1,218
50	150	281	4.15	0.250	78.3	$391	549	$687	$1,078
60	125	234	3.91	0.250	73.7	$368	487	$608	$976
70	107	200	3.71	0.250	70.0	$350	439	$549	$899
80	93.8	175	3.55	0.250	66.9	$335	402	$502	$837
90	83.3	156	3.41	0.250	64.4	$322	371	$464	$786
100	75.0	140	3.30	0.250	62.1	$311	346	$433	$743
110	68.2	128	3.19	0.263	63.4	$317	342	$428	$745
120	62.5	117	3.10	0.279	65.3	$326	342	$428	$754
130	57.7	108	3.02	0.294	67.1	$335	342	$428	$763

Fig. 1 Total cost versus system pressure

Sample Calculation ($p = 20$ psig):

$$W_t = p A_p \; ; \; A_p = \frac{W_t}{p} = 7500\ lbf \times \frac{in.^2}{20\ lbf} = 375\ in.^2$$

$$V_{oil} = A_p L = 375\ in.^2 \times 36\ ft \times \frac{ft^2}{144\ in.^2} \times 7.48\frac{gal}{ft^3} = 701\ gal$$

$$V_{oil} = V_s = \frac{4\pi R_s^3}{3} = \frac{\pi D_s^3}{6} \; ; \; D_s = \left(\frac{6 V_{oil}}{\pi}\right)^{1/3} = \left(\frac{6}{\pi} \times 701\ gal \times \frac{ft^3}{7.48\ gal}\right)^{1/3} = 5.64\ ft$$

From a force balance on the sphere: $p\frac{\pi D_s^2}{4}$; $\pi D_s t \sigma$

Problem 3.47

Thus $p \frac{\pi D_s^2}{4} = \pi D_s t \sigma$, so $t = \frac{p}{\sigma}\frac{D_s}{4} = \frac{1}{4} \times 20 \frac{\text{lbf}}{\text{in.}^2} \times \frac{\text{in.}^2}{4000\,\text{lbf}} \times 5.64\,\text{ft} \times 12 \frac{\text{in.}}{\text{ft}} = 0.0846\,\text{in.}$

Therefore $t = t_{min} = 0.250$ in.

$$A_W = \pi D_s t = \pi \times 5.64\,\text{ft} \times 0.25\,\text{in.} \times 12 \frac{\text{in.}}{\text{ft}} = 106\ \text{in.}^2$$

$$C_W = \frac{\$5.00}{\text{in.}^2} \times 106\ \text{in.}^2 = \$531$$

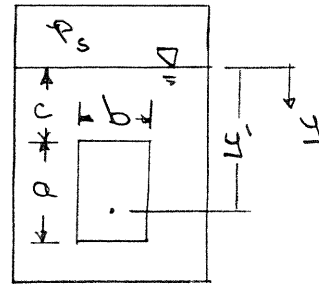
$$M_S = 4\pi R_s^2 t \rho_s = \pi D_s^2 t\, SG_s\, \rho_{H_2O} = \pi \times (5.64)^2\,\text{ft}^2 \times 0.25\,\text{in.} \times 7.8 \times 62.4 \frac{\text{lbm}}{\text{ft}^3} \times \frac{\text{ft}}{12\,\text{in.}} = 1012\,\text{lbm}$$

$$C_S = \frac{\$1.25}{\text{lbm}} \times 1012\ \text{lbm} = \$1265$$

and

$$C_t = C_W + C_S = \$531 + \$1265 = \$1{,}796$$

C_t

Given: Door located in plane vertical wall of water tank as shown
$a = 1.5\,m$, $b = 1\,m$, $c = 1\,m$.
Atmospheric pressure acts on outer surface of door

Find: (a) For $p_s = p_{atm}$, resultant force on door and line of action of force
(b) Resultant force and line of action if $p_s = 0.3$ atm (gage)

Plot: F/F_o and y/y_c over range of p_s/p_{atm}. (F_o is resultant force when $p_s = p_{atm}$; y_c is y coordinate of centroid)

Solution:

Basic equations: $\frac{dp}{dy} = \rho g$; $F_R = \int_A p\,dA$; $y' F_R = \int y p\,dA$

Assumptions: (1) static liquid
(2) incompressible liquid

Note: We will obtain a general expressions for F and y' (needed for the plot) and then simply for cases (a) & (b).

Since $dp = \rho g\,dy$ then $p = p_s + \rho g y$

Because p_{atm} acts on the outside of the door, then p_s is the surface gage pressure

$$F_R = \int p\,dA = \int_c^{c+a} p\,b\,dy = \int_c^{c+a} (p_s + \rho g y)\,b\,dy = b\left[p_s y + \rho g \frac{y^2}{2}\right]_c^{c+a}$$

$$F_R = b\left[p_s a + \frac{\rho g}{2}\{(c+a)^2 - c^2\}\right] = b\left[p_s a + \frac{\rho g}{2}(a^2 + 2ac)\right] \quad \text{---- (1)}$$

$$y' F_R = \int y p\,dA \quad \text{and} \quad y' = \frac{1}{F_R}\int_c^{c+a} y(p_s + \rho g y)\,b\,dy$$

$$y' = \frac{b}{F_R}\left[p_s \frac{y^2}{2} + \rho g \frac{y^3}{3}\right]_c^{c+a}$$

$$y' = \frac{b}{F_R}\left[\frac{p_s}{2}\{(c+a)^2 - c^2\} + \frac{\rho g}{3}\{(c+a)^3 - c^3\}\right] \quad \text{-------- (2)}$$

(a) For $p_s = 0$ (gage) then

from Eq. 1 $\quad F_R = \frac{\rho g b}{2}(a^2 + 2ac)$.

$$F_R = \frac{1}{2} \times 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 1\,m\,\left[(1.5\,m)^2 + 2(1.5\,m)(1\,m)\right]\frac{N \cdot s^2}{kg \cdot m} = 25.7\,kN \quad \longleftarrow F_{R_o}$$

From Eq. 2

$$y' = \frac{b}{F_{R_o}}\frac{\rho g}{3}\left[(c+a)^3 - c^3\right]$$

$$y' = \frac{1\,m}{25.7\,kN} \times \frac{999}{3}\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2}\left[(2.5)^3 - 1\right]m^3\,\frac{N \cdot s^2}{kg \cdot m} \times \frac{kN}{10^3\,N} = 1.86\,m \quad \longleftarrow y'_o$$

b) For $p_s = 0.3$ atm (gage) then

from Eq. 1 $\quad F_R = b\left[p_s a + \frac{\rho g}{2}(a^2 + 2ac)\right]$

$$F_R = 1\,m\left[0.3\,atm \times 1.01\times10^5 \frac{N}{m^2\cdot atm}(1.5\,m) + \frac{1}{2}\times 999\frac{kg}{m^3}\times 9.81\frac{m}{s^2}\left\{(1.5)^2 + 2(1.5)(1)\right\}m^2\cdot\frac{N\cdot s^2}{kg\cdot m}\right]$$

$F_R = 71.2\,kN$ $\quad F_R$

$$y' = \frac{b}{F_R}\left[\frac{p_s}{2}\left\{(c+a)^2 - c^2\right\} + \frac{\rho g}{3}\left\{(c+a)^3 - c^3\right\}\right]$$

$$y' = \frac{1\,m}{71.2\,kN}\left[\frac{1}{2}\times 0.3\,atm\times 1.01\times10^5\frac{N}{m^2\cdot atm}\left\{(2.5)^2 - 1\right\} + \frac{1}{3}\times 999\frac{kg}{m^3}\times 9.81\frac{m}{s^2}\times\left\{(2.5)^3 - 1\right\}m^3 \times \frac{N\cdot s^2}{kg\cdot m}\right]\frac{kN}{10^3\,N}$$

$y' = 1.79\,m$ $\quad y'$

The value of F/F_0 is obtained from Eq. 1 and $F_{R_0} = 25.7\,kN$.

$$\frac{F}{F_0} = \frac{1}{25.7\,kN}\,b\left[p_s a + \frac{\rho g}{2}(a^2 + 2ac)\right] = 0.0389\left[151.5\,p_s + 25.7\right]$$

with p_s in atm. $\quad F/F_0$

For the gate $y_c = c + \frac{a}{2} = 1.75\,m$. Then from Eq. 2

$$y'/y_c = \frac{b}{F_R(1.75)}\left[\frac{p_s}{2}\left\{(c+a)^2 - c^2\right\} + \frac{\rho g}{3}\left\{(c+a)^3 - c^3\right\}\right] = \frac{0.571}{F}\left[265\,p_s + 47.8\right]$$

with F in kN, p_s in atm $\quad y'/y_c$

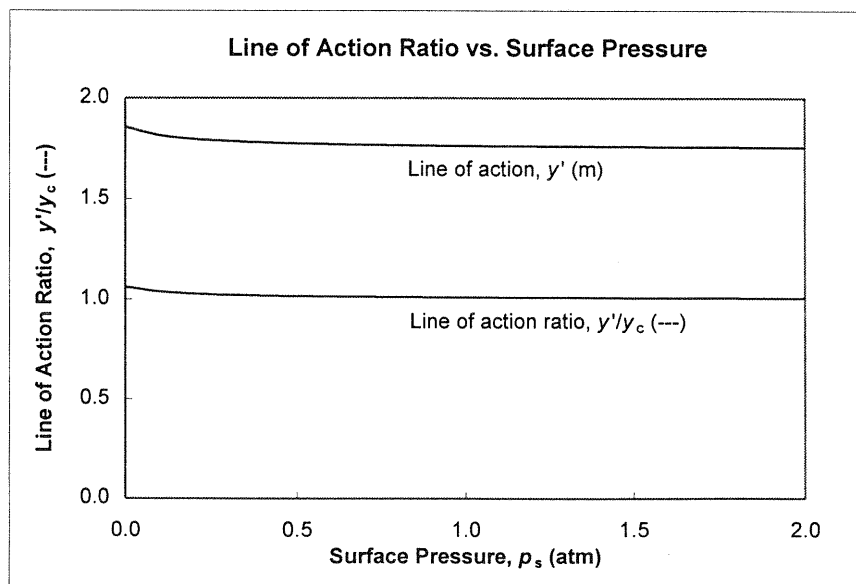
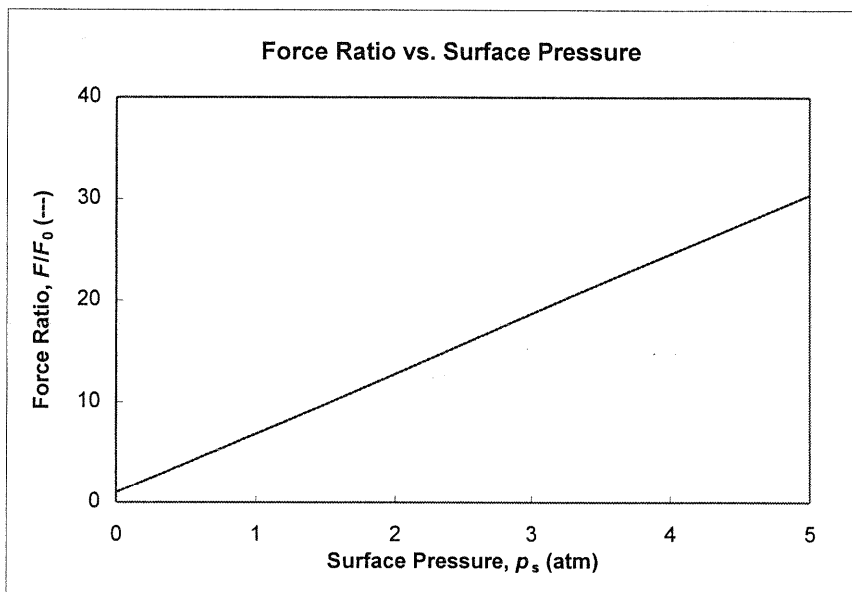
The plots are shown below

Note: The force on the gate increases linearly with increase in surface pressure.

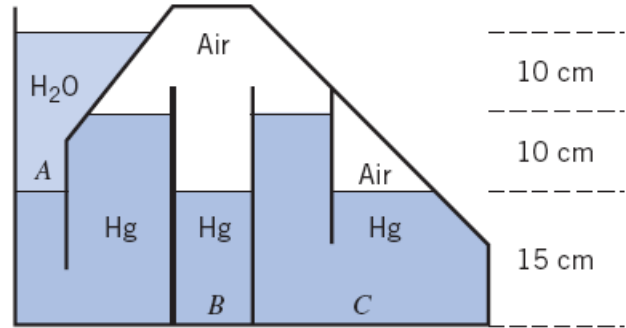
The line of action of the resultant force is always below the centroid of the gate; y'/y_c approaches unity as the surface pressure is increased.

Problem 3.48

Force ratio and line of action ratio vs. surface pressure:

Surface Pressure, p_s (atm)	Force Ratio, F/F_0 (---)	Force, F_0 (kN)	Line of Action Ratio, y'/y_c (---)	Line of Action, y' (m)
0	1.00	25.7	1.0623	1.86
0.1	1.59	40.8	1.0388	1.82
0.2	2.18	56.0	1.0281	1.80
0.3	2.77	71.1	1.0219	1.79
0.5	3.95	101	1.0151	1.78
1.0	6.89	177	1.00822	1.76
2.0	12.8	329	1.00399	1.76
3.0	18.7	480		
4.0	24.6	632		
5.0	30.5	783		

Problem 3.49

[2]

3.49 Find the pressures at points A, B, and C, as shown, and in the two air cavities.

Given: Geometry of chamber system

Find: Pressure at various locations

Solution:

Basic equation $\frac{dp}{dy} = -\rho \cdot g$ or, for constant ρ $\Delta p = \rho \cdot g \cdot \Delta h$ where Δh is height difference

For point A $p_A = p_{atm} + \rho \cdot g \cdot h_1$ or in gage pressure $p_A = \rho \cdot g \cdot h_1$

Here we have $h_1 = 20 \cdot cm$ $h_1 = 0.2\,m$

$$p_A = 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 0.2 \cdot m \times \frac{N \cdot s^2}{kg \cdot m}$$ $p_A = 1962\,Pa$ $p_A = 1.96 \cdot kPa$ (gage)

For the air cavity $p_{air} = p_A - SG_{Hg} \cdot \rho \cdot g \cdot h_2$ where $h_2 = 10 \cdot cm$ $h_2 = 0.1\,m$

From Table A.1 $SG_{Hg} = 13.55$

$$p_{air} = 1962 \cdot \frac{N}{m^2} - 13.55 \times 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 0.1 \cdot m \times \frac{N \cdot s^2}{kg \cdot m}$$ $p_{air} = -11.3 \cdot kPa$ (gage)

Note that p = constant throughout the air pocket

For point B $p_B = p_{atm} + SG_{Hg} \cdot \rho \cdot g \cdot h_3$ where $h_3 = 15 \cdot cm$ $h_3 = 0.15\,m$

$$p_B = -11300 \cdot \frac{N}{m^2} + 13.55 \times 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 0.15 \cdot m \times \frac{N \cdot s^2}{kg \cdot m}$$ $p_B = 8.64 \cdot kPa$ (gage)

For point C $p_C = p_{atm} + SG_{Hg} \cdot \rho \cdot g \cdot h_4$ where $h_4 = 25 \cdot cm$ $h_4 = 0.25\,m$

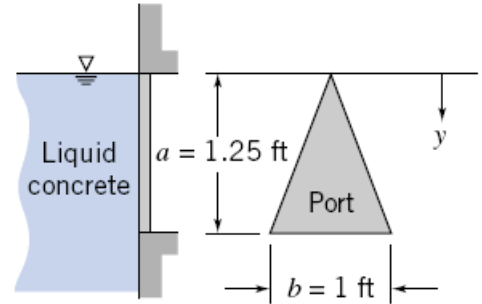
$$p_C = -11300 \cdot \frac{N}{m^2} + 13.55 \times 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 0.25 \cdot m \times \frac{N \cdot s^2}{kg \cdot m}$$ $p_C = 21.93 \cdot kPa$ (gage)

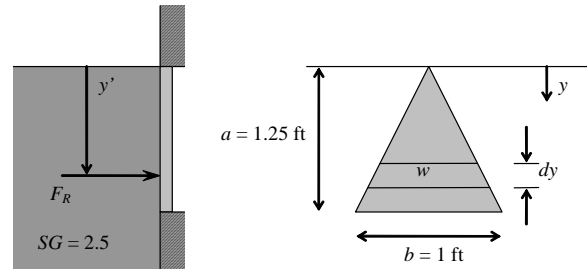
For the second air cavity $p_{air} = p_C - SG_{Hg} \cdot \rho \cdot h_5$ where $h_5 = 15 \cdot cm$ $h_5 = 0.15\,m$

$$p_{air} = 21930 \cdot \frac{N}{m^2} - 13.55 \times 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 0.15 \cdot m \times \frac{N \cdot s^2}{kg \cdot m}$$ $p_{air} = 1.99 \cdot kPa$ (gage)

Problem 3.50

[2]

3.50 A triangular access port must be provided in the side of a form containing liquid concrete. Using the coordinates and dimensions shown, determine the resultant force that acts on the port and its point of application.

Given: Geometry of access port

Find: Resultant force and location

Solution:

Basic equation $\quad F_R = \int p\, dA \qquad \frac{dp}{dy} = \rho \cdot g \qquad \Sigma M_s = y' \cdot F_R = \int y\, dF_R = \int y \cdot p\, dA$

or, use computing equations $\quad F_R = p_c \cdot A \qquad y' = y_c + \frac{I_{xx}}{A \cdot y_c}$

We will show both methods

Assumptions: static fluid; ρ = constant; p_{atm} on other side

$$F_R = \int p\, dA = \int SG \cdot \rho \cdot g \cdot y\, dA \quad \text{but} \quad dA = w \cdot dy \quad \text{and} \quad \frac{w}{b} = \frac{y}{a} \qquad w = \frac{b}{a} \cdot y$$

Hence
$$F_R = \int_0^a SG \cdot \rho \cdot g \cdot y \cdot \frac{b}{a} \cdot y\, dy = \int_0^a SG \cdot \rho \cdot g \cdot \frac{b}{a} \cdot y^2\, dy = \frac{SG \cdot \rho \cdot g \cdot b \cdot a^2}{3}$$

Alternatively
$$F_R = p_c \cdot A \quad \text{and} \quad p_c = SG \cdot \rho \cdot g \cdot y_c = SG \cdot \rho \cdot g \cdot \frac{2}{3} \cdot a \quad \text{with} \quad A = \frac{1}{2} \cdot a \cdot b$$

Hence
$$F_R = \frac{SG \cdot \rho \cdot g \cdot b \cdot a^2}{3}$$

For y'
$$y' \cdot F_R = \int y \cdot p\, dA = \int_0^a SG \cdot \rho \cdot g \cdot \frac{b}{a} \cdot y^3\, dy = \frac{SG \cdot \rho \cdot g \cdot b \cdot a^3}{4} \qquad y' = \frac{SG \cdot \rho \cdot g \cdot b \cdot a^3}{4 \cdot F_R} = \frac{3}{4} \cdot a$$

Alternatively
$$y' = y_c + \frac{I_{xx}}{A \cdot y_c} \quad \text{and} \quad I_{xx} = \frac{b \cdot a^3}{36} \quad \text{(Google it!)}$$

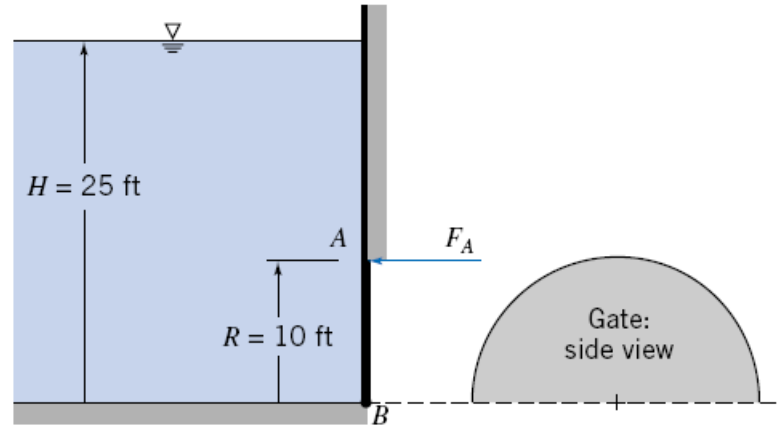
$$y' = \frac{2}{3} \cdot a + \frac{b \cdot a^3}{36} \cdot \frac{2}{a \cdot b} \cdot \frac{3}{2 \cdot a} = \frac{3}{4} \cdot a$$

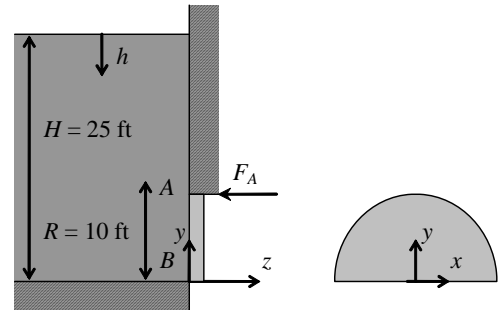
Using given data, and SG = 2.5 (Table A.1)
$$F_R = \frac{2.5}{3} \cdot 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times 1 \cdot ft \times (1.25 \cdot ft)^2 \times \frac{lbf \cdot s^2}{slug \cdot ft} \qquad F_R = 81.3 \cdot lbf$$

and
$$y' = \frac{3}{4} \cdot a \qquad y' = 0.938 \cdot ft$$

Problem 3.51

[3]

3.51 Semicircular plane gate AB is hinged along B and held by horizontal force F_A applied at A. The liquid to the left of the gate is water. Calculate the force F_A required for equilibrium.

Given: Geometry of gate

Find: Force F_A for equilibrium

Solution:

Basic equation $\quad F_R = \int p\, dA \qquad \frac{dp}{dh} = \rho \cdot g \qquad \Sigma M_z = 0$

or, use computing equations $\quad F_R = p_c \cdot A \qquad y' = y_c + \frac{I_{xx}}{A \cdot y_c}$ where y would be measured from the free surface

Assumptions: static fluid; ρ = constant; p_{atm} on other side; door is in equilibrium

Instead of using either of these approaches, we note the following, using y as in the sketch

$\Sigma M_z = 0 \qquad F_A \cdot R = \int y \cdot p\, dA$ with $p = \rho \cdot g \cdot h$ (Gage pressure, since p = p_{atm} on other side)

$F_A = \frac{1}{R} \cdot \int y \cdot \rho \cdot g \cdot h\, dA$ with $dA = r \cdot dr \cdot d\theta$ and $y = r \cdot \sin(\theta)$ $\quad h = H - y$

Hence

$$F_A = \frac{1}{R} \cdot \int_0^{\pi} \int_0^{R} \rho \cdot g \cdot r \cdot \sin(\theta) \cdot (H - r \cdot \sin(\theta)) \cdot r\, dr\, d\theta = \frac{\rho \cdot g}{R} \cdot \int_0^{\pi} \left(\frac{H \cdot R^3}{3} \cdot \sin(\theta) - \frac{R^4}{4} \cdot \sin(\theta)^2 \right) d\theta$$

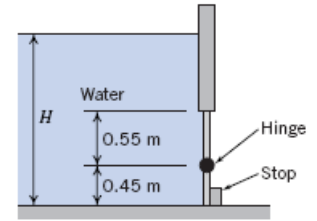
$$F_R = \frac{\rho \cdot g}{R} \cdot \left(\frac{2 \cdot H \cdot R^3}{3} - \frac{\pi \cdot R^4}{8} \right) = \rho \cdot g \cdot \left(\frac{2 \cdot H \cdot R^2}{3} - \frac{\pi \cdot R^3}{8} \right)$$

Using given data

$$F_R = 1.94 \cdot \frac{\text{slug}}{\text{ft}^3} \times 32.2 \cdot \frac{\text{ft}}{\text{s}^2} \times \left[\frac{2}{3} \times 25 \cdot \text{ft} \times (10 \cdot \text{ft})^2 - \frac{\pi}{8} \times (10 \cdot \text{ft})^3 \right] \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}} \qquad F_R = 7.96 \times 10^4 \cdot \text{lbf}$$

Problem 3.52

[3]

3.52 A rectangular gate (width $w = 2$ m) is hinged as shown, with a stop on the lower edge. At what depth H will the gate tip?

Given: Gate geometry

Find: Depth *H* at which gate tips

Solution:

This is a problem with atmospheric pressure on both sides of the plate, so we can first determine the location of the center of pressure with respect to the free surface, using Eq.3.11c (assuming depth *H*)

$$y' = y_c + \frac{I_{xx}}{A \cdot y_c} \qquad \text{and} \qquad I_{xx} = \frac{w \cdot L^3}{12} \qquad \text{with} \qquad y_c = H - \frac{L}{2}$$

where $L = 1$ m is the plate height and w is the plate width

Hence $$y' = \left(H - \frac{L}{2}\right) + \frac{w \cdot L^3}{12 \cdot w \cdot L \cdot \left(H - \frac{L}{2}\right)} = \left(H - \frac{L}{2}\right) + \frac{L^2}{12 \cdot \left(H - \frac{L}{2}\right)}$$

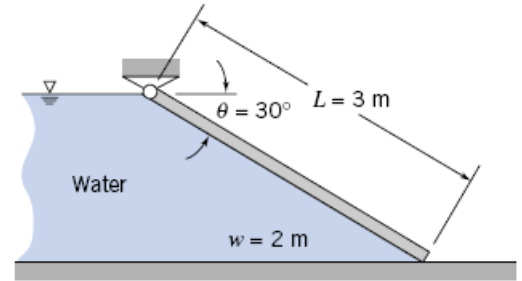
But for equilibrium, the center of force must always be at or below the level of the hinge so that the stop can hold the gate in place. Hence we must have

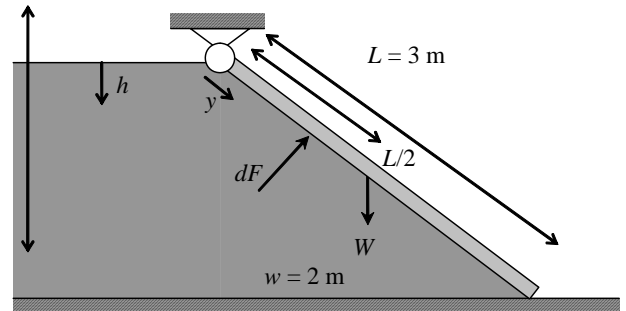
$$y' > H - 0.45 \cdot m$$

Combining the two equations $$\left(H - \frac{L}{2}\right) + \frac{L^2}{12 \cdot \left(H - \frac{L}{2}\right)} \geq H - 0.45 \cdot m$$

Solving for *H* $$H \leq \frac{L}{2} + \frac{L^2}{12 \cdot \left(\frac{L}{2} - 0.45 \cdot m\right)} \qquad H \leq \frac{1 \cdot m}{2} + \frac{(1 \cdot m)^2}{12 \times \left(\frac{1 \cdot m}{2} - 0.45 \cdot m\right)} \qquad H \leq 2.17 \cdot m$$

Problem 3.53

3.53 A plane gate of uniform thickness holds back a depth of water as shown. Find the minimum weight needed to keep the gate closed.

Given: Geometry of plane gate

Find: Minimum weight to keep it closed

Solution:

Basic equation $F_R = \int p\,dA$ $\quad \frac{dp}{dh} = \rho \cdot g$ $\quad \Sigma M_O = 0$

or, use computing equations $F_R = p_c \cdot A$ $\quad y' = y_c + \frac{I_{xx}}{A \cdot y_c}$

Assumptions: static fluid; ρ = constant; p_{atm} on other side; door is in equilibrium

Instead of using either of these approaches, we note the following, using y as in the sketch

$$\Sigma M_O = 0 \qquad W \cdot \frac{L}{2} \cdot \cos(\theta) = \int y\,dF$$

We also have $dF = p \cdot dA$ with $p = \rho \cdot g \cdot h = \rho \cdot g \cdot y \cdot \sin(\theta)$ (Gage pressure, since $p = p_{atm}$ on other side)

Hence
$$W = \frac{2}{L \cdot \cos(\theta)} \cdot \int y \cdot p\,dA = \frac{2}{L \cdot \cos(\theta)} \cdot \int y \cdot \rho \cdot g \cdot y \cdot \sin(\theta) \cdot w\,dy$$

$$W = \frac{2}{L \cdot \cos(\theta)} \cdot \int y \cdot p\,dA = \frac{2 \cdot \rho \cdot g \cdot w \cdot \tan(\theta)}{L} \cdot \int_0^L y^2\,dy = \frac{2}{3} \cdot \rho \cdot g \cdot w \cdot L^2 \cdot \tan(\theta)$$

Using given data
$$W = \frac{2}{3} \cdot 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 2 \cdot m \times (3 \cdot m)^2 \times \tan(30 \cdot deg) \times \frac{N \cdot s^2}{kg \cdot m} \qquad W = 68 \cdot kN$$

Given: Semi-cylindrical trough, partly filled with water to depth d.

Find: (a) General expressions for F_R and y' on end of trough, if open to atmosphere.
(b) Plots of results vs. d/R for $0 \le d/R \le 1$.

Solution: Apply basic equations for hydrostatics of incompressible liquid.

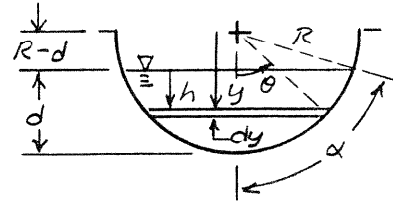
Computing equations: $p = \rho g h \quad F_R = \int_A p\,dA \quad y'F_R = \int_A y\,p\,dA$

Assumptions: (1) Static liquid
(2) ρ = constant

$$p = \rho g h = \rho g[y - (R-d)]$$

$$= \rho g R\left[\frac{y}{R} - \left(1 - \frac{d}{R}\right)\right] = \rho g R(\cos\theta - \cos\alpha)$$

$$h = y - (R-d)$$

$$dA = w\,dy = 2R\sin\theta\,dy\,; \quad y = R\cos\theta$$

$$\cos\alpha = \frac{R-d}{R} = 1 - \frac{d}{R}$$

$$dy = -R\sin\theta\,d\theta$$

$$w = 2R\sin\theta$$

$$F_R = \int_{R-d}^{R} p\,w\,dy = \int_{R-d}^{R} \rho g R(\cos\theta - \cos\alpha)\,2R\sin\theta\,(-R\sin\theta)\,d\theta$$

The new limits are $y = R \rightarrow \theta = 0$ and $y = R-d \rightarrow \theta = \alpha$, so

$$F_R = 2\rho g R^3 \int_\alpha^0 (-\sin^2\theta\cos\theta + \sin^2\theta\cos\alpha)\,d\theta = 2\rho g R^3 \int_0^\alpha (\sin^2\theta\cos\theta - \sin^2\theta\cos\alpha)\,d\theta$$

$$= 2\rho g R^3\left[\frac{\sin^3\theta}{3} - \cos\alpha\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)\right]_0^\alpha = 2\rho g R^3\left[\frac{\sin^3\theta}{3} - \cos\alpha\left(\frac{\theta}{2} - \frac{\sin\theta\cos\theta}{2}\right)\right]_0^\alpha$$

$$F_R = 2\rho g R^3\left[\frac{\sin^3\alpha}{3} - \cos\alpha\left(\frac{\alpha}{2} - \frac{\sin\alpha\cos\alpha}{2}\right)\right] \qquad F_R$$

$$y'F_R = \int_{R-d}^{R} y\,p\,w\,dy = \int_{R-d}^{R} R\cos\theta\,\rho g R(\cos\theta - \cos\alpha)\,2R\sin\theta(-R\sin\theta)\,d\theta$$

$$= 2\rho g R^4 \int_0^\alpha \sin^2\theta\cos\theta(\cos\theta - \cos\alpha)\,d\theta = 2\rho g R^4 \int_0^\alpha (\sin^2\theta\cos^2\theta - \cos\alpha\sin^2\theta\cos\theta)\,d\theta$$

$$= 2\rho g R^4\left[\frac{1}{8}\left(\theta - \frac{\sin 4\theta}{4}\right) - \cos\alpha\frac{\sin^3\theta}{3}\right]_0^\alpha$$

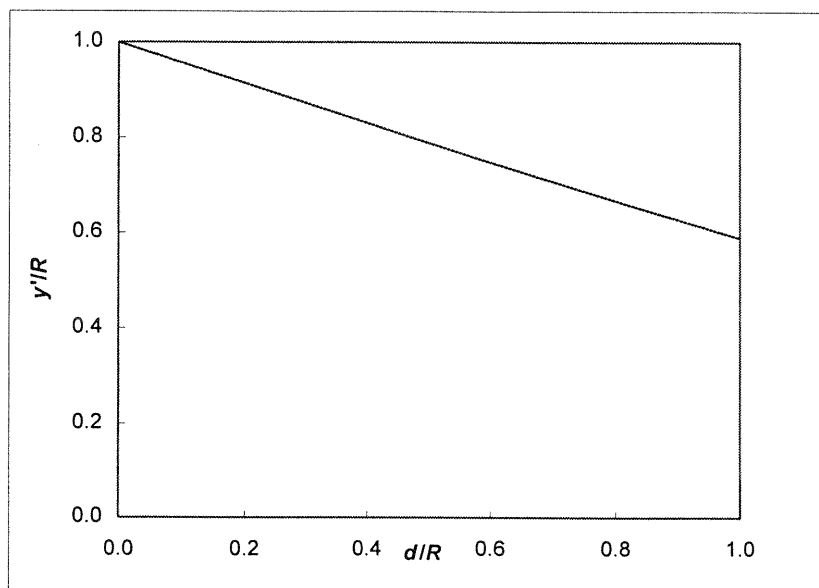
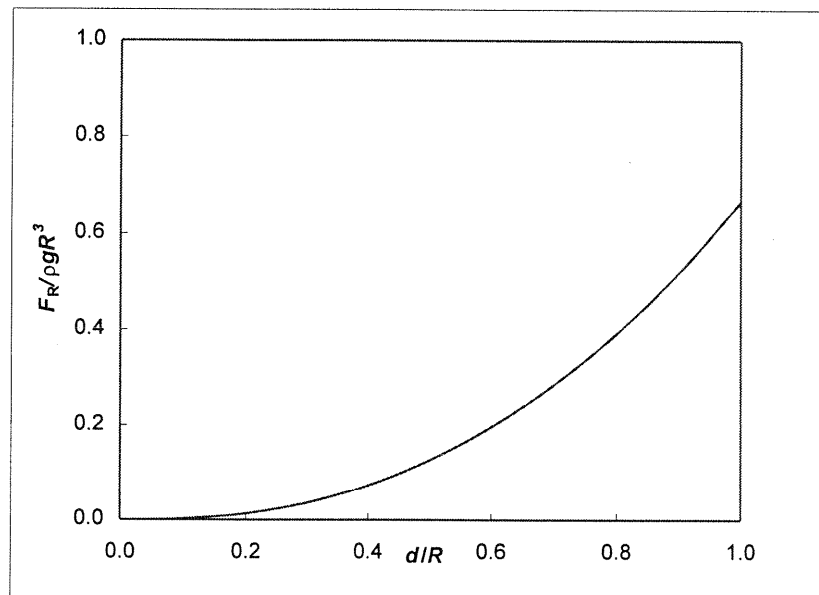
$$y'F_R = 2\rho g R^4\left[\frac{1}{8}\left(\alpha - \frac{\sin 4\alpha}{4}\right) - \cos\alpha\frac{\sin^3\alpha}{3}\right] \qquad y'F_R$$

and

$$y' = \frac{y'F_R}{F_R} \quad \text{or} \quad y'/R = \frac{y'F_R}{R F_R} \qquad y'$$

Problem 3.54

Resultant force and line of action on end of semi-cylindrical water trough:

d/R	α (rad)	α (deg)	$F_R/\rho g R^3$	$y'F_R/\rho g R^4$	y'/R
0	0.001	0.08	7.54E-16	7.54E-16	1.000
0.05	0.318	18.2	0.000419	0.000410	0.979
0.1	0.451	25.8	0.00236	0.00226	0.957
0.2	0.644	36.9	0.0132	0.0121	0.915
0.3	0.795	45.6	0.0360	0.0314	0.873
0.4	0.927	53.1	0.0730	0.0606	0.831
0.5	1.05	60.0	0.126	0.0994	0.790
0.6	1.16	66.4	0.196	0.147	0.749
0.7	1.27	72.5	0.285	0.202	0.708
0.8	1.37	78.5	0.392	0.262	0.668
0.9	1.47	84.3	0.520	0.326	0.628
1.0	1.57	90.0	0.667	0.393	0.589

Problem 3.55

3.55 For a mug of tea (2½ in. diameter), imagine it cut symmetrically in half by a vertical plane. Find the force that each half experiences due to a 3-in. depth of tea.

Given: Geometry of cup

Find: Force on each half of cup

Solution:

Basic equation $F_R = \int p\,dA \qquad \frac{dp}{dh} = \rho \cdot g$

or, use computing equation $F_R = p_c \cdot A$

Assumptions: static fluid; ρ = constant; p_{atm} on other side; cup does not crack!

The force on the half-cup is the same as that on a rectangle of size $h = 3 \cdot in$ and $w = 2.5 \cdot in$

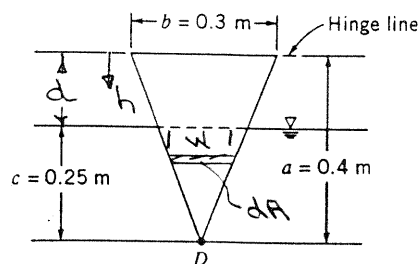
$$F_R = \int p\,dA = \int \rho \cdot g \cdot y\,dA \qquad \text{but} \qquad dA = w \cdot dy$$

Hence $F_R = \int_0^h \rho \cdot g \cdot y \cdot w\,dy = \frac{\rho \cdot g \cdot w \cdot h^2}{2}$

Alternatively $F_R = p_c \cdot A$ and $F_R = p_c \cdot A = \rho \cdot g \cdot y_c \cdot A = \rho \cdot g \cdot \frac{h}{2} \cdot h \cdot w = \frac{\rho \cdot g \cdot w \cdot h^2}{2}$

Using given data $F_R = \frac{1}{2} \cdot 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times 2.5 \cdot in \times (3 \cdot in)^2 \times \left(\frac{1 \cdot ft}{12 \cdot in}\right)^3 \times \frac{lbf \cdot s^2}{slug \cdot ft}$ $\qquad F_R = 0.407 \cdot lbf$

Hence a teacup is being forced apart by about 0.4 lbf: not much of a force, so a paper cup works!

Given: Window, in shape of isosceles triangle and hinged at the top is located in the vertical wall of a form that contains concrete.

Find: the minimum force applied at D needed to keep the window closed.

Plot: the results over the range of concrete depth $0 \le c \le a$

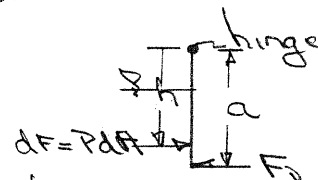
Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $F = \int p\,dA$, $\Sigma M = 0$

Assumptions: (1) static fluid (2) ρ = constant
(3) p_{atm} acts at the free surface and on the outside of the window.

Then $dp = \rho g\,dh$ gives $p = \rho g(h-d)$ for $h > d$
and $p = 0$ for $h < d$
where $d = a - c$

Summing moments about the hinge

$$F_D = \frac{1}{a}\int h\,p\,dA = \frac{1}{a}\int_d^a h\,\rho g(h-d)\,w\,dh$$

From the law of similar triangles $\frac{w}{b} = \frac{a-h}{a}$; $w = \frac{b}{a}(a-h)$

$$F_D = \frac{b}{a^2}\rho g\int_d^a h(h-d)(a-h)\,dh \qquad \{\rho = SG_{concrete}\,\rho_{H_2O}\}$$

$$F_D = \frac{b}{a^2}\rho g\int_d^a \left[-h^3 + h^2(a+d) - adh\right]dh$$

$$F_D = \frac{b}{a^2}\rho g\left[-\frac{h^4}{4} + \frac{h^3}{3}(a+d) - \frac{1}{2}adh^2\right]_d^a$$

$$F_D = \frac{b}{a^2}\rho g\left[-\frac{1}{4}(a^4-d^4) + \frac{1}{3}(a^3-d^3)(a+d) - \frac{1}{2}ad(a^2-d^2)\right]$$

$$F_D = b\rho g a^2\left[-\frac{1}{4}\left(1-\frac{d^4}{a^4}\right) + \frac{1}{3}\left(1-\frac{d^3}{a^3}\right)\left(1+\frac{d}{a}\right) - \frac{1}{2}\frac{d}{a}\left(1-\frac{d^2}{a^2}\right)\right] \quad (1)$$

Evaluating with $\rho = SG_{conc}\,\rho_{H_2O}$ ($SG = 2.5$ - Table A.1)

$$b\rho g a^2 = 0.3\,m \times 2.5 \times 10^3\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times (0.4)^2\,m^2 \times \frac{N\cdot s^2}{kg\cdot m} = 1177\,N$$

For $a = 0.4\,m$, $c = 0.25\,m$, $d = a - c = 0.15\,m$, $\frac{d}{a} = 0.375$

the term [ ] in Eq. 1 has a value of 0.0280

Problem 3.56

Then for the conditions given

$$F_D = 1177\,N \times 0.0280 = 33.0\,N$$

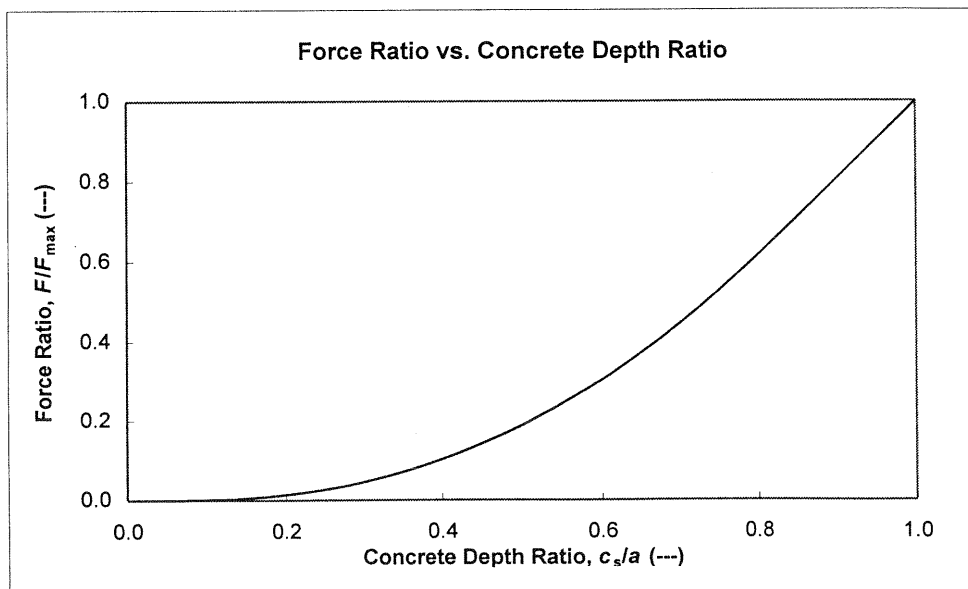
To plot F_D vs c/a for $0 \le c \le a$, recognize. Since $d = a - c$, then $\frac{d}{a} = 1 - \frac{c}{a}$

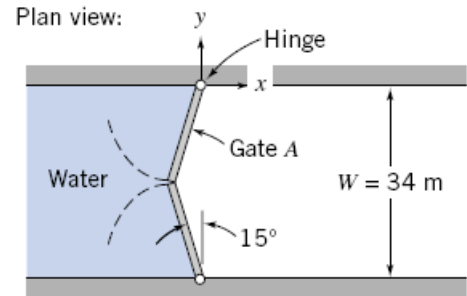
and

$$F_D = 1177\,N\left[-\frac{1}{4}\left\{1-\left(\frac{d}{a}\right)^4\right\} + \frac{1}{3}\left\{1-\left(\frac{d}{a}\right)^3\right\}\left(1+\frac{d}{a}\right) - \frac{1}{2}\frac{d}{a}\left\{1-\left(\frac{d}{a}\right)^2\right\}\right]$$

The results are plotted below

Hinge force vs. concrete depth ratio:

Depth Ratio, c/a (---)	Depth Ratio, d/a (---)	Force Ratio, F/F_{max} (---)
0	1.0	0.0000
0.1	0.9	0.0019
0.2	0.8	0.0144
0.3	0.7	0.0459
0.4	0.6	0.102
0.5	0.5	0.187
0.6	0.4	0.302
0.625	0.375	0.336
0.7	0.3	0.446
0.8	0.2	0.614
0.9	0.1	0.802
1.0	0.0	1.000

Problem 3.57

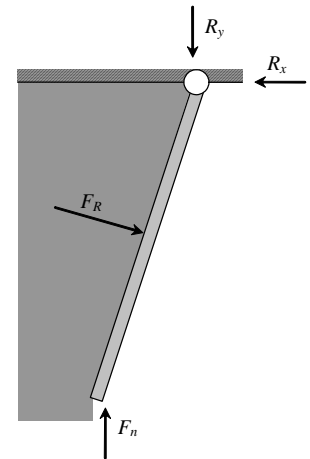
3.57 Gates in the Poe Lock at Sault Ste. Marie, Michigan, close a channel $W = 34$ m wide, $L = 360$ m long, and $D = 10$ m deep. The geometry of one pair of gates is shown; each gate is hinged at the channel wall. When closed, the gate edges are forced together at the center of the channel by water pressure. Evaluate the force exerted by the water on gate A. Determine the magnitude and direction of the force components exerted by the gate on the hinge. (Neglect the weight of the gate.)

Given: Geometry of lock system

Find: Force on gate; reactions at hinge

Solution:

Basic equation $\quad F_R = \int p\,dA \qquad \frac{dp}{dh} = \rho \cdot g$

or, use computing equation $\quad F_R = p_c \cdot A$

Assumptions: static fluid; ρ = constant; p_{atm} on other side

The force on each gate is the same as that on a rectangle of size $\quad h = D = 10 \cdot m \quad$ and $\quad w = \frac{W}{2 \cdot \cos(15 \cdot deg)}$

$$F_R = \int p\,dA = \int \rho \cdot g \cdot y\,dA \qquad \text{but} \qquad dA = w \cdot dy$$

Hence $\quad F_R = \int_0^h \rho \cdot g \cdot y \cdot w\,dy = \frac{\rho \cdot g \cdot w \cdot h^2}{2}$

Alternatively $\quad F_R = p_c \cdot A \quad$ and $\quad F_R = p_c \cdot A = \rho \cdot g \cdot y_c \cdot A = \rho \cdot g \cdot \frac{h}{2} \cdot h \cdot w = \frac{\rho \cdot g \cdot w \cdot h^2}{2}$

Using given data $\quad F_R = \frac{1}{2} \cdot 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times \frac{34 \cdot m}{2 \cdot \cos(15 \cdot deg)} \times (10 \cdot m)^2 \times \frac{N \cdot s^2}{kg \cdot m} \qquad F_R = 8.63 \cdot MN$

For the force components R_x and R_y we do the following

$$\Sigma M_{hinge} = 0 = F_R \cdot \frac{w}{2} - F_n \cdot w \cdot \sin(15 \cdot deg) \qquad F_n = \frac{F_R}{2 \cdot \sin(15 \cdot deg)} \qquad F_n = 16.7 \cdot MN$$

$$\Sigma F_x = 0 = F_R \cdot \cos(15 \cdot deg) - R_x = 0 \qquad R_x = F_R \cdot \cos(15 \cdot deg) \qquad R_x = 8.34 \cdot MN$$

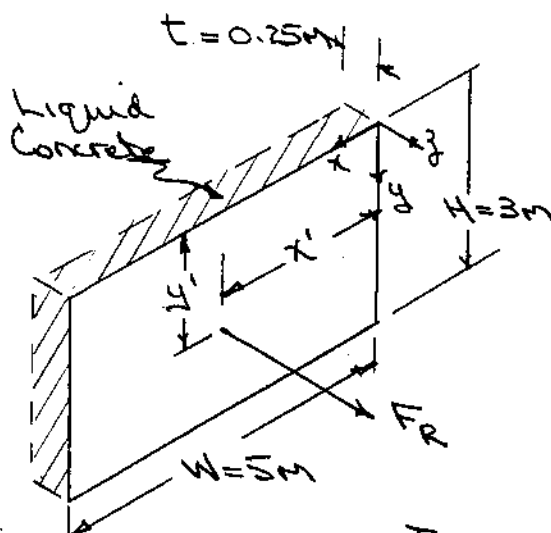
$$\Sigma F_y = 0 = -R_y - F_R \cdot \sin(15 \cdot deg) + F_n = 0 \qquad R_y = F_n - F_R \cdot \sin(15 \cdot deg) \qquad R_y = 14.4 \cdot MN$$

$$R = (8.34 \cdot MN, 14.4 \cdot MN) \qquad R = 16.7 \cdot MN$$

Problem 3.58

Given: Liquid concrete poured between vertical forms as shown

Find: (a) Resultant force on form
(b) Line of application

Liquid Concrete
$t = 0.25\,m$
$H = 3\,m$
$W = 5\,m$
x, y, z, x', y', F_R

Solution:

Basic equation: $\frac{dp}{dy} = \rho g$

Computing equations:

$F_R = p_c A$ (3.14) ; $y' = y_c + \frac{I_{\hat{x}\hat{x}}}{A y_c}$ (3.15a); $x' = x_c + \frac{I_{\hat{x}\hat{y}}}{A y_c}$

For the rectangular plate: $x_c = 2.5\,m$, $y_c = 1.5\,m$.
$I_{\hat{x}\hat{x}} = \frac{1}{12} W H^3$, $I_{\hat{x}\hat{y}} = 0$

Assumptions: (1) static liquid (2) incompressible liquid
(3) p_{atm} acts at free surface and on the vertical form.

Then on integrating $dp = \rho g\,dy$, we obtain $p = \rho g y$

$F_R = p_c A = \rho g y_c A = \rho g y_c W H = SG_{con} \rho_{H_2O} y_c W H$

$F_R = 2.5 \times 10^3 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 1.5\,m \times 5\,m \times 3\,m \times \frac{N \cdot s^2}{kg \cdot m}$ {SG = 2.5, Table A.1}

$F_R = 552\,kN$ ← F_R

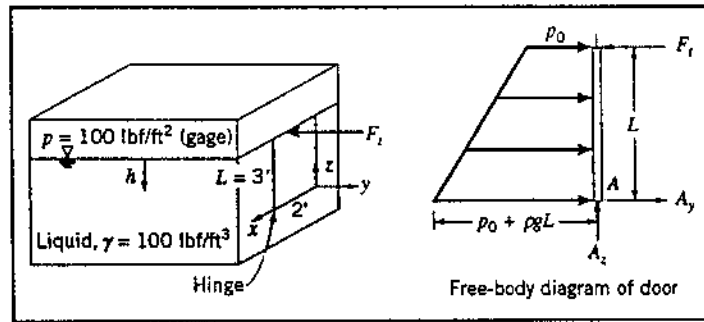
$y' = y_c + \frac{I_{\hat{x}\hat{x}}}{A y_c} = y_c + \frac{1}{12}\frac{W H^3}{W H y_c} = y_c + \frac{1}{12}\frac{H^2}{y_c} = 1.5\,m + \frac{1}{12}\frac{(3\,m)^2}{1.5\,m} = 2.0\,m$

$x' = x_c = 2.5\,m$

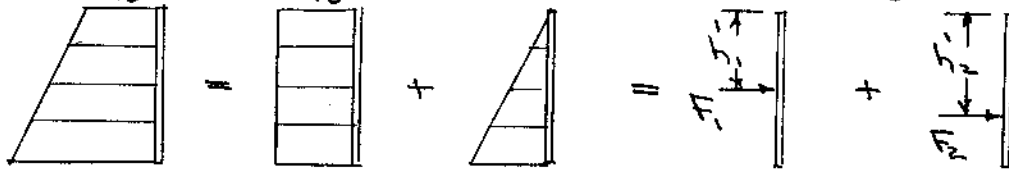
Line of application is through $(x', y') = (2.5, 2.0)\,m$ ← (x', y')

Problem 3.59

Given: Door as shown in the figure; x axis is along the hinge
From Ex. Prob 3.6, pressure in liquid is $p = p_{o\,gage} + \rho g h$

Free-body diagram of door

Find: Force required to keep door shut (by considering the distributed force to be the sum of a force F_1 caused by uniform gage pressure, and force F_2 caused by the liquid)

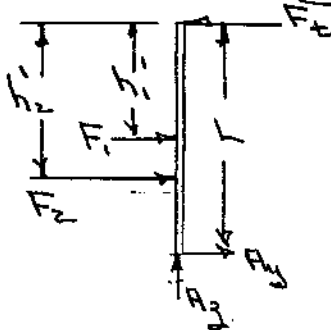
Solution:

Computing equations: $F_R = p_c A$; $y' = y_c + \frac{I_{\hat{x}\hat{x}}}{y_c A}$; $I_{\hat{x}\hat{x}} = \frac{bL^3}{12}$

$F_1 = p_o A = 100 \frac{lbf}{ft^2} \times 3ft \times 2ft = 600\ lb$ { applied at $(x', z') = (1.0, 1.5)$ ft

$F_2 = p_c A = \rho g h_c L b = \gamma h_c L b = 100 \frac{lb}{ft^3} \times 1.5 ft \times 3 ft \times 2 ft = 900\ lbf$.

For the rectangular door $I_{\hat{x}\hat{x}} = \frac{1}{12} b L^3$

$h_2' = h_c + \frac{I_{\hat{x}\hat{x}}}{A h_c} = h_c + \frac{1}{12}\frac{bL^3}{bL h_c} = h_c + \frac{1}{12}\frac{L^2}{h_c} = 1.5m + \frac{1}{12}\frac{(3m)^2}{1.5m} = 2.0m$

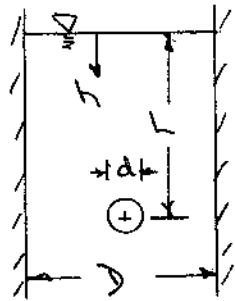
The free-body diagram of the door is then

$\Sigma M_{Ax} = 0 = L F_t - F_1(L - h_1') - F_2(L - h_2')$.

$F_t = F_1\left(1 - \frac{h_1'}{L}\right) + F_2\left(1 - \frac{h_2'}{L}\right)$

$= 600 lb\left(1 - \frac{1.5}{3.0}\right) + 900 lb\left(1 - \frac{2}{3}\right)$

$F_t = 600\ lb$ ← F_t

Problem 3.60

Given: Circular access port, of diameter $d = 0.6\,m$, in side of water standpipe, of diameter $D = 7\,m$, is held in place by eight bolts evenly spaced around circumference of the port.
Center of the port is located at distance $L = 12\,m$ below the free surface of the water

Find: (a) Total force on the port
(b) Appropriate bolt diameter

Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $\sigma = \frac{F}{A}$

Computing equation: $F_R = p_c A$

Assumptions: (1) static fluid
(2) incompressible
(3) force distributed uniformly over the bolts
(4) appropriate working stress for steel bolts is $\sigma = 100$ MPa
(5) p_{atm} acts at free surface and on the outside of the port

Then on integrating $dp = \rho g\,dh$ we obtain $p = \rho g h$

$F_R = p_c A = \rho g h_c \pi R^2 = \rho g L \pi R^2$

$F_R = 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 12\,m \times \pi \times (0.3\,m)^2 \times \frac{N \cdot s^2}{kg \cdot m} = 33.3\,kN$ — F_R

$\sigma = \frac{F}{A}$ where A (total area of bolts) $= 8 \times \frac{\pi d_b^2}{4}$

Then

$\sigma = \frac{F}{2\pi d_b^2}$

$d_b = \left[\frac{F}{2\pi\sigma}\right]^{1/2} = \left[\frac{33.3 \times 10^3\,N}{2\pi} \times 10^{-8}\frac{m^2}{N} \times 10^6\frac{mm^2}{m^2}\right]^{1/2} = 7.28\,mm$ — d_b

Problem 3.61

[1]

3.61 What holds up a car on its rubber tires? Most people would tell you that it is the air pressure inside the tires. However, the air pressure is the same all around the hub (inner wheel), and the air pressure inside the tire therefore pushes down from the top as much as it pushes up from below, having no net effect on the hub. Resolve this paradox by explaining where the force is that keeps the car off the ground.

Given: Description of car tire

Find: Explanation of lift effect

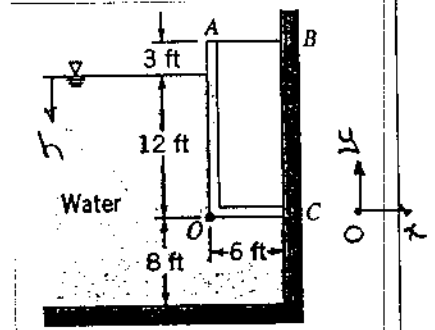
Solution:

The explanation is as follows: It is true that the pressure in the entire tire is the same everywhere. However, the tire at the top of the hub will be essentially circular in cross-section, but at the bottom, where the tire meets the ground, the cross section will be approximately a flattened circle, or elliptical. Hence we can explain that the lower cross section has greater upward force than the upper cross section has downward force (providing enough lift to keep the car up) two ways. First, the horizontal projected area of the lower ellipse is larger than that of the upper circular cross section, so that net pressure times area is upwards. Second, any time you have an elliptical cross section that's at high pressure, that pressure will always try to force the ellipse to be circular (thing of a round inflated balloon - if you squeeze it it will resist!). This analysis ignores the stiffness of the tire rubber, which also provides a little lift.

Problem 3.62

Given: Gate AOC, hinged along O, has width $b = 6$ ft; weight of gate may be neglected. Gate is sealed at C.

Find: Force in bar AB

Solution:

Basic equations: $\frac{dp}{dh} = \rho g$; $\Sigma M_z = 0$

Computing equations: $F_R = p_c A$; $y' = y_c + \frac{I_{\hat{x}\hat{x}}}{y_c A}$; $I_{\hat{x}\hat{x}} = \frac{bL^3}{12}$

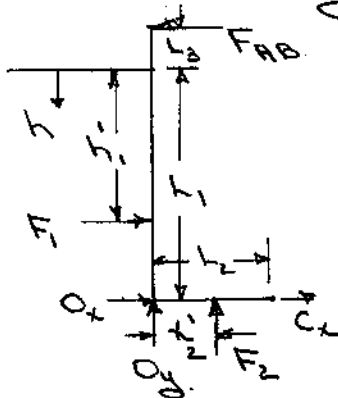
Assumptions: (1) static liquid (2) ρ = constant

(3) p_{atm} acts at free surface and on outside of gate

(4) no resisting moment in hinge along O

(5) no vertical resisting force at C

Then on integrating $dp = \rho g dh$, we obtain $p = \rho g h$

The free body diagram of the gate is as shown.

F_1 is resultant of distributed force on L_1

F_2 " " " uniform force on L_2

F_{AB} is force of bar

C_x is force from seal at C

$F_1 = p_c A_1 = \rho g h_{c_1} b L_1$

$F_1 = 1.94 \frac{slug}{ft^3} \times 32.2 \frac{ft}{s^2} \times 6ft \times 6ft \times 12ft \times \frac{lbf \cdot s^2}{ft \cdot slug} = 27.0 \times 10^3 lbf$

$h_1' = h_{c_1} + \frac{bL_1^3}{12 h_{c_1} b L_1} = \frac{L_1}{2} + \frac{L_1^2}{12} \frac{2}{L_1} = \frac{L_1}{2} + \frac{L_1}{6} = \frac{2}{3} L_1 = \frac{2}{3} \times 12 ft = 8 ft$

$F_2 = p_c A_2 = \rho g h_c b L_2 = \rho g L_1 b L_2$

$F_2 = 1.94 \frac{slug}{ft^3} \times 32.2 \frac{ft}{s^2} \times 12ft \times 6ft \times 6ft = 27.0 \times 10^3 lbf$

Since the pressure is uniform over surface (2), the force F_2 acts at the centroid of the surface, i.e. $x_2' = L_2/2 = 3 ft$

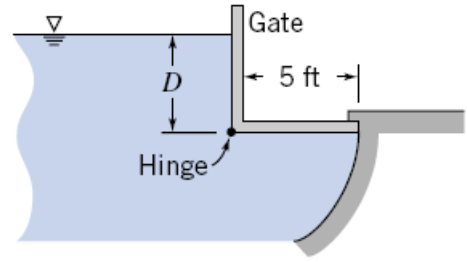
Then summing moments about O gives

$\Sigma M_{O_z} = 0 = (L_1 + L_3) F_{AB} + x_2' F_2 - (L_1 - h_1') F_1$

$F_{AB} = \frac{1}{(L_1 + L_3)} \left[(L_1 - h_1') F_1 - x_2' F_2 \right] = \frac{1}{15 ft} \left[(12-8) ft \times 27{,}000 lbf - 3 ft \times 27{,}000 lbf \right]$

$F_{AB} = 1800 lbf$ ← F_{AB}

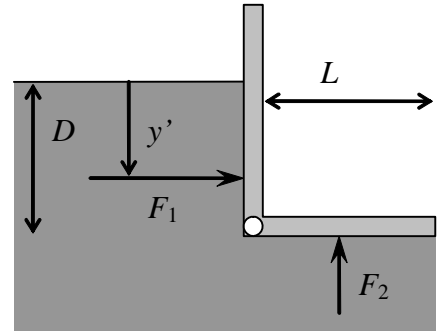
Thus bar AB is in compression

3.63 As water rises on the left side of the rectangular gate, the gate will open automatically. At what depth above the hinge will this occur? Neglect the mass of the gate.

Given: Geometry of rectangular gate

Find: Depth for gate to open

Solution:

Basic equation $\frac{dp}{dh} = \rho \cdot g$ $\Sigma M_z = 0$

Computing equations $F_R = p_c \cdot A$ $y' = y_c + \frac{I_{xx}}{A \cdot y_c}$ $I_{xx} = \frac{b \cdot D^3}{12}$

Assumptions: static fluid; ρ = constant; p_{atm} on other side; no friction in hinge

For incompressible fluid $p = \rho \cdot g \cdot h$ where p is gage pressure and h is measured downwards

The force on the vertical gate (gate 1) is the same as that on a rectangle of size h = D and width w

Hence $$F_1 = p_c \cdot A = \rho \cdot g \cdot y_c \cdot A = \rho \cdot g \cdot \frac{D}{2} \cdot D \cdot w = \frac{\rho \cdot g \cdot w \cdot D^2}{2}$$

The location of this force is $$y' = y_c + \frac{I_{xx}}{A \cdot y_c} = \frac{D}{2} + \frac{w \cdot D^3}{12} \times \frac{1}{w \cdot D} \times \frac{2}{D} = \frac{2}{3} \cdot D$$

The force on the horizontal gate (gate 2) is due to constant pressure, and is at the centroid

$$F_2 = p(y = D) \cdot A = \rho \cdot g \cdot D \cdot w \cdot L$$

Summing moments about the hinge $$\Sigma M_{hinge} = 0 = -F_1 \cdot (D - y') + F_2 \cdot \frac{L}{2} = -F_1 \cdot \left(D - \frac{2}{3} \cdot D\right) + F_2 \cdot \frac{L}{2}$$

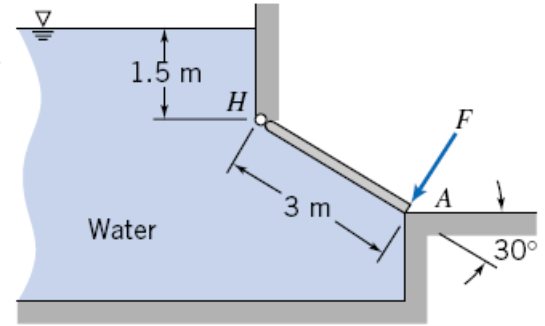
$$F_1 \cdot \frac{D}{3} = \frac{\rho \cdot g \cdot w \cdot D^2}{2} \cdot \frac{D}{3} = F_2 \cdot \frac{L}{2} = \rho \cdot g \cdot D \cdot w \cdot L \cdot \frac{L}{2}$$

$$\frac{\rho \cdot g \cdot w \cdot D^3}{6} = \frac{\rho \cdot g \cdot D \cdot w \cdot L^2}{2}$$

$$D = \sqrt{3} \cdot L = \sqrt{3} \times 5ft$$

$$D = 8.66 \cdot ft$$

Problem 3.64

3.64 The gate shown is hinged at *H*. The gate is 3 m wide normal to the plane of the diagram. Calculate the force required at *A* to hold the gate closed.

Given: Geometry of gate

Find: Force at A to hold gate closed

Solution:

Basic equation $\frac{dp}{dh} = \rho \cdot g$ $\quad \Sigma M_z = 0$

Computing equations $F_R = p_c \cdot A$ $\quad y' = y_c + \frac{I_{xx}}{A \cdot y_c}$ $\quad I_{xx} = \frac{w \cdot L^3}{12}$

Assumptions: static fluid; ρ = constant; p_{atm} on other side; no friction in hinge

For incompressible fluid $p = \rho \cdot g \cdot h$ where p is gage pressure and h is measured downwards

The hydrostatic force on the gate is that on a rectangle of size L and width w.

Hence
$$F_R = p_c \cdot A = \rho \cdot g \cdot h_c \cdot A = \rho \cdot g \cdot \left(D + \frac{L}{2} \cdot \sin(30 \cdot deg) \right) \cdot L \cdot w$$

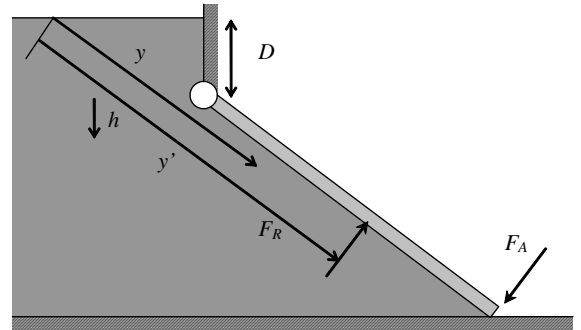
$$F_R = 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times \left(1.5 + \frac{3}{2} \sin(30 \cdot deg) \right) \cdot m \times 3 \cdot m \times 3 \cdot m \times \frac{N \cdot s^2}{kg \cdot m} \qquad F_R = 199 \cdot kN$$

The location of this force is given by $y' = y_c + \frac{I_{xx}}{A \cdot y_c}$ where y' and y_c are measured along the plane of the gate to the free surface

$$y_c = \frac{D}{\sin(30 \cdot deg)} + \frac{L}{2} \qquad y_c = \frac{1.5 \cdot m}{\sin(30 \cdot deg)} + \frac{3 \cdot m}{2} \qquad y_c = 4.5\,m$$

$$y' = y_c + \frac{I_{xx}}{A \cdot y_c} = y_c + \frac{w \cdot L^3}{12} \cdot \frac{1}{w \cdot L} \cdot \frac{1}{y_c} = y_c + \frac{L^2}{12 \cdot y_c} = 4.5 \cdot m + \frac{(3 \cdot m)^2}{12 \cdot 4.5 \cdot m} \qquad y' = 4.67\,m$$

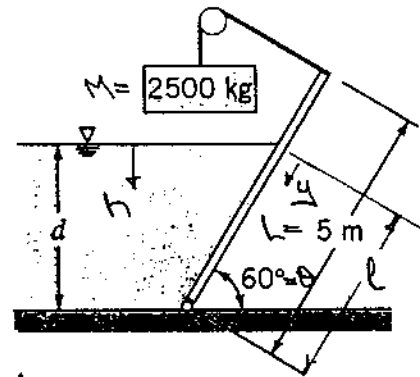
Taking moments about the hinge
$$\Sigma M_H = 0 = F_R \cdot \left(y' - \frac{D}{\sin(30 \cdot deg)} \right) - F_A \cdot L$$

$$F_A = F_R \cdot \frac{\left(y' - \frac{D}{\sin(30 \cdot deg)} \right)}{L} \qquad F_A = 199 \cdot kN \cdot \frac{\left(4.67 - \frac{1.5}{\sin(30 \cdot deg)} \right)}{3} \qquad F_A = 111 \cdot kN$$

Problem 3.65

Given: Gate shown has width $b = 3\,m$; mass of gate is negligible.
Gate is in equilibrium

M= 2500 kg, L= 5 m, 60°=θ

Find: Water depth, d

Solution:

Basic equation: $\frac{dp}{dh} = \rho g \qquad \Sigma M_z = 0$

Computing equations: $F_R = p_c A$; $y' = y_c + \frac{I_{\hat{x}\hat{x}}}{y_c A}$; $I_{\hat{x}\hat{x}} = \frac{bL^3}{12}$

Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts at free surface and on underside of gate.

Then on integrating $dp = \rho g dh$, we obtain $p = \rho g h$

$F_R = p_c A = \rho g h_c A \qquad h_c = \frac{d}{2}, \; A = b \times \frac{d}{\sin\theta}$

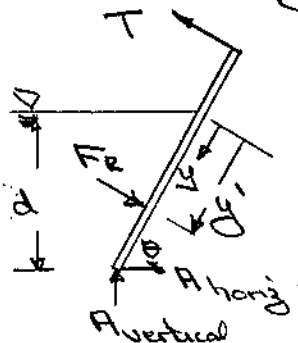
$F_R = \rho g \frac{d}{2} \frac{db}{\sin\theta} = \frac{\rho g b d^2}{2\sin\theta}$

$y' = y_c + \frac{I_{\hat{x}\hat{x}}}{y_c A} = y_c + \frac{1}{12}\frac{b l^3}{y_c l b}$ where l is length of gate in contact with the water

$y' = y_c + \frac{l^2}{12 y_c} \qquad l = \frac{d}{\sin\theta}, \; y_c = \frac{l}{2} = \frac{d}{2\sin\theta}$

$y' = \frac{d}{2\sin\theta} + \frac{1}{12}\left(\frac{d}{\sin\theta}\right)^2 \frac{2\sin\theta}{d} = \frac{d}{2\sin\theta} + \frac{d}{6\sin\theta} = \frac{2d}{3\sin\theta}$

The free body diagram of the gate is as shown.

T, F_R, y, y', d, θ, A horiz, A vertical

Summing moments about A

$\Sigma M_z = 0 = TL - (l - y')F_R \qquad T = Mg$

$MgL = (l - y')F_R = \left(\frac{d}{\sin\theta} - \frac{2d}{3\sin\theta}\right)\frac{\rho g b d^2}{2\sin\theta}$

$MgL = \frac{1}{3}\frac{d}{\sin\theta} \times \frac{\rho g b d^2}{2\sin\theta} = \frac{\rho g b d^3}{6 \sin^2\theta}$

$d^3 = \frac{6\sin^2\theta\, ML}{\rho b}$

$d = \left[6 \times \sin^2 60° \times 2500\,kg \times 5\,m \times \frac{m^3}{999\,kg} \times \frac{1}{3\,m}\right]^{1/3} = 2.66\,m$ ← d

Problem 3.66

[4]

3.66 A solid concrete dam is to be built to hold back a depth D of water. For ease of construction the walls of the dam must be planar. Your supervisor asks you to consider the following dam cross-sections: a rectangle, a right triangle with the hypotenuse in contact with the water, and a right triangle with the vertical in contact with the water. She wishes you to determine which of these would require the least amount of concrete. What will your report say? You decide to look at one more possibility: a nonright triangle, as shown. Develop and plot an expression for the cross-section area A as a function of α, and find the minimum cross-sectional area.

Given: Various dam cross-sections

Find: Which requires the least concrete; plot cross-section area A as a function of α

Solution:

For each case, the dam width b has to be large enough so that the weight of the dam exerts enough moment to balance the moment due to fluid hydrostatic force(s). By doing a moment balance this value of b can be found

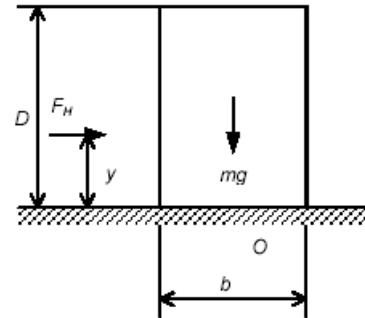
a) Rectangular dam

Straightforward application of the computing equations of Section 3-5 yields

$$F_H = p_c \cdot A = \rho \cdot g \cdot \frac{D}{2} \cdot w \cdot D = \frac{1}{2} \cdot \rho \cdot g \cdot D^2 \cdot w$$

$$y' = y_c + \frac{I_{xx}}{A \cdot y_c} = \frac{D}{2} + \frac{w \cdot D^3}{12 \cdot w \cdot D \cdot \frac{D}{2}} = \frac{2}{3} \cdot D$$

so $$y = D - y' = \frac{D}{3}$$

Also $$m = \rho_{cement} \cdot g \cdot b \cdot D \cdot w = SG \cdot \rho \cdot g \cdot b \cdot D \cdot w$$

Taking moments about O $$\sum M_{0.} = 0 = -F_H \cdot y + \frac{b}{2} \cdot m \cdot g$$

so $$\left(\frac{1}{2} \cdot \rho \cdot g \cdot D^2 \cdot w\right) \cdot \frac{D}{3} = \frac{b}{2} \cdot (SG \cdot \rho \cdot g \cdot b \cdot D \cdot w)$$

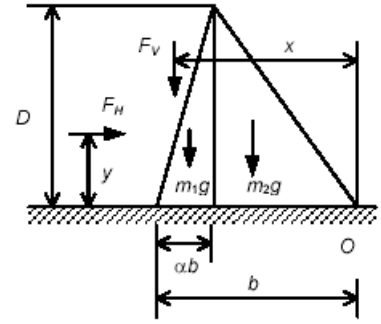
Solving for b $$b = \frac{D}{\sqrt{3 \cdot SG}}$$

The minimum rectangular cross-section area is $$A = b \cdot D = \frac{D^2}{\sqrt{3 \cdot SG}}$$

For concrete, from Table A.1, SG = 2.4, so $$A = \frac{D^2}{\sqrt{3 \cdot SG}} = \frac{D^2}{\sqrt{3 \times 2.4}}$$ $$A = 0.373 \cdot D^2$$

a) Triangular dams

Instead of analysing right-triangles, a general analysis is made, at the end of which right triangles are analysed as special cases by setting $\alpha = 0$ or 1.

Straightforward application of the computing equations of Section 3-5 yields

$$F_H = p_c \cdot A = \rho \cdot g \cdot \frac{D}{2} \cdot w \cdot D = \frac{1}{2} \cdot \rho \cdot g \cdot D^2 \cdot w$$

$$y' = y_c + \frac{I_{xx}}{A \cdot y_c} = \frac{D}{2} + \frac{w \cdot D^3}{12 \cdot w \cdot D \cdot \frac{D}{2}} = \frac{2}{3} \cdot D$$

so $$y = D - y' = \frac{D}{3}$$

Also $$F_V = \rho \cdot V \cdot g = \rho \cdot g \cdot \frac{\alpha \cdot b \cdot D}{2} \cdot w = \frac{1}{2} \cdot \rho \cdot g \cdot \alpha \cdot b \cdot D \cdot w \qquad x = (b - \alpha \cdot b) + \frac{2}{3} \cdot \alpha \cdot b = b \cdot \left(1 - \frac{\alpha}{3}\right)$$

For the two triangular masses

$$m_1 = \frac{1}{2} \cdot SG \cdot \rho \cdot g \cdot \alpha \cdot b \cdot D \cdot w \qquad x_1 = (b - \alpha \cdot b) + \frac{1}{3} \cdot \alpha \cdot b = b \cdot \left(1 - \frac{2 \cdot \alpha}{3}\right)$$

$$m_2 = \frac{1}{2} \cdot SG \cdot \rho \cdot g \cdot (1 - \alpha) \cdot b \cdot D \cdot w \qquad x_2 = \frac{2}{3} \cdot b(1 - \alpha)$$

Taking moments about *O*

$$\sum M_{0.} = 0 = -F_H \cdot y + F_V \cdot x + m_1 \cdot g \cdot x_1 + m_2 \cdot g \cdot x_2$$

so

$$-\left(\frac{1}{2} \cdot \rho \cdot g \cdot D^2 \cdot w\right) \cdot \frac{D}{3} + \left(\frac{1}{2} \cdot \rho \cdot g \cdot \alpha \cdot b \cdot D \cdot w\right) \cdot b \cdot \left(1 - \frac{\alpha}{3}\right) \ldots$$
$$+ \left(\frac{1}{2} \cdot SG \cdot \rho \cdot g \cdot \alpha \cdot b \cdot D \cdot w\right) \cdot b \cdot \left(1 - \frac{2 \cdot \alpha}{3}\right) + \left[\frac{1}{2} \cdot SG \cdot \rho \cdot g \cdot (1 - \alpha) \cdot b \cdot D \cdot w\right] \cdot \frac{2}{3} \cdot b(1 - \alpha) = 0$$

Solving for *b* $$b = \frac{D}{\sqrt{\left(3 \cdot \alpha - \alpha^2\right) + SG \cdot (2 - \alpha)}}$$

For a right triangle with the hypotenuse in contact with the water, $\alpha = 1$, and

$$b = \frac{D}{\sqrt{3 - 1 + SG}} = \frac{D}{\sqrt{3 - 1 + 2.4}} \qquad b = 0.477 \cdot D$$

The cross-section area is $$A = \frac{b \cdot D}{2} = 0.238 \cdot D^2 \qquad A = 0.238 \cdot D^2$$

For a right triangle with the vertical in contact with the water, $\alpha = 0$, and

$$b = \frac{D}{\sqrt{2 \cdot SG}} = \frac{D}{\sqrt{2 \cdot 2.4}} \qquad b = 0.456 \cdot D$$

The cross-section area is

$$A = \frac{b \cdot D}{2} = 0.228 \cdot D^2 \qquad A = 0.228 \cdot D^2$$

For a general triangle

$$A = \frac{b \cdot D}{2} = \frac{D^2}{2 \cdot \sqrt{\left(3 \cdot \alpha - \alpha^2\right) + SG \cdot (2 - \alpha)}} \qquad A = \frac{D^2}{2 \cdot \sqrt{\left(3 \cdot \alpha - \alpha^2\right) + 2.4 \cdot (2 - \alpha)}}$$

The final result is

$$A = \frac{D^2}{2 \cdot \sqrt{4.8 + 0.6 \cdot \alpha - \alpha^2}}$$

From the corresponding Excel workbook, the minimum area occurs at $\alpha = 0.3$

$$A_{min} = \frac{D^2}{2 \cdot \sqrt{4.8 + 0.6 \times 0.3 - 0.3^2}} \qquad A = 0.226 \cdot D^2$$

The final results are that a triangular cross-section with $\alpha = 0.3$ uses the least concrete; the next best is a right triangle with the vertical in contact with the water; next is the right triangle with the hypotenuse in contact with the water; and the cross-section requiring the most concrete is the rectangular cross-section.

Problem 3.66

3.66 A solid concrete dam is to be built to hold back a depth D of water. For ease of construction the walls of the dam must be planar. Your supervisor asks you to consider the following dam cross-sections: a rectangle, a right triangle with the hypotenuse in contact with the water, and a right triangle with the vertical in contact with the water. She wishes you to determine which of these would require the least amount of concrete. What will your report say? You decide to look at one more possibility: a nonright triangle, as shown. Develop and plot an expression for the cross-section area A as a function of α, and find the minimum cross-sectional area.

Given: Various dam cross-sections

Find: Which requires the least concrete; plot cross-section area A as a function of α

Solution:

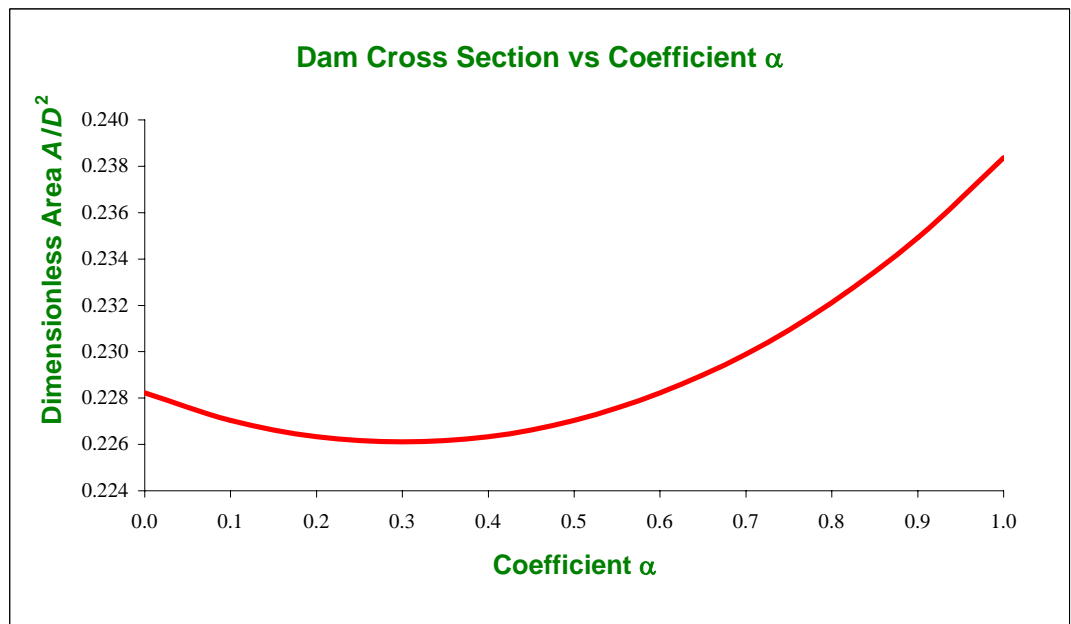
The triangular cross-sections are considered in this workbook

The final result is
$$A = \frac{D^2}{2\cdot\sqrt{4.8 + 0.6\cdot\alpha - \alpha^2}}$$

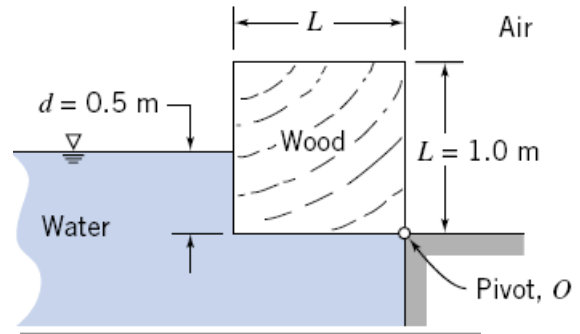
The dimensionless area, A/D^2, is plotted

α	A/D^2
0.0	0.2282
0.1	0.2270
0.2	0.2263
0.3	0.2261
0.4	0.2263
0.5	0.2270
0.6	0.2282
0.7	0.2299
0.8	0.2321
0.9	0.2349
1.0	0.2384

Solver can be used to find the minimum area

α	A/D^2
0.30	0.2261

Problem 3.67

[3]

3.67 A long, square wooden block is pivoted along one edge. The block is in equilibrium when immersed in water to the depth shown. Evaluate the specific gravity of the wood, if friction in the pivot is negligible.

Given: Block hinged and floating

Find: SG of the wood

Solution:

Basic equation $\frac{dp}{dh} = \rho \cdot g$ $\qquad \Sigma M_z = 0$

Computing equations $F_R = p_c \cdot A$ $\qquad y' = y_c + \frac{I_{xx}}{A \cdot y_c}$

Assumptions: static fluid; ρ = constant; p_{atm} on other side; no friction in hinge

For incompressible fluid $p = \rho \cdot g \cdot h$ where p is gage pressure and h is measured downwards

The force on the vertical section is the same as that on a rectangle of height d and width L

Hence $F_1 = p_c \cdot A = \rho \cdot g \cdot y_c \cdot A = \rho \cdot g \cdot \frac{d}{2} \cdot d \cdot L = \frac{\rho \cdot g \cdot L \cdot d^2}{2}$

The location of this force is $y' = y_c + \frac{I_{xx}}{A \cdot y_c} = \frac{d}{2} + \frac{L \cdot d^3}{12} \times \frac{1}{L \cdot d} \times \frac{2}{d} = \frac{2}{3} \cdot d$

The force on the horizontal section is due to constant pressure, and is at the centroid

$$F_2 = p(y = d) \cdot A = \rho \cdot g \cdot d \cdot L \cdot L$$

Summing moments about the hinge $\Sigma M_{hinge} = 0 = -F_1 \cdot (d - y') - F_2 \cdot \frac{L}{2} + M \cdot g \cdot \frac{L}{2}$

Hence $F_1 \cdot \left(d - \frac{2}{3} \cdot d\right) + F_2 \cdot \frac{L}{2} = SG \cdot \rho \cdot L^3 \cdot g \cdot \frac{L}{2}$

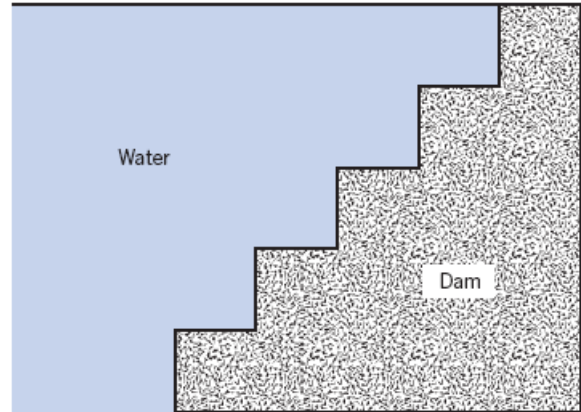
$$\frac{SG \cdot \rho \cdot g \cdot L^4}{2} = \frac{\rho \cdot g \cdot L \cdot d^2}{2} \cdot \frac{d}{3} + \rho \cdot g \cdot d \cdot L^2 \cdot \frac{L}{2}$$

$$SG = \frac{1}{3} \cdot \left(\frac{d}{L}\right)^3 + \frac{d}{L}$$

$$SG = \frac{1}{3} \cdot \left(\frac{0.5}{1}\right)^3 + \frac{0.5}{1}$$

$SG = 0.542$

Problem 3.68

3.68 For the geometry shown, what is the vertical force on the dam? The steps are 1 ft high, 1 ft deep, and 10 ft wide.

Given:
Geometry of dam

Find:
Vertical force on dam

Solution:

Basic equation $$\frac{dp}{dh} = \rho \cdot g$$

Assumptions: static fluid; ρ = constant

For incompressible fluid $p = p_{atm} + \rho \cdot g \cdot h$ where h is measured downwards from the free surface

The force on each horizontal section (depth d = 1 ft and width w = 10 ft) is

$$F = p \cdot A = \left(p_{atm} + \rho \cdot g \cdot h\right) \cdot d \cdot w$$

Hence the total force is $$F_T = \left[p_{atm} + \left(p_{atm} + \rho \cdot g \cdot h\right) + \left(p_{atm} + \rho \cdot g \cdot 2 \cdot h\right) + \left(p_{atm} + \rho \cdot 3 \cdot g \cdot h\right) + \left(p_{atm} + \rho \cdot g \cdot 4 \cdot h\right)\right] \cdot d \cdot w$$

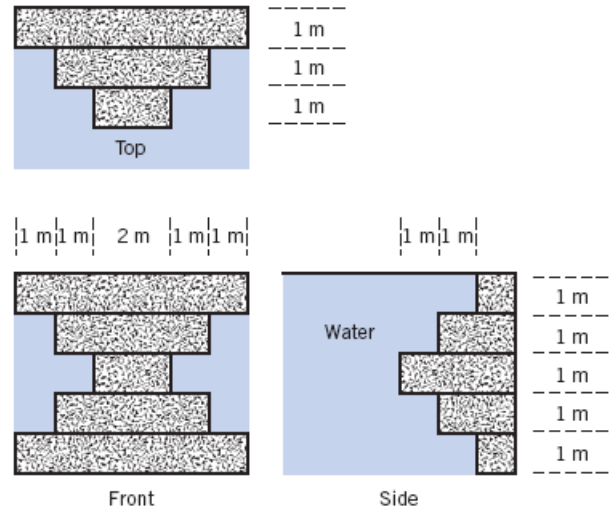
where we have used h as the height of the steps

$$F_T = d \cdot w \cdot \left(5 \cdot p_{atm} + 10 \cdot \rho \cdot g \cdot h\right)$$

$$F_T = 1 \cdot ft \times 10 \cdot ft \times \left[5 \times 14.7 \cdot \frac{lbf}{in^2} \times \left(\frac{12 \cdot in}{1 \cdot ft}\right)^2 + 10 \times 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times 1 \cdot ft \times \frac{lbf \cdot s^2}{slug \cdot ft}\right]$$

$$F_T = 1.12 \times 10^5 \cdot lbf$$

Problem 3.69

3.69 For the dam shown, what is the vertical force of the water on the dam?

Given: Geometry of dam

Find: Vertical force on dam

Solution:

Basic equation $\frac{dp}{dh} = \rho \cdot g$

Assumptions: static fluid; ρ = constant; since we are asked for the force of water, we use gage pressures

For incompressible fluid $p = \rho \cdot g \cdot h$ where p is gage pressure and h is measured downwards from the free surface

The force on each horizontal section (depth d and width w) is

$$F = p \cdot A = \rho \cdot g \cdot h \cdot d \cdot w$$

Hence the total force is (allowing for the fact that some faces experience an upwards (negative) force)

$$F_T = p \cdot A = \Sigma \rho \cdot g \cdot h \cdot d \cdot w = \rho \cdot g \cdot d \cdot \Sigma h \cdot w$$

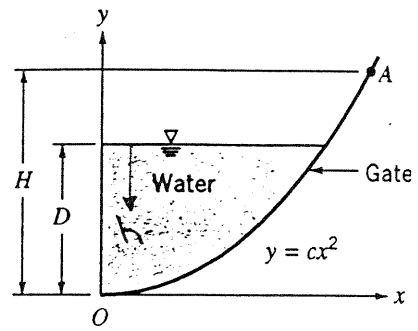
Starting with the top and working downwards

$$F_T = 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 1 \cdot m \times [(1 \cdot m \times 4 \cdot m) + (2 \cdot m \times 2 \cdot m) - (3 \cdot m \times 2 \cdot m) - (4 \cdot m \times 4 \cdot m)] \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F_T = -137 \cdot kN$$

The negative sign indicates a net upwards force (it's actually a buoyancy effect on the three middle sections)

Given: Parabolic gate, hinged at O, has width $b = 2$ m.
$c = 0.25\ m^{-1}$, $D = 2$ m, $H = 3$ m

Find: (a) Magnitude and line of action of vertical force on gate due to water
(b) Horizontal force applied at A needed for equilibrium
(c) Vertical force applied at A needed for equilibrium

Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $\Sigma M_{O_z} = 0$, $F_V = \int p\, dA_y$, $x' F_V = \int x\, dF_V$

Computing equations $F_H = p_c A$, $h' = h_c + \frac{I_{xx}}{h_c A}$

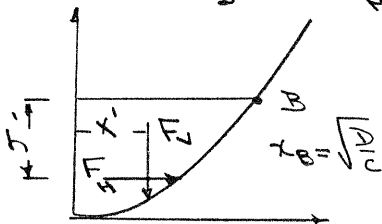
Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts on the surface of the water and along the outside surface of the gate

Then on integrating $dp = \rho g\, dh$, we obtain $p = \rho g h$

(a) $F_V = \int p\, dA_y = \int_0^{\sqrt{D/c}} \rho g h\, b\, dx = \int_0^{\sqrt{D/c}} \rho g (D-y)\, b\, dx = \int_0^{\sqrt{D/c}} \rho g (D - cx^2)\, b\, dx$

$F_V = \rho g b \left[Dx - \frac{cx^3}{3} \right]_0^{\sqrt{D/c}} = \rho g b \left[\frac{D^{3/2}}{c^{1/2}} - \frac{c}{3}\left(\frac{D}{c}\right)^{3/2} \right] = \frac{2}{3} \frac{\rho g b}{c^{1/2}} D^{3/2}$ ___ (1)

$F_V = \frac{2}{3} \times 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 2m \times (2m)^{3/2} \left(\frac{m}{0.25}\right)^{0.5} \times \frac{N \cdot s^2}{kg \cdot m} = 73.9\ kN$ ← F_V

[Sketch: gate with water to point B, forces F_V at x' and F_H at y'; $x_B = \sqrt{D/c}$]

$x' = \frac{1}{F_V} \int x\, dF_V = \frac{1}{F_V} \int x\, p\, dA_y = \frac{1}{F_V} \int_0^{\sqrt{D/c}} x\, \rho g h\, b\, dx$

$x' = \frac{1}{F_V} \int_0^{\sqrt{D/c}} x \rho g (D - cx^2)\, b\, dx$

$x' = \frac{b\rho g}{F_V} \left[\frac{Dx^2}{2} - \frac{cx^4}{4} \right]_0^{\sqrt{D/c}} = \frac{b\rho g}{F_V} \left[\frac{D}{2}\frac{D}{c} - \frac{c}{4}\frac{D^2}{c^2} \right] = \frac{b\rho g D^2}{F_V\, 4c}$

Substituting for F_V from Eq. 1

$x' = \frac{b\rho g D^2}{4c} \times \frac{3}{2} \frac{c^{1/2}}{\rho g b D^{3/2}} = \frac{3}{8}\left(\frac{D}{c}\right)^{1/2} = \frac{3}{8}\left[2m \times \frac{m}{0.25} \right]^{1/2} = 1.06\ m$ ← x'

In order to sum moments about point O to find the required force at A required for equilibrium, we need to find the horizontal force of the water on the gate and its line of action

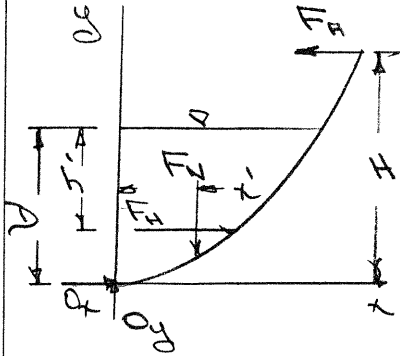
$$F_H = p_c A = \rho g h_c bD = \rho g b \frac{D^2}{2} \qquad \{ h_c = D/2 \}$$

$$F_H = 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 2\,m \times \frac{(2\,m)^2}{2} \times \frac{N \cdot s^2}{kg \cdot m} = 39.2\,kN$$

$$h' = h_c + \frac{I_{\hat{x}\hat{x}}}{A h_c} = h_c + \frac{D^2}{12 h_c} \qquad \{ I_{\hat{x}\hat{x}} = \frac{bD^3}{12} \text{ and } A = bD \}.$$

$$h' = \frac{D}{2} + \frac{D^2}{12} \times \frac{2}{D} \qquad \{ h_c = \frac{D}{2} \}$$

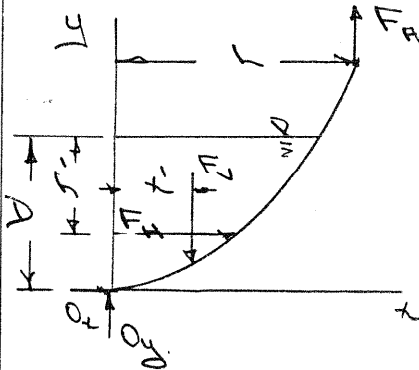
$$h' = \frac{2}{3} D = \frac{4}{3}\,m$$

(b) Horizontal force applied at A for equilibrium

$$\Sigma M_o = 0 = F_A H - F_V x' - F_H (D - h')$$

$$F_A = \frac{1}{H}\left[F_V x' + F_H (D - h') \right]$$

$$= \frac{1}{3\,m}\left[73.9\,kN \times 1.06\,m + 39.2\,kN \times \left(2 - \frac{4}{3}\right) m \right]$$

$$F_{A_H} = 34.8\,kN \qquad F_{A_H}$$

(c) Vertical force applied at A for equilibrium

$$\Sigma M_o = 0 = F_A L - F_V x' - F_H (D - h').$$

$$F_A = \frac{1}{L}\left[F_V x' + F_H (D - h') \right]$$

$L = x$ @ $y = H$. Since $y = cx^2$

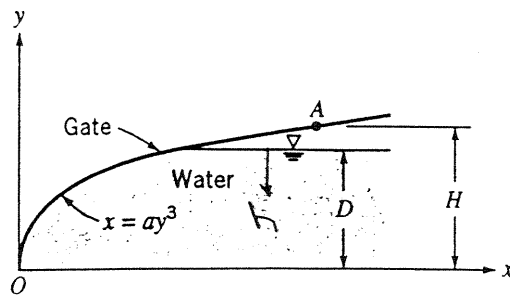
$$L = \sqrt{\frac{H}{c}} = \left[3\,m \times \frac{m}{0.25} \right]^{1/2} = 3.46\,m$$

$$F_A = \frac{1}{3.46\,m}\left[73.9\,kN \times 1.06\,m + 39.2\,kN \times \left(2 - \frac{4}{3}\right) m \right]$$

$$F_{A_V} = 30.2\,kN \qquad F_{A_V}$$

Problem 3.71

Given: Gate, hinged at O, has width $b = 1.5$ m

$a = 1.0\ \text{m}^{-2}$, $D = 1.20$ m, $H = 1.40$ m

Find: (a) Magnitude and moment about O of vertical force on gate due to water

(b) Horizontal force applied at A needed for equilibrium

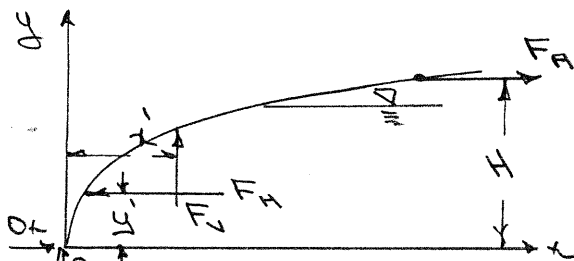
Solution

Basic equations: $\frac{dp}{dh} = \rho g$, $F_V = \int p\,dA_y$, $x' F_V = \int x\,dF_V$

$y' F_H = \int y\,dF_H$, $F_H = \int p\,dA_x$, $\Sigma M_{O_z} = 0$

Assumptions: (1) static liquid (2) ρ = constant

(3) p_{atm} acts on the surface of the water and along the top surface of the gate

Then on integrating $dp = \rho g\,dh$, we obtain $p = \rho g h$

$$F_V = \int p\,dA_y = \int \rho g h\,b\,dx$$

$$h = D - y \qquad x = ay^3 \qquad dx = 3ay^2\,dy$$

$$F_V = \int_0^D \rho g (D - y)\,b\,3ay^2\,dy$$

$$F_V = 3\rho g b a \left[D\frac{y^3}{3} - \frac{y^4}{4}\right]_0^D = 3\rho g b a \frac{D^4}{12} = \rho g b a \frac{D^4}{4}$$

$$F_V = 999\frac{\text{kg}}{\text{m}^3} \times 9.81\frac{\text{m}}{\text{s}^2} \times 1.5\,\text{m} \times \frac{1.0}{\text{m}^2} \times \frac{(1.20\,\text{m})^4}{4} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} = 7.62\ \text{kN} \quad F_V$$

The moment of F_V about O is given by

$$x' F_V = \int x\,dF_V = \int x\,p\,dA_y = \int x\,\rho g h\,b\,dx$$

$$= \rho g b \int_0^D ay^3 (D - y)\,3ay^2\,dy = 3\rho g b a^2 \int_0^D y^5 (D - y)\,dy$$

$$= 3\rho g b a^2 \left[D\frac{y^6}{6} - \frac{y^7}{7}\right]_0^D = \frac{\rho g b a^2 D^7}{14}$$

$$x' F_V = 999\frac{\text{kg}}{\text{m}^3} \times 9.81\frac{\text{m}}{\text{s}^2} \times 1.5\,\text{m} \times \frac{(1.0)^2}{\text{m}^4} \times \frac{(1.20\,\text{m})^7}{14} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$x' F_V = 3.76\ \text{kN}\cdot\text{m} \quad \text{(counterclockwise)} \quad x' F_V$$

From the free body diagram of the gate

$$\Sigma M_{oz} = x' F_V + y' F_H - H F_A$$

$$y' F_H = \int y \, dF_H = \int y \, p \, dA_x = \int y \, \rho g h \, b \, dy = \rho g b \int_0^D y(D-y) \, dy$$

$$= \rho g b \left[\frac{D y^2}{2} - \frac{y^3}{3} \right]_0^D = \rho g b \frac{D^3}{6}$$

$$y' F_H = \frac{1}{6} \times 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 1.5 \, m \times (1.20 \, m)^3 \times \frac{N \cdot s^2}{kg \cdot m} = 4.23 \, kN \cdot m$$

(counterclockwise)

Then

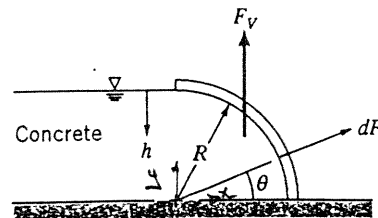
$$F_A = \frac{1}{H} \left[x' F_V + y' F_H \right] = \frac{1}{1.40 \, m} \left[3.76 + 4.23 \right] kN \cdot m$$

$$F_A = 5.71 \, kN$$

F_A

Problem 3.72

Given: Liquid concrete is poured into form shown; width $w = 4.25\,m$

Find: Magnitude and line of action of vertical force on form

Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $F_V = \int p\, dA_y$, $x' F_V = \int x\, dF_V$

Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts on the liquid surface and along the outside of the form.

Then on integrating $dp = \rho g\, dh$, we obtain $p = \rho g h$

$$F_V = \int p\, dA_y = \int \rho g h\, dA \sin\theta \qquad dA = wR\,d\theta,\ h = R - y = R - R\sin\theta$$

$$F_V = \int_0^{\pi/2} \rho g R(1-\sin\theta)\sin\theta\, wR\, d\theta = \rho g R^2 w \int_0^{\pi/2} (\sin\theta - \sin^2\theta)\, d\theta$$

$$F_V = \rho g R^2 w \left[-\cos\theta - \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = \rho g R^2 w \left[-0 + 1 - \frac{\pi}{4} + 0 + 0 - 0\right]$$

$$F_V = \rho g R^2 w \left(1 - \frac{\pi}{4}\right) \qquad \left\{ \rho = SG\, \rho_{H_2O};\ SG = 2.5 \text{ (Table A.1)} \right.$$

$$F_V = 2.5 \times 1000 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times (0.313\,m)^2 \times 4.25\,m \left(1 - \frac{\pi}{4}\right) \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F_V = 2.19\ kN \qquad F_V$$

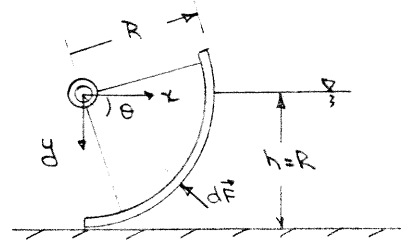
$$x' F_V = \rho g R^2 w \int_0^{\pi/2} x(\sin\theta - \sin^2\theta)\, d\theta = \rho g R^2 w \int_0^{\pi/2} R\cos\theta(\sin\theta - \sin^2\theta)\, d\theta$$

$$= \rho g R^3 w \int_0^{\pi/2} (\sin\theta\cos\theta - \sin^2\theta\cos\theta)\, d\theta = \rho g R^3 w \left[\frac{\sin^2\theta}{2} - \frac{\sin^3\theta}{3}\right]_0^{\pi/2}$$

$$x' F_V = \rho g R^3 w \left[\frac{1}{2} - \frac{1}{3}\right] = \frac{\rho g R^3 w}{6}$$

$$x' = \frac{\rho g R^3 w}{6 F_V} = \frac{\rho g R^3 w}{6} \times \frac{1}{\rho g R^2 w (1 - \frac{\pi}{4})} = \frac{R}{6(1 - \frac{\pi}{4})} = \frac{0.313\,m}{6(1 - \pi/4)}$$

$$x' = 0.243\ m \qquad x'$$

Given: Gate formed in the shape of a circular arc has width of w meters. Liquid is water; depth $h = R$

Find: (a) magnitude and direction of the net vertical force component due to fluids acting on the gate
(b) line of action of vertical component of the force.

Solution

Basic equations: $\vec{F}_R = -\int P\, d\vec{A}$ $\quad \frac{dP}{dy} = \rho g \quad x' F_{R_y} = \int x\, dF$

Assumptions: (1) static fluid
(2) ρ = constant
(3) y is measured positive downward from free surface

$$\vec{F}_{R_y} = \vec{F}_R \cdot \hat{j} = \int d\vec{F} \cdot \hat{j} = -\int P\, d\vec{A} \cdot \hat{j} = -\int P\, dA \sin\theta = -\int_0^{\pi/2} P \sin\theta\, wR\, d\theta$$

We can obtain an expression for P as a function of y

$$\frac{dP}{dy} = \rho g \qquad dP = \rho g\, dy \qquad \text{and} \quad P - P_0 = \int_{P_0}^{P} dP = \int_0^y \rho g\, dy = \rho g y$$

Since atmospheric pressure acts at the free surface and on the back surface of the gate, then the appropriate expression for P is $P = \rho g y$
Along the surface of the gate,

$$y = R \sin\theta \quad \text{and hence} \quad P = \rho g R \sin\theta$$

Thus,

$$F_{R_y} = -\int_0^{\pi/2} P \sin\theta\, wR\, d\theta = -\rho g w R^2 \int_0^{\pi/2} \sin^2\theta\, d\theta = -\rho g w R^2 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$F_{R_y} = -\frac{\rho g w R^2 \pi}{4} \qquad \{F_{R_y} \text{ acts upward}\} \qquad F_{R_y}$$

For any element of surface area, $d\vec{A}$, the force, $d\vec{F}$, acts normal to the surface. Thus each $d\vec{F}$ has a line of action through the origin. Consequently, the line of action of F_R must also be through the origin.

We can find the line of action of F_{R_y} by recognizing that the moment of F_{R_y} about an axis through the origin must be equal to the sum of the moments of dF_y about the same axis.

$$x' F_{R_y} = \int x\, dF_y = \int x(-P\, dA \sin\theta) = -\int x P\, dA \sin\theta$$

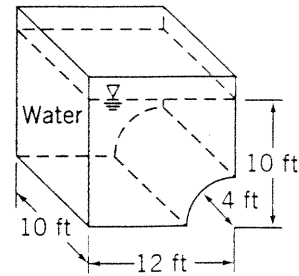
$$x' F_{R_y} = -\int_0^{\pi/2} R\cos\theta\, \rho g R \sin\theta\, wR\, d\theta \sin\theta = -\rho g w R^3 \int_0^{\pi/2} \sin^2\theta \cos\theta\, d\theta$$

$$x' = \frac{-\rho g w R^3}{F_{R_y}} \int_0^{\pi/2} \sin^2\theta \cos\theta\, d\theta = \frac{-\rho g w R^3}{-\frac{\rho g w R^2 \pi}{4}} \left[\frac{1}{3}\sin^3\theta\right]_0^{\pi/2}$$

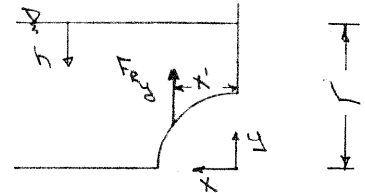
$$x' = \frac{4R}{3\pi}$$

Problem 3.74

Given: Open tank as shown
width of curved surface $b = 10\,ft$

Find: (a) vertical force component, F_{R_y}, on curved surface
(b) line of action of F_{R_y}

Solution:

Basic equations: $\vec{F}_R = -\int P\,d\vec{A}$ $\quad \frac{dP}{dh} = \gamma$ $\quad \vec{r}' \times \vec{F}_R = \int \vec{r} \times d\vec{F} = -\int \vec{r} \times P\,d\vec{A}$

Assumptions: (1) static fluid
(2) gravity is only body force
(3) $\gamma = \text{constant} = 62.4\ lbf/ft^3$
(4) h is measured positive downward from free surface

$F_{R_y} = \vec{F}_R \cdot \hat{j} = -\int P\,d\vec{A}\cdot\hat{j} = -\int P\,dA_y = -\int P b\,dx$

We can obtain an expression for P as a function of y

$\frac{dP}{dh} = \gamma \qquad dP = \gamma\,dh \qquad P - P_0 = \int_{P_0}^{P} dP = \int_0^h \gamma\,dh = \gamma h$

Since atmospheric pressure acts at the free surface and on the underside of the curved surface, then the appropriate expression for P is $P = \gamma h$

Now, $h = L - y \quad \therefore P = \gamma(L-y)$

$F_{R_y} = -\int P b\,dx = -\int \gamma(L-y)b\,dx$. Along the surface $y = (R^2 - x^2)^{1/2}$ and so

$F_{R_y} = -\gamma b \int_0^R \{L - (R^2-x^2)^{1/2}\}\,dx = -\gamma b\left[Lx - \frac{1}{2}\left(x\sqrt{R^2-x^2} + R^2 \arcsin\frac{x}{R}\right)\right]_0^R$

$= -\gamma b\left\{LR - \frac{1}{2}(R^2 \arcsin 1) + \frac{1}{2}R^2 \arcsin 0\right\} = \gamma b R\left\{L - \frac{R}{2}\arcsin 1\right\}$

$= -\gamma b R\left\{L - R\frac{\pi}{4}\right\}$

$F_{R_y} = -62.4\,\frac{lbf}{ft^3} \times 10\,ft \times 4\,ft \times \left\{10\,ft - 4\,ft \times \frac{\pi}{4}\right\} = -17{,}100\ lbf$ (ie acts downward) $\quad F_{R_y}$

$x'\hat{i} \times F_{R_y}\hat{j} = \int x\hat{i} \times dF_{R_y}\hat{j} = \int x\hat{i} \times (-P\,dA_y\hat{j}) = -\int x\hat{i} \times P b\,dx\,\hat{j}$

$x' F_{R_y}\hat{k} = -\hat{k}\int x P b\,dx$

$x' = -\frac{1}{F_{R_y}}\int_0^R x P b\,dx = -\frac{1}{F_{R_y}}\int_0^R x\gamma(L-y)b\,dx = -\frac{\gamma b}{F_{R_y}}\int_0^R x\{L - (R^2-x^2)^{1/2}\}\,dx$

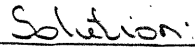
$= -\frac{\gamma b}{F_{R_y}}\left[L\frac{x^2}{2} + \frac{1}{3}\sqrt{(R^2-x^2)^3}\right]_0^R = -\frac{\gamma b}{F_{R_y}}\left[L\frac{R^2}{2} - \frac{1}{3}R^3\right] = -\frac{\gamma b R^2}{F_{R_y}}\left[\frac{L}{2} - \frac{R}{3}\right]$

$x' = -62.4\,\frac{lbf}{ft^3} \times 10\,ft \times (4)^2\,ft^2 \times \frac{1}{(-17{,}100)\,lbf}\left[\frac{10\,ft}{2} - \frac{4\,ft}{3}\right]$

$x' = 2.14\ ft \quad x'$

Problem 3.75

Given: Dam with cross-section shown (width $b = 50\,m$)

Find: (a) Magnitude and line of action of vertical force on dam due to water.
(b) If it is possible for water force to overturn the dam

Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $F_v = \int p\,dA_y$, $x' F_v = \int x\,dF_v$, $\sum M_z = 0$

Computing equations: $F_H = p_c A$, $h' = h_c + \frac{I_{\hat{x}\hat{x}}}{h_c A}$

Assumptions: (1) static fluid (2) ρ = constant
(3) p_{atm} acts on the surface of the water and on the back side of the dam.

Then on integrating $dp = \rho g\,dh$ we obtain $p = \rho g h$

(Sketch: y, 0.7m, (A), F_v, h', x', F_H, (B), y', x)

$$F_v = \int p\,dA_y = \int_{x_A}^{x_B} \rho g h b\,dx = \rho g b \int_{x_A}^{x_B} (H - y)\,dx$$

$$y(x - A) = B \quad \text{so} \quad y = \frac{B}{(x - A)}$$

$$F_v = \rho g b \int_{x_A}^{x_B} \left(H - \frac{B}{(x-A)}\right) dx$$

$$= \rho g b \left[Hx - B \ln(x - A) \right]_{x_A}^{x_B}$$

$$F_v = \rho g b \left[H(x_B - x_A) - B \ln \frac{(x_B - A)}{(x_A - A)} \right] \quad \text{--- (1)}$$

$$F_v = 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 50m \left[2.5m(2.2 - 0.76)m - 0.9m^2 \ln \frac{(2.2 - 0.4)}{(0.76 - 0.4)} \right] \frac{N \cdot s^2}{kg \cdot m}$$

$$F_v = 1.05 \times 10^6\,N \quad \longleftarrow F_v$$

$$x' F_v = \int x\,dF_v = \int_{x_A}^{x_B} x \rho g b \left(H - \frac{B}{(x-A)}\right) dx = \rho g b \int_{x_A}^{x_B} \left[Hx - \frac{Bx}{(x-A)} \right] dx$$

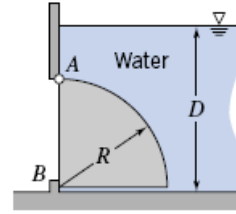
$$x' F_v = \rho g b \left[H \frac{x^2}{2} - Bx - BA \ln(x - A) \right]_{x_A}^{x_B}$$

$$x' F_v = \rho g b \left[\frac{H}{2}(x_B^2 - x_A^2) - B(x_B - x_A) - BA \ln \frac{(x_B - A)}{(x_A - A)} \right] \quad \text{--- (2)}$$

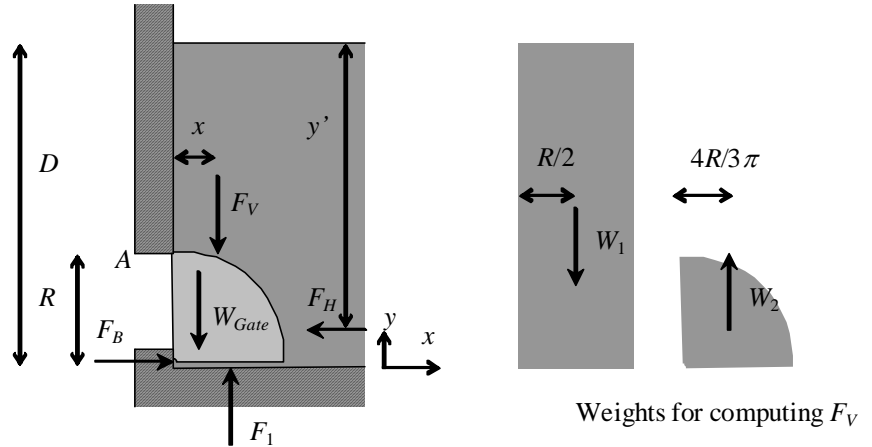
$$x' = 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 50m \left\{ \frac{2.5m}{2} \left[(2.2)^2 m^2 - (0.76)^2 m^2 - 0.9m^2 (2.2 - 0.76) m \right. \right.$$
$$\left. - 0.9m^2 \times 0.4m \ln \frac{2.2 - 0.4}{0.76 - 0.4} \right\} \frac{N \cdot s^2}{kg \cdot m} \times \frac{1}{1.05 \times 10^6\,N}$$

$$x' = 1.61\,m \quad \longleftarrow x'$$

Problem 3.76

3.76 A gate, in the shape of a quarter-cylinder, hinged at A and sealed at B, is 3 m wide. The bottom of the gate is 4.5 m below the water surface. Determine the force on the stop at B if the gate is made of concrete; $R = 3$ m.

Given: Gate geometry

Find: Force on stop B

Solution:

Weights for computing F_V

Basic equations $\frac{dp}{dh} = \rho \cdot g$

$\Sigma M_A = 0$

Assumptions: static fluid; ρ = constant; p_{atm} on other side

For incompressible fluid $p = \rho \cdot g \cdot h$ where p is gage pressure and h is measured downwards

We need to compute force (including location) due to water on curved surface and underneath. For curved surface we could integrate pressure, but here we use the concepts that F_V (see sketch) is equivalent to the weight of fluid above, and F_H is equivalent to the force on a vertical flat plate. Note that the sketch only shows forces that will be used to compute the moment at A

For F_V $\quad F_V = W_1 - W_2$

with

$$W_1 = \rho \cdot g \cdot w \cdot D \cdot R = 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 3 \cdot m \times 4.5 \cdot m \times 3 \cdot m \times \frac{N \cdot s^2}{kg \cdot m} \qquad W_1 = 397 \cdot kN$$

$$W_2 = \rho \cdot g \cdot w \cdot \frac{\pi \cdot R^2}{4} = 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 3 \cdot m \times \frac{\pi}{4} \times (3 \cdot m)^2 \times \frac{N \cdot s^2}{kg \cdot m} \qquad W_2 = 208 \cdot kN$$

$$F_V = W_1 - W_2 \qquad F_V = 189 \cdot kN$$

with x given by $\quad F_V \cdot x = W_1 \cdot \frac{R}{2} - W_2 \cdot \frac{4 \cdot R}{3 \cdot \pi} \quad$ or $\quad x = \frac{W_1}{F_V} \cdot \frac{R}{2} - \frac{W_2}{F_V} \cdot \frac{4 \cdot R}{3 \cdot \pi}$

$$x = \frac{397}{189} \times \frac{3 \cdot m}{2} - \frac{208}{189} \times \frac{4}{3 \cdot \pi} \times 3 \cdot m \qquad x = 1.75\,m$$

For F_H $\quad$ Computing equations $\quad F_H = p_c \cdot A \quad y' = y_c + \frac{I_{xx}}{A \cdot y_c}$

Hence $$F_H = p_c \cdot A = \rho \cdot g \cdot \left(D - \frac{R}{2}\right) \cdot w \cdot R$$

$$F_H = 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times \left(4.5 \cdot m - \frac{3 \cdot m}{2}\right) \times 3 \cdot m \times 3 \cdot m \times \frac{N \cdot s^2}{kg \cdot m} \qquad F_H = 265 \cdot kN$$

The location of this force is

$$y' = y_c + \frac{I_{xx}}{A \cdot y_c} = \left(D - \frac{R}{2}\right) + \frac{w \cdot R^3}{12} \times \frac{1}{w \cdot R \cdot \left(D - \frac{R}{2}\right)} = D - \frac{R}{2} + \frac{R^2}{12 \cdot \left(D - \frac{R}{2}\right)}$$

$$y' = 4.5 \cdot m - \frac{3 \cdot m}{2} + \frac{(3 \cdot m)^2}{12 \times \left(4.5 \cdot m - \frac{3 \cdot m}{2}\right)} \qquad y' = 3.25 m$$

The force F_1 on the bottom of the gate is $F_1 = p \cdot A = \rho \cdot g \cdot D \cdot w \cdot R$

$$F_1 = 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 4.5 \cdot m \times 3 \cdot m \times 3 \cdot m \times \frac{N \cdot s^2}{kg \cdot m} \qquad F_1 = 397 \cdot kN$$

For the concrete gate (SG = 2.4 from Table A.2)

$$W_{Gate} = SG \cdot \rho \cdot g \cdot w \cdot \frac{\pi \cdot R^2}{4} = 2.4 \cdot 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 3 \cdot m \times \frac{\pi}{4} \times (3 \cdot m)^2 \times \frac{N \cdot s^2}{kg \cdot m} \qquad W_{Gate} = 499 \cdot kN$$

Hence, taking moments about A $$F_B \cdot R + F_1 \cdot \frac{R}{2} - W_{Gate} \cdot \frac{4 \cdot R}{3 \cdot \pi} - F_V \cdot x - F_H \cdot [y' - (D - R)] = 0$$

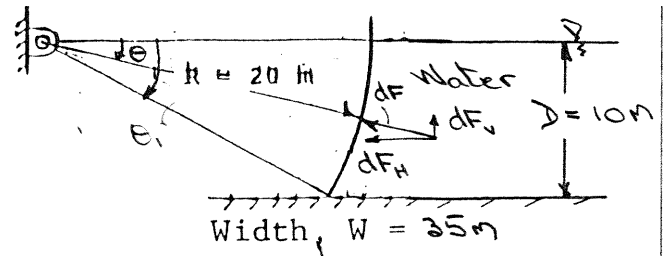
$$F_B = \frac{4}{3 \cdot \pi} \cdot W_{Gate} + \frac{x}{R} \cdot F_V + \frac{[y' - (D - R)]}{R} \cdot F_H - \frac{1}{2} \cdot F_1$$

$$F_B = \frac{4}{3 \cdot \pi} \times 499 \cdot kN + \frac{1.75}{3} \times 189 \cdot kN + \frac{[3.25 - (4.5 - 3)]}{3} \times 265 \cdot kN - \frac{1}{2} \times 397 \cdot kN$$

$$F_B = 278 \cdot kN$$

Problem 3.77

Given: Tainter gate as shown

Find: Force of the water acting on the gate

Solution:

Basic equations: $dF = p\,dA$; $\frac{dp}{dh} = \rho g$

Assumptions: (1) static fluid
(2) ρ = constant
(3) p_{atm} acts at free surface and on surface of gate

For ρ = const, $\int dp = \int \rho g\,dh$ yields $p - p_{atm} = \rho g h = \rho g R \sin\theta$ (using (3))

$dF_H = dF\cos\theta = p\,dA\cos\theta = \rho g R\sin\theta\, wR\,d\theta\,\cos\theta$ $\{dA = wR\,d\theta\}$

$F_H = \int dF_H = \int_0^{\theta_1} \rho g w R^2 \sin\theta\cos\theta\,d\theta$ where $\theta_1 = \sin^{-1}\frac{10}{20} = 30°$

$F_H = \rho g w R^2 \int_0^{30°} \sin\theta\cos\theta\,d\theta = \rho g w R^2 \left[\frac{\sin^2\theta}{2}\right]_0^{30°} = \frac{\rho g w R^2}{8}$

$F_H = \frac{1}{8} \times 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{sec^2} \times 35\,m \times (20\,m)^2 \times \frac{N\cdot sec^2}{kg\cdot m} = 1.72 \times 10^7\,N$

$dF_V = dF\sin\theta = p\,dA\sin\theta = \rho g R\sin\theta\, wR\,d\theta\,\sin\theta$

$F_V = \int dF_V = \rho g w R^2 \int_0^{30°} \sin^2\theta\,d\theta = \rho g w R^2 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/6}$

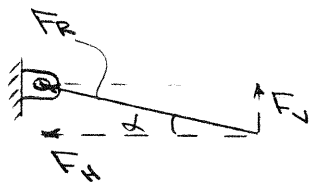
$F_V = \rho g w R^2 \left[\frac{\pi}{12} - \frac{0.866}{4}\right] = 0.0453\,\rho g w R^2$

$F_V = 0.0453 \times 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 35\,m \times (20\,m)^2 \times \frac{N\cdot s^2}{kg\cdot m} = 6.22 \times 10^6\,N$

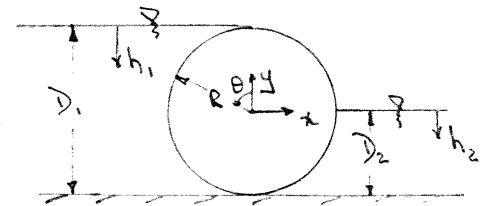
Since the gate surface in contact with the water is a circular arc, all elements dF, of the force and hence the line of action of the resultant force must pass through the pivot. Thus

$F_R = \left[F_H^2 + F_V^2\right]^{1/2} = \left[(17.2\times10^6)^2 + (6.22\times10^6)^2\right]^{1/2} = 1.83 \times 10^7\,N$ F_R

$\alpha = \tan^{-1}\frac{F_V}{F_R} = \tan^{-1}\frac{6.22}{17.2}$

$\alpha = 19.9°$ α

F_R passes through pivot at angle α to the horizontal

Problem 3.78

Given: Cylindrical weir of radius, $R = 1.5\text{ m}$ and length, $L = 6\text{ m}$ as shown
Liquid is water
$D_1 = 3\text{ m}$ $D_2 = 1.5\text{ m}$

Find: Magnitude and direction of resultant force of water on the weir

Solution:

Basic equations: $\vec{F}_R = -\int P\,d\vec{A}$ $\frac{dP}{dh} = \rho g$

Assumptions: (1) static fluid
(2) ρ = constant
(3) h is measured positive down from free surface

$F_{R_x} = \int dF_x = \vec{F}_R\cdot\hat{i} = \int d\vec{F}\cdot\hat{i} = -\int P\,d\vec{A}\cdot\hat{i} = -\int P\,dA\cos(90+\theta) = \int P\,dA\sin\theta$

$F_{R_y} = \int dF_y = \vec{F}_R\cdot\hat{j} = \int d\vec{F}\cdot\hat{j} = -\int P\,d\vec{A}\cdot\hat{j} = -\int P\,dA\cos\theta$

Since $dA = LR\,d\theta$,

$$F_{R_x} = \int_0^{3\pi/2} PLR\sin\theta\,d\theta \quad\text{and}\quad F_{R_y} = -\int_0^{3\pi/2} PLR\cos\theta\,d\theta$$

We can obtain an expression for P as a function of h.

$$\frac{dP}{dh} = \rho g \qquad dP = \rho g\,dh \quad\text{and}\quad P - P_0 = \int_{P_0}^{P} dP = \int_0^h \rho g\,dh = \rho g h$$

Since atmospheric pressure acts over the first quadrant of the cylinder and both free surfaces, the appropriate expression for P is $P = \rho g h$.

For

$0 \le \theta \le \pi$, $h_1 = R - R\cos\theta = R(1-\cos\theta)$ and hence $P_1 = \rho g R(1-\cos\theta)$

$\pi \le \theta \le \frac{3\pi}{2}$, $h_2 = -R\cos\theta$ and hence $P_2 = -\rho g R\cos\theta$

$$F_{R_x} = \int_0^{3\pi/2} PLR\sin\theta\,d\theta = \int_0^{\pi} \rho g R(1-\cos\theta)LR\sin\theta\,d\theta + \int_{\pi}^{3\pi/2}(-\rho g R\cos\theta)LR\sin\theta\,d\theta$$

$$= \rho g R^2 L\int_0^{\pi}(1-\cos\theta)\sin\theta\,d\theta - \rho g R^2 L\int_{\pi}^{3\pi/2}\cos\theta\sin\theta\,d\theta$$

$$= \rho g R^2 L\left[-\cos\theta - \tfrac{1}{2}\sin^2\theta\right]_0^{\pi} - \rho g R^2 L\left[\tfrac{1}{2}\sin^2\theta\right]_{\pi}^{3\pi/2} = \rho g R^2 L\left[2 - \tfrac{1}{2}\right] = \tfrac{3}{2}\rho g R^2 L$$

$$F_{R_x} = \tfrac{3}{2}\times 999\,\frac{\text{kg}}{\text{m}^3}\times 9.81\,\frac{\text{m}}{\text{s}^2}\times(1.5)^2\,\text{m}^2\times 6\,\text{m}\times\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} = 198\text{ kN}$$

$$F_{R_y} = -\int_0^{3\pi/2} PLR\cos\theta = -\int_0^{\pi}\rho g R(1-\cos\theta)LR\cos\theta\,d\theta - \int_{\pi}^{3\pi/2}(-\rho g R\cos\theta)LR\cos\theta\,d\theta$$

$$= -\rho g R^2 L\int_0^{\pi}(1-\cos\theta)\cos\theta\,d\theta + \rho g R^2 L\int_{\pi}^{3\pi/2}\cos^2\theta\,d\theta$$

$$= -\rho g R^2 L\left[\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi} + \rho g R^2 L\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{\pi}^{3\pi/2} = \rho g R^2 L\left[\frac{\pi}{2} + \frac{3\pi}{4} - \frac{\pi}{2}\right] = \frac{3\pi}{4}\rho g R^2 L$$

$$F_{R_y} = \frac{3\pi}{4}\times 999\,\frac{\text{kg}}{\text{m}^3}\times 9.81\,\frac{\text{m}}{\text{s}^2}\times(1.5)^2\,\text{m}^2\times 6\,\text{m}\times\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} = 312\text{ kN}$$

$$\vec{F}_R = \hat{i}F_{R_x} + \hat{j}F_{R_y} = 198\hat{i} + 312\hat{j}\text{ kN}$$

$$F_R = \sqrt{F_{R_x}^2 + F_{R_y}^2} = \left[(198)^2 + (312)^2\right]^{1/2}\text{ kN} = 370\text{ kN}$$ ← F_R

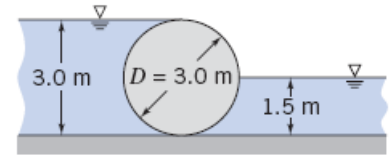
Since all elements of force $d\vec{F}$ are normal to the surface, the direction α,

$$\alpha = \tan^{-1} F_{R_y}/F_{R_x} = \tan^{-1} 312/198 = 57.6^\circ$$ ← α

(sketch: force triangle with F_{R_y}, F_{R_x}, angle α)

Problem 3.79

[4]

3.79 Consider the cylindrical weir of diameter 3 m and length 6 m. If the fluid on the left has a specific gravity of 1.6, and on the right has a specific gravity of 0.8, find the magnitude and direction of the resultant force.

Given: Sphere with different fluids on each side

Find: Resultant force and direction

Solution:

The horizontal and vertical forces due to each fluid are treated separately. For each, the horizontal force is equivalent to that on a vertical flat plate; the vertical force is equivalent to the weight of fluid "above".

For horizontal forces, the computing equation of Section 3-5 is $F_H = p_c \cdot A$ where A is the area of the equivalent vertical plate.

For vertical forces, the computing equation of Section 3-5 is $F_V = \rho \cdot g \cdot V$ where V is the volume of fluid above the curved surface.

The data is

For water $\rho = 999 \cdot \frac{kg}{m^3}$

For the fluids $SG_1 = 1.6$ $\quad SG_2 = 0.8$

For the weir $D = 3 \cdot m$ $\quad L = 6 \cdot m$

(a) Horizontal Forces

For fluid 1 (on the left)

$$F_{H1} = p_c \cdot A = \left(\rho_1 \cdot g \cdot \frac{D}{2}\right) \cdot D \cdot L = \frac{1}{2} \cdot SG_1 \cdot \rho \cdot g \cdot D^2 \cdot L$$

$$F_{H1} = \frac{1}{2} \cdot 1.6 \cdot 999 \cdot \frac{kg}{m^3} \cdot 9.81 \cdot \frac{m}{s^2} \cdot (3 \cdot m)^2 \cdot 6 \cdot m \cdot \frac{N \cdot s^2}{kg \cdot m} \qquad F_{H1} = 423\,kN$$

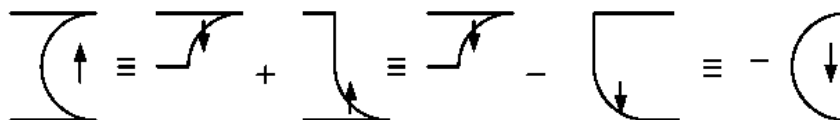
For fluid 2 (on the right)

$$F_{H2} = p_c \cdot A = \left(\rho_2 \cdot g \cdot \frac{D}{4}\right) \cdot \frac{D}{2} \cdot L = \frac{1}{8} \cdot SG_2 \cdot \rho \cdot g \cdot D^2 \cdot L$$

$$F_{H2} = \frac{1}{8} \cdot 0.8 \cdot 999 \cdot \frac{kg}{m^3} \cdot 9.81 \cdot \frac{m}{s^2} \cdot (3 \cdot m)^2 \cdot 6 \cdot m \cdot \frac{N \cdot s^2}{kg \cdot m} \qquad F_{H2} = 52.9\,kN$$

The resultant horizontal force is $F_H = F_{H1} - F_{H2}$ $\qquad F_H = 370\,kN$

(b) Vertical forces

For the left geometry, a "thought experiment" is needed to obtain surfaces with fluid "above"

Hence $$F_{V1} = SG_1 \cdot \rho \cdot g \cdot \frac{\frac{\pi \cdot D^2}{4}}{2} \cdot L$$

$$F_{V1} = 1.6 \times 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times \frac{\pi \cdot (3 \cdot m)^2}{8} \times 6 \cdot m \times \frac{N \cdot s^2}{kg \cdot m}$$ $$F_{V1} = 333\,kN$$

(Note: Use of buoyancy leads to the same result!)

For the right side, using a similar logic

$$F_{V2} = SG_2 \cdot \rho \cdot g \cdot \frac{\frac{\pi \cdot D^2}{4}}{4} \cdot L$$

$$F_{V2} = 0.8 \times 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times \frac{\pi \cdot (3 \cdot m)^2}{16} \times 6 \cdot m \times \frac{N \cdot s^2}{kg \cdot m}$$ $$F_{V2} = 83.1\,kN$$

The resultant vertical force is $$F_V = F_{V1} + F_{V2}$$ $$F_V = 416\,kN$$

Finally the resultant force and direction can be computed

$$F = \sqrt{F_H^{\,2} + F_V^{\,2}}$$ $$F = 557\,kN$$

$$\alpha = \operatorname{atan}\left(\frac{F_V}{F_H}\right)$$ $$\alpha = 48.3\,deg$$

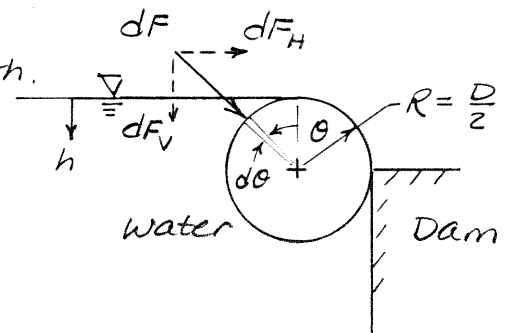
Problem 3.80

Given: Cylindrical log floating against dam.

Find: (a) Mass per unit length

(b) Contact force per unit length.

Solution: Use hydrostatic equations

Basic equations: $\frac{dp}{dh} = \rho g \qquad dF = p\,dA$

Assumptions: (1) Static liquid
(2) Incompressible
(3) Neglect p_{atm} (it acts everywhere)

Then

$$p - p_0 \overset{(3)}{=} \rho g h = \rho g R(1-\cos\theta)$$

$$dF = p\,dA = p w R\,d\theta, \quad dF_H = dF\sin\theta, \quad dF_V = dF\cos\theta$$

$$F_H = \int_0^{3\pi/2} \rho g R(1-\cos\theta)\, wR\sin\theta\, d\theta = \rho g w R^2\left[-\cos\theta - \frac{\sin^2\theta}{2}\right]_0^{3\pi/2} = \rho g w R^2\left[-\frac{(-1)^2}{2} - (-1)\right]$$

$$F_H = \frac{1}{2}\rho g w R^2 \qquad \frac{F_H}{w} = \frac{1}{2}\rho g R^2$$

$\frac{F_H}{w}$

$$F_V = \int_0^{3\pi/2} \rho g R(1-\cos\theta)\, wR\cos\theta\, d\theta = \int_0^{3\pi/2} \rho g w R^2\left(\cos\theta - \frac{1+\cos 2\theta}{2}\right) d\theta$$

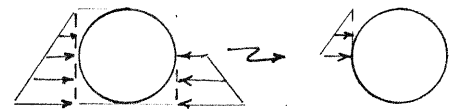
$$F_V = \rho g w R^2\left[\sin\theta - \frac{\theta + \frac{1}{2}\sin 2\theta}{2}\right]_0^{3\pi/2} = \rho g w R^2\left[-1 - \frac{3\pi}{4}\right] = -\rho g w R^2\left[1 + \frac{3\pi}{4}\right]$$

From a free-body diagram of the log

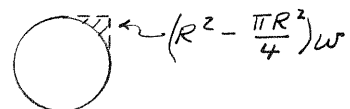
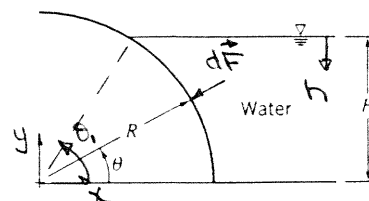
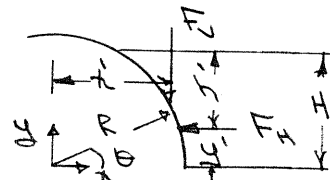
$$\Sigma F_y = -mg - F_V = 0 \qquad m = -\frac{F_V}{g} = \rho w R^2\left[1 + \frac{3\pi}{4}\right]$$

$$\frac{m}{w} = \rho R^2\left[1 + \frac{3\pi}{4}\right]$$

$\frac{m}{w}$

Check: $F_H = p_c A = \rho g \frac{R}{2} wR = \frac{1}{2}\rho g w R^2$ ✓

$$F_V = -\rho g \forall - \rho g\left[R^2 - \frac{\pi R^2}{4}\right]w = -\rho g w\left[-\pi R^2 - R^2 + \frac{\pi R^2}{4}\right] = -\rho g w R^2\left[1 + \frac{3\pi}{4}\right] \checkmark$$

$\left(R^2 - \frac{\pi R^2}{4}\right)w$

Problem 3.81

Given: Curved surface, in shape of quarter cylinder, with radius $R = 0.750$ m and width $w = 3.55$ m; water stands to depth $H = 0.650$ m

Find: Magnitude and line of action of:
(a) vertical force, and
(b) horizontal force
on the curved surface.

Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $F_V = \int p\, dA_y$, $x' F_V = \int x\, dF_V$

Computing equations: $F_H = p_c A$, $h' = h_c + \frac{I_{xx}}{h_c A}$

Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts at free surface of the water

Then on integrating $dp = \rho g\, dh$, we obtain $p = \rho g h$

From the geometry $h = H - R\sin\theta$, $y = R\sin\theta$, $x = R\cos\theta$

$\theta_1 = \sin^{-1} H/R$, $dA = wR\, d\theta$

$$F_V = \int p\, dA_y = \int \rho g h\, dA \sin\theta = \int_0^{\theta_1} \rho g (H - R\sin\theta)\sin\theta\, wR\, d\theta$$

$$F_V = \rho g w R \int_0^{\theta_1} (H\sin\theta - R\sin^2\theta)\, d\theta = \rho g w R \left[-H\cos\theta - R\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)\right]_0^{\theta_1}$$

$$F_V = \rho g w R \left[H(1-\cos\theta_1) - R\left(\frac{\theta_1}{2} - \frac{\sin 2\theta_1}{4}\right)\right] \quad (1)$$

Evaluating for $\theta_1 = \sin^{-1}\frac{H}{R} = \sin^{-1}\frac{0.650}{0.750} = 60°$ $(\pi/3)$.

$$F_V = 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 3.55\,m \times 0.75\,m \left[0.65\,m(1-\cos 60°) - 0.75\,m\left(\frac{\pi}{6} - \frac{\sin 120°}{4}\right)\right]\frac{N\cdot s^2}{kg\cdot m}$$

$F_V = 2.47$ kN ← F_V

$$x' F_V = \rho g w R \int_0^{\theta_1} R\cos\theta (H\sin\theta - R\sin^2\theta)\, d\theta = \rho g w R^2 \int_0^{\theta_1} (H\sin\theta\cos\theta - R\sin^2\theta\cos\theta)\, d\theta$$

$$x' F_V = \rho g w R^2 \left[H\frac{\sin^2\theta}{2} - R\frac{\sin^3\theta}{3}\right]_0^{\theta_1}$$

$$x' = \frac{\rho g w R^2}{F_V}\left[\frac{H}{2}\sin^2\theta_1 - \frac{R}{3}\sin^3\theta_1\right] \quad (2)$$

$$x' = 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 3.55\,m \times (0.75\,m)^2 \times \frac{1}{2.47\times10^3\,N}\left[\frac{0.650\,m}{2}\sin^2 60° - \frac{0.750\,m}{3}\sin^3 60°\right]\frac{N\cdot s^2}{kg\cdot m}$$

$x' = 0.645$ m ← x'

$$F_H = p_c A = \rho g h_c H w = \rho g \frac{H}{2} H w = \frac{\rho g H^2 w}{2} \quad (3)$$

$$F_H = \frac{1}{2} \times 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times (0.65\,m)^2 \times 3.55\,m \times \frac{N\cdot s^2}{kg\cdot m} = 7.35\text{ kN}$$ ← F_H

$$h' = h_c + \frac{I_{\hat{x}\hat{x}}}{h_c A} = h_c + \frac{1}{12}\frac{WH^3}{h_c A} = \frac{H}{2} + \frac{1}{12}\frac{WH^3}{\frac{H}{2}HW} = \frac{H}{2} + \frac{H}{6} = \frac{2}{3}H$$

$$y' = H - h' = H - \frac{2}{3}H = \frac{1}{3}H \qquad (4)$$

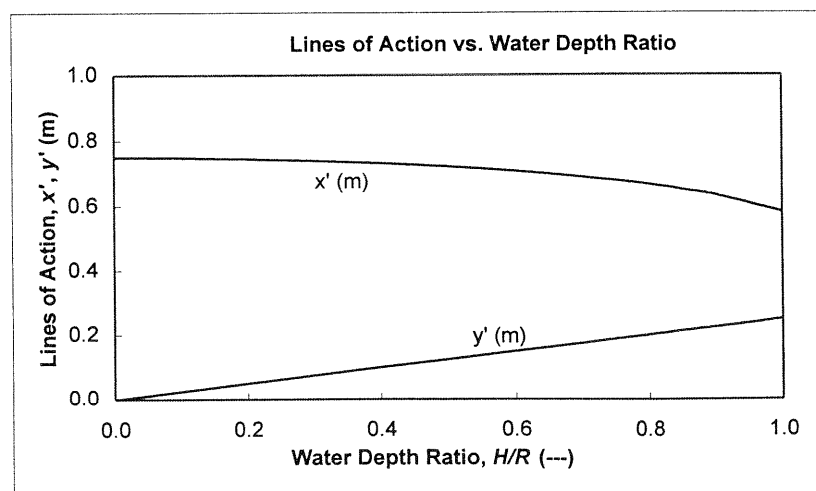
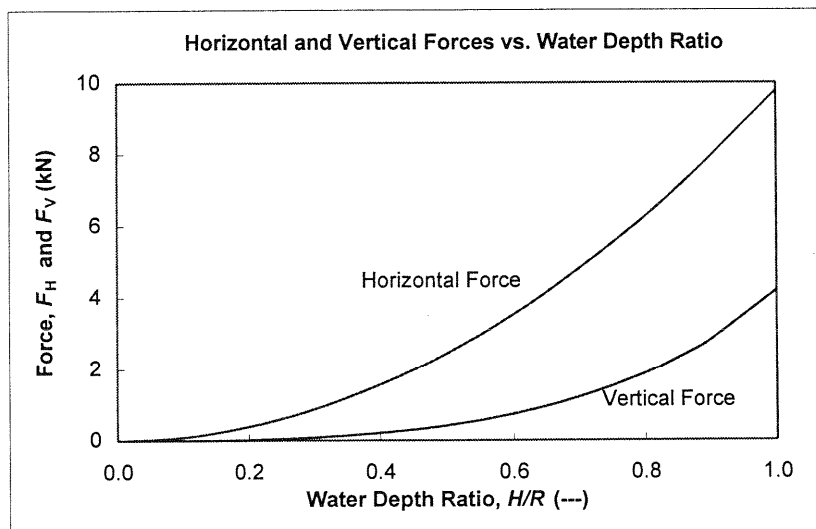
$$y' = \frac{1}{3}H = \frac{1}{3} \times 0.650\,m = 0.217\,m \qquad y'$$

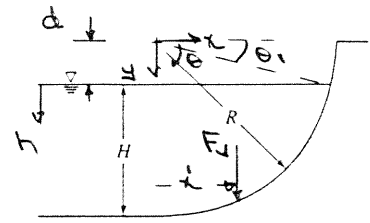
The computing equations for the plot are:

$$\theta_1 = \sin^{-1} H/R$$

$$F_V = \rho g W R^2 \left[\frac{H}{R}(1 - \cos\theta_1) - \frac{\theta_1}{2} + \frac{\sin 2\theta_1}{4} \right]$$

$$x' = \frac{\rho g W R^3 \sin^2\theta_1}{F_V} \left[\frac{1}{2}\frac{H}{R} - \frac{1}{3}\sin\theta_1 \right]$$

$$F_H = \frac{\rho g H^2 W}{2}$$

$$y' = \frac{H}{3}$$

Given: Curved surface, in shape of quarter cylinder, with radius $R = 0.3\text{ m}$ and width $w = 1.25\text{ m}$ is filled to depth $H = 0.24\text{ m}$ with liquid concrete.

Find: (a) Magnitude, and (b) line of action, of the vertical force on the form from the concrete.

Plot: F_V and x' over the range of depth $0 \le H \le R$

Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $F_V = \int p\,dA_y$, $x' F_V = \int x\,dF_V$

Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts at surface of concrete

Then on integrating $dp = \rho g\,dh$, we obtain $p = \rho g h$

$F_V = \int p\,dA_y = \int \rho g h\,dA \sin\theta \qquad dA = wR\,d\theta$

From the geometry: $y = R\sin\theta$, $h = y - d$, $d = R - H$

$$F_V = \int_{\theta_1}^{\pi/2} \rho g (R\sin\theta - d)\sin\theta\, wR\,d\theta \qquad \text{where } \theta_1 = \sin^{-1}\frac{d}{R}$$

$$F_V = \rho g R w \int_{\theta_1}^{\pi/2} (R\sin^2\theta - d\sin\theta)\,d\theta = \rho g R w\left[R\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) + d\cos\theta\right]_{\theta_1}^{\pi/2}$$

$$F_V = \rho g R w\left[R\left(\frac{\pi}{4} - \frac{\theta_1}{2} + \frac{\sin 2\theta_1}{4}\right) - d\cos\theta_1\right] \qquad (1)$$

Evaluating, $\theta_1 = \sin^{-1}\frac{d}{R} = \sin^{-1}\frac{0.3 - 0.24}{0.30} = 11.5°$

$\rho = SG\,\rho_{H_2O} \qquad (SG = 2.50,\ \text{Table A.1})$

$$F_V = 1000\,\frac{\text{kg}}{\text{m}^3} \times 2.5 \times 9.81\,\frac{\text{m}}{\text{s}^2} \times 0.3\,\text{m} \times 1.25\,\text{m} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}\left[0.3\,\text{m}\left(\frac{\pi}{4} - 0.0639\frac{\pi}{2} + \frac{\sin 23°}{4}\right) - 0.06\,\text{m}\cos 11.5°\right]$$

$F_V = 1.62\text{ kN}$ ← F_V

$$x' F_V = \rho g R w \int_{\theta_1}^{\pi/2} x(R\sin^2\theta - d\sin\theta)\,d\theta = \rho g R^2 w \int_{\theta_1}^{\pi/2} (R\sin^2\theta\cos\theta - d\sin\theta\cos\theta)\,d\theta$$

$$= \rho g R^2 w\left[R\,\frac{\sin^3\theta}{3} + \frac{d\cos^2\theta}{2}\right]_{\theta_1}^{\pi/2}$$

$$x' F_V = \rho g R^2 w\left[\frac{R}{3}(1 - \sin^3\theta_1) - \frac{d}{2}\cos^2\theta_1\right]$$

$$x' = SG\,\rho_{H_2O}\,\frac{gR^2 w}{F_V}\left[\frac{R}{3}\left(1-\sin^3\theta_1\right)-\frac{d}{2}\cos^2\theta_1\right] \qquad (2)$$

$$x' = 2.5\times 1000\,\frac{kg}{m^3}\times 9.81\,\frac{m}{s^2}\times (0.3\,m)^2\times 1.25\,m\times\frac{1}{1.62\times10^3\,N}\times\frac{N\cdot s^2}{kg\cdot m}\times$$

$$\left[\frac{0.3\,m}{3}\left(1-\sin^3 11.5^\circ\right)-\frac{0.06\,m}{2}\cos^2 11.5^\circ\right]$$

$$x' = 0.120\,m \qquad x'$$

The computing equations for the required plots are:

$$\theta_1 = \sin^{-1}\frac{R-H}{R} = \sin^{-1}\left(1-\frac{H}{R}\right) \qquad (3)$$

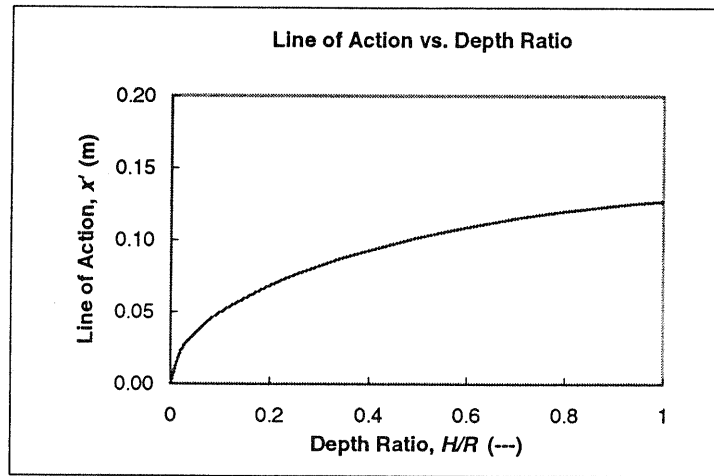
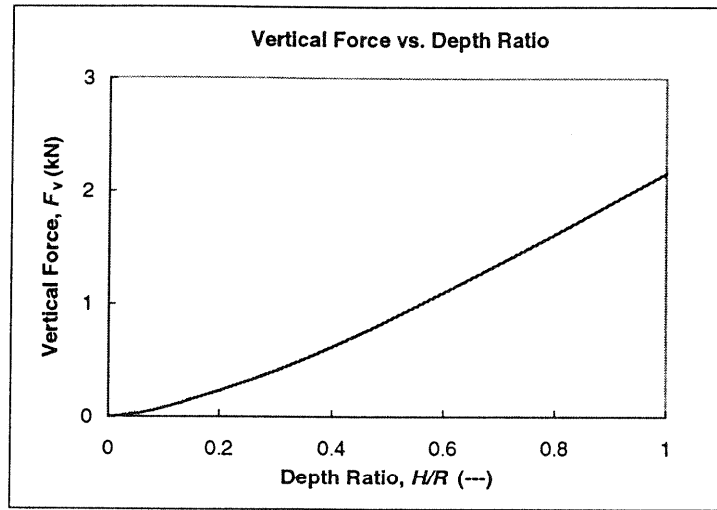
$$F_V = SG\,\rho_{H_2O}\,gR^2 w\left[\frac{\pi}{4}-\frac{\theta_1}{2}+\frac{\sin 2\theta_1}{4}-\left(1-\frac{H}{R}\right)\cos\theta_1\right] \qquad (1a)$$

$$x' = SG\,\frac{\rho_{H_2O}\,gR^3 w}{F_V}\left[\frac{1}{3}\left(1-\sin^3\theta_1\right)-\frac{1}{2}\left(1-\frac{H}{R}\right)\cos^2\theta_1\right] \qquad (2a)$$

Force and line of action vs. liquid concrete depth:

Radius:	***R* =**	**0.3**	**m**
Specific gravity:	***SG* =**	**2.5**	**---**
Width:	***W* =**	**1.25**	**m**

Depth Ratio, H/R (---)	Concrete Depth, H (m)	Angle, θ_1 (deg)	Vertical Force, F_V (kN)	Line of Action, x' (m)
0	0	90.0	0	0
0.02	0.006	78.5	0.00734	0.0224
0.05	0.015	71.8	0.0289	0.0352
0.1	0.03	64.2	0.0810	0.0494
0.2	0.06	53.1	0.226	0.0685
0.3	0.09	44.4	0.408	0.0822
0.4	0.12	36.9	0.617	0.0930
0.5	0.15	30.0	0.847	0.102
0.6	0.18	23.6	1.09	0.109
0.7	0.21	17.5	1.35	0.115
0.8	0.24	11.5	1.62	0.120
0.9	0.27	5.7	1.89	0.124
1.0	0.30	0.0	2.17	0.127

Problem 3.82

Vertical Force vs. Depth Ratio
Vertical Force, F_V (kN)
3
2
1
0
0
0.2
0.4
0.6
0.8
1
Depth Ratio, H/R (---)

Line of Action vs. Depth Ratio
Line of Action, x' (m)
0.20
0.15
0.10
0.05
0.00
0
0.2
0.4
0.6
0.8
1
Depth Ratio, H/R (---)

Problem 3.83

Given: Model cross section of canoe by $y = ax^2$, where $a = 3.89\ m^{-1}$; coordinates are in meters. Assume constant width $W = 0.6m$ over entire length $L = 5.25\ m$.

Find: Expression relating total mass of canoe and contents to distance d; determine maximum allowable total mass without swamping the canoe.

Solution:

At any value of d the weight of the canoe and its contents is balanced by the net vertical force of the water on the canoe.

Basic equations: $\frac{dp}{dh} = \rho g$, $F_V = \int p\,dA_y$

Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts at free surface of the water and on inner surface of canoe.

Then on integrating $dp = \rho g\,dh$, we obtain $p = \rho g h$

$F_V = \int p\,dA_y = \int \rho g h L\,dx$ where $h = (H-d) - y$

$y = ax^2$, At surface $y = H - d$ $\therefore$ $x = \sqrt{\frac{H-d}{a}}$

$$F_V = 2\int_0^{\sqrt{\frac{(H-d)}{a}}} \rho g\left[(H-d) - ax^2\right]L\,dx = 2\rho g L\left[(H-d)x - a\frac{x^3}{3}\right]_0^{\sqrt{\frac{H-d}{a}}}$$

$$F_V = 2\rho g L\left[\frac{(H-d)^{3/2}}{\sqrt{a}} - \frac{a}{3}\frac{(H-d)^{3/2}}{a^{3/2}}\right] = \frac{2\rho g L}{\sqrt{a}}(H-d)^{3/2}\left[1 - \frac{1}{3}\right]$$

$$F_V = \frac{4}{3}\frac{\rho g L}{\sqrt{a}}(H-d)^{3/2} = Mg$$

$$\therefore M = \frac{4\rho L (H-d)^{3/2}}{3\sqrt{a}}$$ At $d = 0$, $x = W/2$, $y = H = 0.35m$

For $d = 0$, $M = \frac{4}{3} \times 999\frac{kg}{m^3} \times 5.25m \times (0.35m)^{3/2} \times \left(\frac{m}{3.89}\right)^{1/2} = 734\ kg$

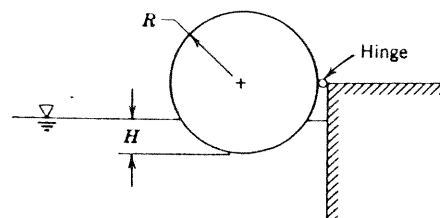
This does not provide any cushion from swamping

Set $d = 0.050\ m$

$M = \frac{4}{3} \times 999\frac{kg}{m^3} \times 5.25m \times (0.30m)^{3/2} \times \left(\frac{m}{3.89}\right)^{1/2} = 583\ kg$ ← M

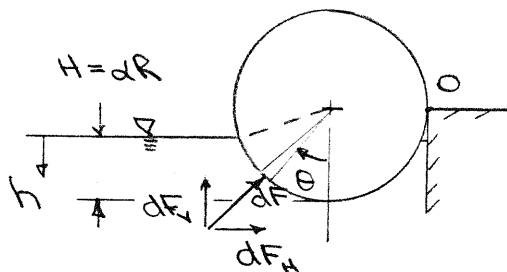
The answer clearly depends on the allowed risk of swamping!

Given: Cylinder, of mass M, length L, and radius R, is hinged along it's length and immersed in an incompressible liquid to depth H.

Find: a general expression for the cylinder specific gravity as a function of $\alpha = H/R$ needed to hold the cylinder in equilibrium for $0 \le \alpha \le 1$.

Solution: Apply fluid statics

Basic eqs.: $\frac{dp}{dh} = \rho g$, $F = \int p\,dA$, $\Sigma M = 0$

Assumptions: (1) static liquid
(2) ρ = constant
$\therefore p = \rho g h$

For $0 \le \alpha \le 1$, F_H causes no net moment about O

$$dF_v = dF\cos\theta = p\,dA\cos\theta = \rho g h\, w R\, d\theta \cos\theta$$

$$h + R(1-\cos\theta) = H, \quad \therefore h = H - R(1-\cos\theta).$$

$$dF_v = \rho g\left[H - R(1-\cos\theta)\right] w R\cos\theta\, d\theta = \rho g w R^2\left[\frac{H}{R} - (1-\cos\theta)\right]\cos\theta\, d\theta$$

$$dF_v = \rho g w R^2\left[(\alpha-1)\cos\theta + \cos^2\theta\right]d\theta = \rho g w R^2\left[(\alpha-1)\cos\theta + \frac{1+\cos 2\theta}{2}\right]$$

For $\alpha \le 1$, $F_H = 0$, and

$$F_v = \int_{-\theta_{max}}^{\theta_{max}} dF_v = 2\int_0^{\theta_{max}} dF_v \quad \text{where } \cos\theta_{max} = \frac{R-H}{R} = 1-\alpha$$

$$\theta_{max} = \cos^{-1}(1-\alpha)$$

$$F_v = 2\rho g w R^2 \int_0^{\theta_{max}}\left[(\alpha-1)\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right]d\theta$$

$$F_v = 2\rho g w R^2\left[(\alpha-1)\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\theta_{max}}$$

$$\sin\theta_{max} = \sqrt{1-\cos^2\theta_{max}} = \left[1-(1-\alpha)^2\right]^{1/2} = \left[1-1+2\alpha-\alpha^2\right]^{1/2} = \sqrt{\alpha(2-\alpha)}.$$

$$\sin 2\theta_{max} = 2\sin\theta_{max}\cos\theta_{max} = 2\sqrt{\alpha(2-\alpha)}\,(1-\alpha)$$

Then,

$$F_v = 2\rho g w R^2\left[(\alpha-1)\sqrt{\alpha(2-\alpha)} + \frac{1}{2}\cos^{-1}(1-\alpha) + \frac{1}{2}(1-\alpha)\sqrt{\alpha(2-\alpha)}\right]$$

$$F_v = 2\rho g w R^2\left[\frac{1}{2}\cos^{-1}(1-\alpha) - \frac{1}{2}(1-\alpha)\sqrt{\alpha(2-\alpha)}\right.$$

$$F_v = \rho g w R^2\left[\cos^{-1}(1-\alpha) - (1-\alpha)\sqrt{\alpha(2-\alpha)}\right] \ldots$$

The line of action of the vertical force due to the liquid is through the centroid of the displaced liquid, i.e. through the center of the cylinder

The weight of the cylinder is given by

$$W = mg = \rho_c \forall g = SG \rho \pi R^2 w g$$

where $SG = \rho_c/\rho$ and the gravity force acts through the center of the cylinder

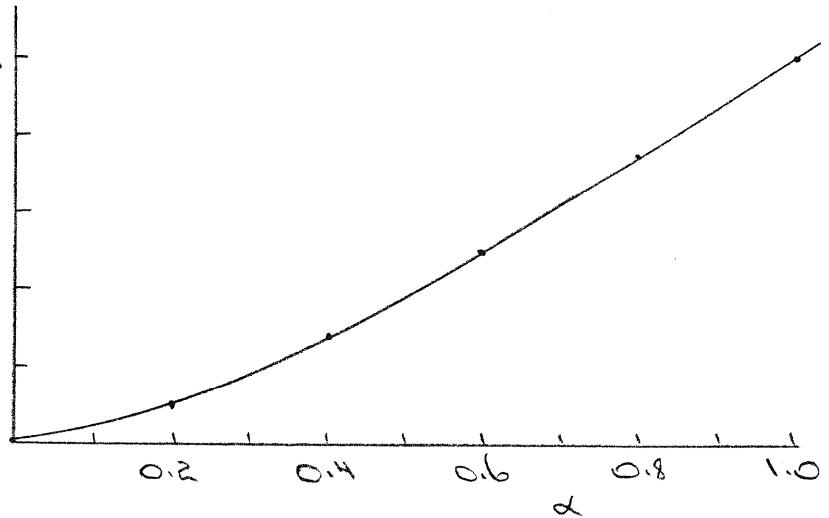
$$\Sigma M_o = WR - F_V R = 0 \quad \therefore \quad W = F_V \text{ and}$$

$$SG \rho \pi R^2 w g = \rho g w R^2 \left[\cos^{-1}(1-\alpha) - (1-\alpha)\sqrt{\alpha(2-\alpha)}\right]$$

$$SG = \frac{1}{\pi}\left[\cos^{-1}(1-\alpha) + (\alpha-1)\sqrt{\alpha(2-\alpha)}\right] \longleftarrow SG(0 \le \alpha \le 1)$$

Tabulating values

α	SG
0	0
0.2	0.052
0.4	0.142
0.6	0.252
0.8	0.374
1.0	0.500

Problem 3.85

Given: Canoe, modelled as a right circular semi-cylindrical shell, floats in water of depth, d. The shell has outer radius, $R = 0.35$ m and length, $L = 5.25$ m.

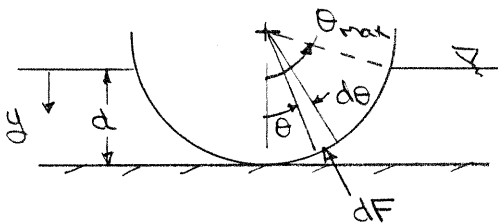
Find: (a) a general algebraic expression for the maximum total mass that can be floated, as a function of depth and

(b) evaluate for the given conditions with $d = 0.245$ m

Plot: the results over the range of water depth $0 \le d \le R$.

Solution:

Basic equations: $\frac{dP}{dy} = \rho g$; $P = P_{atm} + \rho g y$; $F_R = \int P\,dA$

End view of canoe

Assumptions: (1) static liquid
(2) P_{atm} acts on both inside & outside surfaces.

Geometry $y = y(\theta)$ for given d

$$y = d - (R - R\cos\theta) = d - R + R\cos\theta$$

$$\theta_{max} = \cos^{-1}\frac{R-d}{R}$$

A fbd of the canoe gives $\Sigma F_y = 0 = Mg - F_V$
where F_V is the vertical force of the water on the canoe

$$F_V = \int dF_V = \int dF\cos\theta = \int_A P\,dA\cos\theta = \int_{-\theta_{max}}^{\theta_{max}} \rho g y\, LR\,d\theta\cos\theta$$

$$F_V = 2\int_0^{\theta_{max}} \rho g LR\left[(d-R)\cos\theta + R\cos^2\theta\right]d\theta$$

$$F_V = 2\rho g LR\left[(d-R)\sin\theta + R\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\right]_0^{\theta_{max}}$$

$$F_V = 2\rho g LR\left[(d-R)\sin\theta_{max} + R\left(\frac{\theta_{max}}{2} + \frac{\sin 2\theta_{max}}{4}\right)\right]$$

where $\theta_{max} = \cos^{-1}\left(\frac{R-d}{R}\right)$.

Since $M = F_V/g$

$$M = 2\rho LR\left[(d-R)\sin\theta_{max} + R\left(\frac{\theta_{max}}{2} + \frac{\sin 2\theta_{max}}{4}\right)\right] \quad M(d)$$

For $R = 0.35$ m, $L = 5.25$ m and $d = 0.245$ m,

$$\theta_{max} = \cos^{-1}\frac{(R-d)}{R} = \cos^{-1}\frac{(0.35 - 0.245)}{0.35} = \cos^{-1} 0.30 = 72.5°$$

$$\theta_{max} = 0.403\,\pi$$

$$M = 2 \times 999\frac{kg}{m^3} \times 5.25\,m \times 0.35\,m\left[(0.245 - 0.35)\sin 72.5 + 0.35\left(\frac{0.403\pi}{2} + \frac{1}{4}\sin 145\right)\right]m$$

$$M = 631\ kg \quad M$$

Problem 3.85

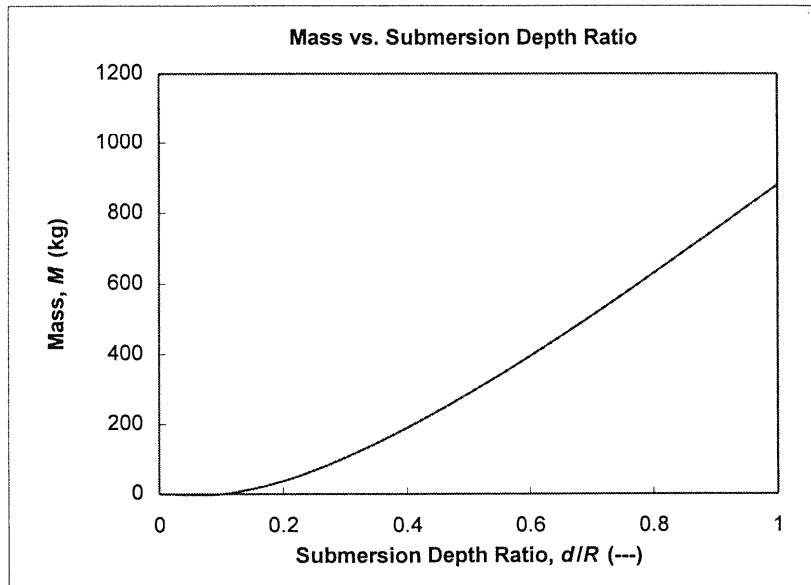
The computing equations for the plot are.

$$\theta_{max} = \cos^{-1}\left(1 - \frac{d}{R}\right)$$

$$M = 2\rho L R^2 \left[\frac{\theta_{max}}{2} + \frac{\sin 2\theta_{max}}{4} - \left(1 - \frac{d}{R}\right)\sin\theta_{max} \right]$$

Mass of canoe vs. depth of submersion ratio:

Density:	ρ =	999	kg/m^3
Length:	L =	5.25	m
Radius:	R =	0.35	m

d (m)	d/R (---)	θ_{max} (rad)	θ_{max} (deg)	Mass (kg)
0	0	0	0	0
0.035	0.10	0.45	25.8	37.7
0.070	0.20	0.64	36.9	105
0.105	0.30	0.80	45.6	190
0.140	0.40	0.93	53.1	287
0.175	0.50	1.05	60.0	395
0.210	0.60	1.16	66.4	509
0.245	0.70	1.27	72.5	630
0.280	0.80	1.37	78.5	754
0.315	0.90	1.47	84.3	881
0.350	1.00	1.57	90.0	1009

Problem 3.86

3.86 A glass observation room is to be installed at the corner of the bottom of an aquarium. The aquarium is filled with seawater to a depth of 10 m. The glass is a segment of a sphere, radius 1.5 m, mounted symmetrically in the corner. Compute the magnitude and direction of the net force on the glass structure.

Given: Geometry of glass observation room

Find: Resultant force and direction

Solution:

The *x*, *y* and *z* components of force due to the fluid are treated separately. For the *x*, *y* components, the horizontal force is equivalent to that on a vertical flat plate; for the *z* component, (vertical force) the force is equivalent to the weight of fluid above.

For horizontal forces, the computing equation of Section 3-5 is $F_H = p_c \cdot A$ where A is the area of the equivalent vertical plate.

For the vertical force, the computing equation of Section 3-5 is $F_V = \rho \cdot g \cdot V$ where V is the volume of fluid above the curved surface.

The data is For water $\rho = 999 \cdot \frac{kg}{m^3}$

For the fluid (Table A.2) $SG = 1.025$

For the aquarium $R = 1.5 \cdot m$ $H = 10 \cdot m$

(a) Horizontal Forces

Consider the *x* component

The center of pressure of the glass is $y_c = H - \frac{4 \cdot R}{3 \cdot \pi}$ $y_c = 9.36\,m$

Hence $F_{Hx} = p_c \cdot A = (SG \cdot \rho \cdot g \cdot y_c) \cdot \frac{\pi \cdot R^2}{4}$

$$F_{Hx} = 1.025 \times 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 9.36 \cdot m \times \frac{\pi \cdot (1.5 \cdot m)^2}{4} \times \frac{N \cdot s^2}{kg \cdot m} \qquad F_{Hx} = 166\,kN$$

The *y* component is of the same magnitude as the *x* component

$$F_{Hy} = F_{Hx} \qquad F_{Hy} = 166\,kN$$

The resultant horizontal force (at 45^o to the *x* and *y* axes) is

$$F_H = \sqrt{F_{Hx}^2 + F_{Hy}^2} \qquad F_H = 235\,kN$$

(b) Vertical forces

The vertical force is equal to the weight of fluid above (a volume defined by a rectangular column minus a segment of a sphere)

The volume is $V = \frac{\pi \cdot R^2}{4} \cdot H - \frac{\frac{4 \cdot \pi \cdot R^3}{3}}{8}$ $V = 15.9\,m^3$

Then $F_V = SG \cdot \rho \cdot g \cdot V$ $F_V = 1.025 \times 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 15.9 \cdot m^3 \times \frac{N \cdot s^2}{kg \cdot m}$ $F_V = 160\,kN$

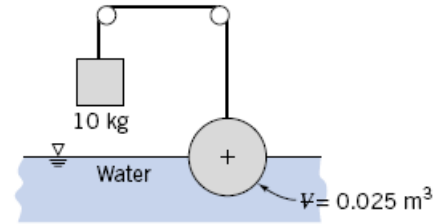
Finally the resultant force and direction can be computed

$$F = \sqrt{F_H^2 + F_V^2} \qquad F = 284\,kN$$

$$\alpha = \operatorname{atan}\left(\frac{F_V}{F_H}\right) \qquad \alpha = 34.2\,deg$$

Note that α is the angle the resultant force makes with the horizontal

Problem *3.87

*3.87 Find the specific weight of the sphere shown if its volume is 0.025 m^3. State all assumptions. What is the equilibrium position of the sphere if the weight is removed?

Given: Data on sphere and weight

Find: SG of sphere; equilibrium position when freely floating

Solution:

Basic equation $F_B = \rho \cdot g \cdot V$ and $\Sigma F_z = 0$ $\quad \Sigma F_z = 0 = T + F_B - W$

T, F_B, W

where $T = M \cdot g$ $\quad M = 10 \cdot kg$ $\quad F_B = \rho \cdot g \cdot \frac{V}{2}$ $\quad W = SG \cdot \rho \cdot g \cdot V$

Hence $M \cdot g + \rho \cdot g \cdot \frac{V}{2} - SG \cdot \rho \cdot g \cdot V = 0$ $\quad SG = \frac{M}{\rho \cdot V} + \frac{1}{2}$

$$SG = 10 \cdot kg \times \frac{m^3}{1000 \cdot kg} \times \frac{1}{0.025 \cdot m^3} + \frac{1}{2} \qquad SG = 0.9$$

The specific weight is $\gamma = \frac{Weight}{Volume} = \frac{SG \cdot \rho \cdot g \cdot V}{V} = SG \cdot \rho \cdot g$ $\quad \gamma = 0.9 \times 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}$ $\quad \gamma = 8829 \cdot \frac{N}{m^3}$

For the equilibriul position when floating, we repeat the force balance with T = 0

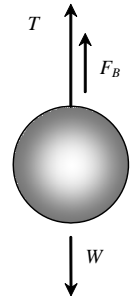
$$F_B - W = 0 \qquad W = F_B \quad \text{with} \quad F_B = \rho \cdot g \cdot V_{submerged}$$

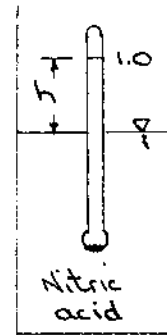
From references (trying Googling "partial sphere volume") $V_{submerged} = \frac{\pi \cdot h^2}{3} \cdot (3 \cdot R - h)$

where h is submerged depth and R is the sphere radius $R = \left(\frac{3 \cdot V}{4 \cdot \pi}\right)^{\frac{1}{3}}$ $\quad R = \left(\frac{3}{4 \cdot \pi} \cdot 0.025 \cdot m^3\right)^{\frac{1}{3}}$ $\quad R = 0.181\,m$

Hence $W = SG \cdot \rho \cdot g \cdot V = F_B = \rho \cdot g \cdot \frac{\pi \cdot h^2}{3} \cdot (3 \cdot R - h)$ $\quad h^2 \cdot (3 \cdot R - h) = \frac{3 \cdot SG \cdot V}{\pi}$

$$h^2 \cdot (3 \cdot 0.181 \cdot m - h) = \frac{3 \cdot 0.9 \cdot .025 \cdot m^3}{\pi} \qquad h^2 \cdot (0.544 - h) = 0.0215$$

This is a cubic equation for h. We can keep guessing h values, manually iterate, or use Excel's Goal Seek to find $h = 0.292 \cdot m$

Problem 3.88

Given: Hydrometer, as shown, submerged in nitric acid, S.G = 1.5
When immersed in water, $h = 0$ and immersed volume is $15\ \text{cm}^3$.
Stem diameter $d = 6\ \text{mm}$.

Find: The distance, h

Solution:

Basic equation: $\sum \vec{F} = m\vec{a} = 0$

Computing equation: $\vec{F}_{buoyancy} = \rho g \forall_d \hat{k}$

Assumptions: (1) static conditions
(2) $\rho =$ constant

$$\sum \vec{F} = 0 = M\vec{g} + \vec{F}_{buoyancy}$$

Using the data given for water, we can calculate M

$$-Mg + F_b = 0 \qquad M = \frac{F_b}{g} = \rho_{H_2O} \forall_{H_2O}$$

When immersed in nitric acid

$$M = \rho_{n.a} \forall_{n.a} \qquad \text{where } \forall_{n.a} = \forall_{H_2O} - \frac{\pi d^2 h}{4}$$

Since the mass is the same in both cases

$$M = \rho_{H_2O} \forall_{H_2O} = \rho_{n.a}\left(\forall_{H_2O} - \frac{\pi d^2 h}{4}\right)$$

$$\frac{\pi d^2 h}{4} = \forall_{H_2O} - \frac{\rho_{H_2O}}{\rho_{n.a}} \forall_{H_2O} = \forall_{H_2O}\left(1 - \frac{1}{S.G_{n.a}}\right)$$

$$h = \frac{4 \forall_{H_2O}}{\pi d^2}\left(1 - \frac{1}{S.G_{n.a}}\right)$$

$$h = \frac{4}{\pi} \times 15\ \text{cm}^3 \times \frac{1}{6^2\ \text{mm}^2}\left(1 - \frac{1}{1.5}\right) \times \frac{1000\ \text{mm}^3}{\text{cm}^3} = 177\ \text{mm}$$

h

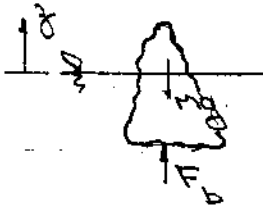
Problem *3.89

[2]

Given: Iceberg floating in sea water

Find: Quantify the statement "only the tip of an iceberg shows"

Solution:

A floating body is buoyed up by a force equal to the weight of the displaced liquid.

$$\Sigma F_z = 0 = F_b - mg$$

$$F_b = \rho_s \forall_{sub}\, g \qquad m = \rho \forall_{total}$$

$$\therefore \rho_s \forall_{sub}\, g = \rho \forall_{tot}\, g$$

$$\therefore \forall_{sub} = \forall_{tot} \frac{\rho}{\rho_{sw}} = \forall_{tot} \frac{\rho/\rho^*}{\rho_{sw}/\rho^*}$$

where $\rho^* = \rho_{H_2O}$ at 4C.

$$\forall_{sub} = \forall_{tot} \frac{SG_{ice}}{SG_{sw}}$$

$$\forall_{not\,sub} = \forall_{tot} - \forall_{sub} = \forall_{tot}\left(1 - \frac{SG_{ice}}{SG_{sw}}\right)$$ {Table A.1, $SG_{ice} = 0.917$; Table A.2, $SG_{sw} = 1.025$}

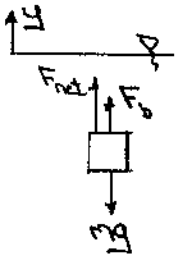
$$\therefore \frac{\forall_{not\,sub}}{\forall_{tot}} = 1 - \frac{SG_{ice}}{SG_s} = 1 - \frac{0.917}{1.025}$$

$$\frac{\forall_{not\,sub}}{\forall_{tot}} = 0.105 \qquad (10\%\text{ shows})$$

Problem *3.90

Given: Specific gravity of a person is to be determined from measurements of weight in air and the net weight when totally immersed in water.

Find: Expression for the specific gravity of a person from the measurements.

Solution:

For equilibrium $\Sigma F_y = 0$

$$F_{net} = mg - F_b$$

$$F_b = \rho_{H_2O}\, g\, \forall \qquad F_{air} = mg$$

$$\therefore F_{net} = F_{air} - \rho_{H_2O}\, g\, \forall \quad \text{and} \quad \forall = \frac{F_{air} - F_{net}}{\rho_{H_2O}\, g}$$

$$F_{air} = mg = \rho \forall g = \frac{\rho}{\rho_{H_2O}}(F_{air} - F_{net})$$

Let $\rho^* = \rho_{H_2O}$ at 4C. Then

$$F_{air} = \frac{\rho/\rho^*}{\rho_{H_2O}/\rho^*}(F_{air} - F_{net}) = \frac{SG}{SG_{H_2O}}(F_{air} - F_{net}).$$

Solving for SG,

$$SG = SG_{H_2O}\frac{F_{air}}{(F_{air} - F_{net})}$$

SG

Problem *3.91

[2]

Given: Experiment performed by Archimedes to identify the material content of King Hero's crown.

Measured weight of crown in air, W_a, and in water, W_w.

Find: Expression for specific gravity of crown as function of W_a and W_w.

Solution: Apply principle of bouyancy to free-body of crown:

Computing equation: $F_B = \rho_{H_2O} g \forall$

Assumptions: (1) Static liquid
(2) Incompressible liquid

Free-body diagram of crown in water:

W_w, z, Mg, F_B

$$\Sigma F_z = W_w - Mg + F_B = ma_z = 0$$

or

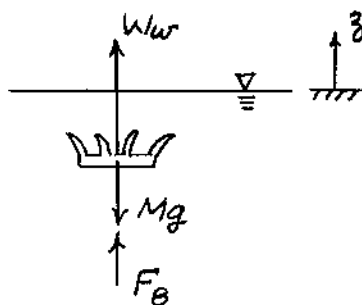
$$W_w - Mg + \rho_{H_2O} \forall g = 0$$

For the crown in air, $W_a = Mg$

Combining, $W_w - W_a + \rho_{H_2O} g \forall$, so $\forall = \dfrac{W_a - W_w}{\rho_{H_2O} g}$

The crown's density is $\rho_c = \dfrac{M}{\forall} = \dfrac{W_a}{g\forall} = \rho_{H_2O} \dfrac{W_a}{W_a - W_w}$

The crown's specific gravity is $SG = \dfrac{\rho_c}{\rho_{H_2O}} = \dfrac{W_a}{W_a - W_w}$ ← SG

{Note: by definition, $SG = \rho / \rho_{H_2O}(4°C)$, so the measured temperature of water and data from Table A.7 or A.8 may be used to correct the density to 4°C.}

Problem *3.92

*3.92 An open tank is filled to the top with water. A steel cylindrical container, wall thickness $\delta = 1$ mm, outside diameter $D = 100$ mm, and height $H = 1$ m, with an open top, is gently placed in the water. What is the volume of water that overflows from the tank? How many 1 kg weights must be placed in the container to make it sink? Neglect surface tension effects.

Given: Geometry of steel cylinder

Find: Volume of water displaced; number of 1 kg wts to make it sink

Solution:

The data is For water $\rho = 999 \cdot \frac{kg}{m^3}$

For steel (Table A.1) $SG = 7.83$

For the cylinder $D = 100 \cdot mm$ $H = 1 \cdot m$ $\delta = 1 \cdot mm$

The volume of the cylinder is $V_{steel} = \delta \cdot \left(\frac{\pi \cdot D^2}{4} + \pi \cdot D \cdot H \right)$ $V_{steel} = 3.22 \times 10^{-4} m^3$

The weight of the cylinder is $W = SG \cdot \rho \cdot g \cdot V_{steel}$

$$W = 7.83 \times 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 3.22 \times 10^{-4} \cdot m^3 \times \frac{N \cdot s^2}{kg \cdot m} \qquad W = 24.7\,N$$

At equilibium, the weight of fluid displaced is equal to the weight of the cylinder

$$W_{displaced} = \rho \cdot g \cdot V_{displaced} = W$$

$$V_{displaced} = \frac{W}{\rho \cdot g} = 24.7 \cdot N \times \frac{m^3}{999 \cdot kg} \times \frac{s^2}{9.81 \cdot m} \times \frac{kg \cdot m}{N \cdot s^2} \qquad V_{displaced} = 2.52\,L$$

To determine how many 1 kg wts will make it sink, we first need to find the extra volume that will need to be dsiplaced

Distance cylinder sank $x_1 = \frac{V_{displaced}}{\left(\frac{\pi \cdot D^2}{4} \right)}$ $x_1 = 0.321\,m$

Hence, the cylinder must be made to sink an additional distance $x_2 = H - x_1$ $x_2 = 0.679\,m$

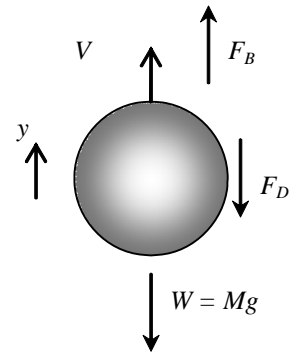
We deed to add n weights so that $1 \cdot kg \cdot n \cdot g = \rho \cdot g \cdot \frac{\pi \cdot D^2}{4} \cdot x_2$

$$n = \frac{\rho \cdot \pi \cdot D^2 \cdot x_2}{4 \times 1 \cdot kg} = 999 \cdot \frac{kg}{m^3} \times \frac{\pi}{4} \times (0.1 \cdot m)^2 \times 0.679 \cdot m \times \frac{1}{1 \cdot kg} \times \frac{N \cdot s^2}{kg \cdot m} \qquad n = 5.33$$

Hence we need $n = 6$ weights to sink the cylinder

Problem *3.93

[2]

*3.93 Hydrogen bubbles are used to visualize water flow streaklines in the video, *Flow Visualization*. A typical hydrogen bubble diameter is $d = 0.001$ in. The bubbles tend to rise slowly in water because of buoyancy; eventually they reach terminal speed relative to the water. The drag force of the water on a bubble is given by $F_D = 3\pi\mu Vd$, where μ is the viscosity of water and V is the bubble speed relative to the water. Find the buoyancy force that acts on a hydrogen bubble immersed in water. Estimate the terminal speed of a bubble rising in water.

Given: Data on hydrogen bubbles

Find: Buoyancy force on bubble; terminal speed in water

Solution:

Basic equation $F_B = \rho \cdot g \cdot V = \rho \cdot g \cdot \frac{\pi}{6} \cdot d^3$ and $\Sigma F_y = M \cdot a_y$ $\quad \Sigma F_y = 0 = F_B - F_D - W$ for terminal speed

$$F_B = 1.94 \cdot \frac{\text{slug}}{\text{ft}^3} \times 32.2 \cdot \frac{\text{ft}}{\text{s}^2} \times \frac{\pi}{6} \times \left(0.001 \cdot \text{in} \times \frac{1 \cdot \text{ft}}{12 \cdot \text{in}}\right)^3 \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}} \qquad F_B = 1.89 \times 10^{-11} \cdot \text{lbf}$$

For terminal speed $F_B - F_D - W = 0$ $\qquad F_D = 3 \cdot \pi \cdot \mu \cdot V \cdot d = F_B$ $\quad$ where we have ignored W, the weight of the bubble (at STP most gases are about 1/1000 the density of water)

Hence $V = \frac{F_B}{3 \cdot \pi \cdot \mu \cdot d}$ with $\mu = 2.10 \times 10^{-5} \cdot \frac{\text{lbf} \cdot \text{s}}{\text{ft}^2}$ from Table A.7 at 68ºF

$$V = 1.89 \times 10^{-11} \cdot \text{lbf} \times \frac{1}{3 \cdot \pi} \times \frac{1}{2.10 \times 10^{-5}} \cdot \frac{\text{ft}^2}{\text{lbf} \cdot \text{s}} \times \frac{1}{0.001 \cdot \text{in}} \times \frac{12 \cdot \text{in}}{1 \cdot \text{ft}}$$

$$V = 1.15 \times 10^{-3} \cdot \frac{\text{ft}}{\text{s}} \qquad V = 0.825 \cdot \frac{\text{in}}{\text{min}}$$

As noted by Professor Kline in the film "Flow Visualization", bubbles rise slowly!

Problem *3.94

[2]

Gas bubbles are released from the regulator of a submerged scuba diver. What happens to the bubbles as they rise through the seawater? Explain.

Open-Ended Problem Statement: Gas bubbles are released from the regulator of a submerged Scuba diver. What happens to the bubbles as they rise through the seawater?

Discussion: Air bubbles released by a submerged diver should be close to ambient pressure at the depth where the diver is swimming. The bubbles are small compared to the depth of submersion, so each bubble is exposed to essentially constant pressure. Therefore the released bubbles are nearly spherical in shape.

The air bubbles are buoyant in water, so they begin to rise toward the surface. The bubbles are quite light, so they reach terminal speed quickly. At low speeds the spherical shape should be maintained. At higher speeds the bubble shape may be distorted.

As the bubbles rise through the water toward the surface, the hydrostatic pressure decreases. Therefore the bubbles expand as they rise. As the bubbles grow larger, one would expect the tendency for distorted bubble shape to be exaggerated.

Problem *3.95

Given: Balloons with hot air, helium, and hydrogen. Claim lift per cubic foot of 0.018, 0.066, and 0.071 lbf/ft³ for respective gases, with air heated to 150°F over ambient.

Find: (a) Evaluate claims
(b) Compare air at 250°F above ambient.

Solution: Assume ambient conditions are STP, $p_{gas} = p_{air}$, and apply ideal gas equation of state.

(Use data from Table A.6.)

Basic equations: $\text{Lift} = \rho_{air} g \forall - \rho_{gas} g \forall$, $p = \rho R T$

Then

$$\text{Lift}/\forall = g(\rho_a - \rho_g) = \rho_a g\left(1 - \frac{\rho_g}{\rho_a}\right) = \rho_a g\left(1 - \frac{R_a T_a}{R_g T_g}\right); \quad \rho_a g = 0.0765 \frac{\text{lbf}}{\text{ft}^3}$$

For helium

$$\frac{L}{\forall} = 0.0765 \frac{\text{lbf}}{\text{ft}^3}\left[1 - 53.33 \frac{\text{ft}\cdot\text{lbf}}{\text{lbm}\cdot R} \times (460+59)R \times \frac{\text{lbm}\cdot R}{386.1\ \text{ft}\cdot\text{lbf}} \times \frac{1}{(460+59\text{F})}\right]$$

$$\frac{L}{\forall} = 0.0659\ \text{lbf}/\text{ft}^3 \quad \text{(rounds to 0.066)}$$

He

For hydrogen

$$\frac{L}{\forall} = 0.0765 \frac{\text{lbf}}{\text{ft}^3}\left(1 - \frac{53.33}{766.5}\right) = 0.0712\ \text{lbf}/\text{ft}^3 \quad \text{(rounds to 0.071)}$$

H_2

For air at 150°F above ambient,

$$\frac{L}{\forall} = 0.0765 \frac{\text{lbf}}{\text{ft}^3}\left[1 - \frac{53.33\,(460+59)}{53.33\,(460+59+150)}\right] = 0.0172\ \text{lbf}/\text{ft}^3$$

Air $\Delta T = 150$

For air at 250°F above ambient,

$$\frac{L}{\forall} = 0.0765 \frac{\text{lbf}}{\text{ft}^3}\left[1 - \frac{53.33\,(460+59)}{53.33\,(460+59+250)}\right] = 0.0249\ \text{lbf}/\text{ft}^3$$

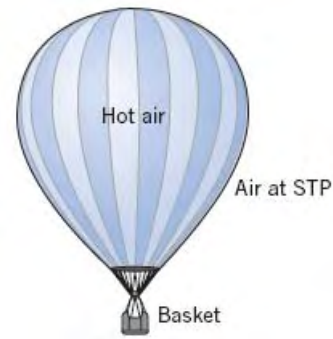
Air $\Delta T = 250$ F

Agreement with claims is good.

Air at $\Delta T = 250°F$ gives 45 percent more lift than at $\Delta T = 150°F$.

{ Hot air balloon needs 40.2 ft³/lbf of lift at $\Delta T = 250°F$! }

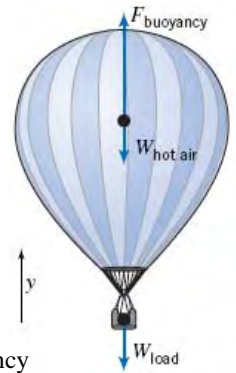
Problem *3.96

***3.96** A hot air balloon is designed to lift a basket, two people, three gallons of fuel, a pair of binoculars, a camera, a GPS, a cell phone, a pair of blankets, twelve candy bars, and the components of the balloon itself (fabric, ropes, and torch). The total mass is estimated at 450 kg. The rides are planned in summer morning hours when the air temperature is about 9°C. The torch will warm the air inside the balloon to a temperature of 70°C. Both inside and outside pressures will be "standard" (101 kPa). What volume of hot air should the balloon hold to create neutral buoyancy? What additional volume will ensure a vertical take-off acceleration of 0.8 m/s^2? For this, consider that both balloon and inside air have to be accelerated, as well as some of the surrounding air (to make way for the balloon). The rule of thumb is that the total mass subject to acceleration is the mass of the balloon, all its appurtenances, and twice its volume of air. Given that the volume of hot air is fixed during flight, what can the balloonists do when they want to go down?

Given: Data on hot air balloon

Find: Volume of balloon for neutral buoyancy; additional volume for initial acceleration of 0.8 m/s^2.

Solution:

Basic equation $F_B = \rho_{atm} \cdot g \cdot V$ and $\Sigma F_y = M \cdot a_y$

Hence $\Sigma F_y = 0 = F_B - W_{hotair} - W_{load} = \rho_{atm} \cdot g \cdot V - \rho_{hotair} \cdot g \cdot V - M \cdot g$ for neutral buoyancy

$$V = \frac{M}{\rho_{atm} - \rho_{hotair}} = \frac{M}{\dfrac{p_{atm}}{R \cdot T_{atm}} - \dfrac{p_{atm}}{R \cdot T_{hotair}}} = \frac{M \cdot R}{p_{atm}} \cdot \left(\frac{1}{\dfrac{1}{T_{atm}} - \dfrac{1}{T_{hotair}}} \right)$$

$$V = 450 \cdot kg \times 286.9 \cdot \frac{N \cdot m}{kg \cdot K} \times \frac{1}{101 \times 10^3} \cdot \frac{m^2}{N} \times \left[\frac{1}{\dfrac{1}{(9 + 273) \cdot K} - \dfrac{1}{(70 + 273) \cdot K}} \right] \qquad V = 2027 \cdot m^3$$

Initial acceleration $\Sigma F_y = F_B - W_{hotair} - W_{load} = (\rho_{atm} - \rho_{hotair}) \cdot g \cdot V_{new} - M \cdot g = M_{accel} \cdot a = (M + 2 \cdot \rho_{hotair} \cdot V_{new}) \cdot a$

Solving for V_{new} $(\rho_{atm} - \rho_{hotair}) \cdot g \cdot V_{new} - M \cdot g = (M + 2 \cdot \rho_{hotair} \cdot V_{new}) \cdot a$

$$V_{new} = \frac{M \cdot g + M \cdot a}{(\rho_{atm} - \rho_{hotair}) \cdot g - 2 \cdot \rho_{hotair} \cdot a} = \frac{M \cdot \left(1 + \dfrac{a}{g}\right) \cdot R}{p_{atm} \cdot \left[\left(\dfrac{1}{T_{atm}} - \dfrac{1}{T_{hotair}} \right) - \dfrac{2}{T_{hotair}} \cdot \dfrac{a}{g} \right]}$$

$$V_{new} = 450 \cdot kg \times \left(1 + \frac{0.8}{9.81}\right) \times 286.9 \cdot \frac{N \cdot m}{kg \cdot K} \times \frac{1}{101 \times 10^3} \cdot \frac{m^2}{N} \times \frac{1}{\left(\dfrac{1}{9 + 273} - \dfrac{1}{70 + 273} - \dfrac{2}{70 + 273} \cdot \dfrac{0.8}{9.81} \right)} \cdot K$$

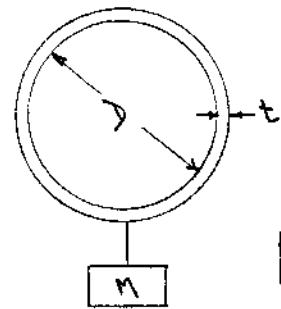
$V_{new} = 8911 \cdot m^3$ Hence $\Delta V = V_{new} - V$ $\Delta V = 6884 \cdot m^3$

To make the balloon move up or down during flight, the air needs to be heated to a higher temperature, or let cool (or let in ambient air).

Problem *3.97

Given: Spherical balloon of diameter D, and skin thickness, $t = 0.013$ mm, filled with helium lifted a payload of mass $M = 230$ kg to an altitude of 49 km. At altitude,

$P = 0.95$ mbar and $T = -20°C$.

The helium temperature is $-10°C$. The specific gravity of the skin material is 1.28

Find: The diameter and mass of the balloon.

Solution: Basic equation $\Sigma \vec{F} = m\vec{a} = 0$

Assumptions (1) static equilibrium at altitude of 49 km
(2) air and helium exhibit ideal gas behavior.

$$\Sigma F_z = 0 = F_{buoy} - M_{He}g - M_s g - Mg = \rho_{air} g \forall_b - \rho_{He} g \forall_b - \rho_s \forall_s - Mg$$

$$0 = \forall_b(\rho_{air} - \rho_{He}) - \rho_s A_s t - M = \frac{4}{3}\pi R^3(\rho_{air} - \rho_{He}) - \rho_s 4\pi R^2 t - M$$

$$0 = \frac{\pi D^3}{6}(\rho_{air} - \rho_{He}) - \rho_s \pi D^2 t - M$$

This is a cubic equation which requires an iterative solution

$$\pi D^2\left[\frac{D}{6}(\rho_{air} - \rho_{He}) - \rho_s t\right] - M = 0 \qquad \text{Solving for } D,$$

$$D = \frac{6}{(\rho_{air} - \rho_{He})}\left[\frac{M}{\pi D^2} + \rho_s t\right] = 6\left[\frac{M}{\pi D^2(\rho_{air} - \rho_{He})} + \frac{\rho_s t}{(\rho_{air} - \rho_{He})}\right]$$

From the ideal gas law,

$$\rho_{air} = \frac{P}{RT} = 0.95 \times 10^{-3}\,\text{bar} \times \frac{\text{kg}\cdot\text{K}}{287\,\text{J}} \times \frac{1}{253\,\text{K}} \times \frac{10^5\,\text{Pa}}{\text{bar}} \times \frac{\text{N}}{\text{Pa}\cdot\text{m}^2} \times \frac{\text{J}}{\text{N}\cdot\text{m}} = 1.31 \times 10^{-3}\,\frac{\text{kg}}{\text{m}^3}$$

$$\rho_{He} = \frac{P}{RT} = 0.95 \times 10^{-3}\,\text{bar} \times \frac{\text{kg}\cdot\text{K}}{2080} \times \frac{1}{263\,\text{K}} \times \frac{10^5\,\text{Pa}}{\text{bar}} \times \frac{\text{N}}{\text{Pa}\cdot\text{m}^2} \times \frac{\text{J}}{\text{N}\cdot\text{m}} = 1.74 \times 10^{-4}\,\frac{\text{kg}}{\text{m}^3}$$

Substituting into the expression for D

$$D = 6\left[\frac{1}{\pi D^2} \times 230\,\text{kg} \times \frac{\text{m}^3}{11.4 \times 10^{-4}\,\text{kg}} + (1.28)999\,\frac{\text{kg}}{\text{m}^3} \times 1.3 \times 10^{-5}\,\text{m} \times \frac{\text{m}^3}{11.4 \times 10^{-4}\,\text{kg}}\right]$$

$$D = \left[\frac{38.5 \times 10^4}{D^2} + 87.5\right] \qquad \text{where } D \text{ is in meters}$$

Organizing Calculations: Guess D (m) = 100 120 116

RHS = 126 114 116.1

$\therefore D = 116$ m ← D

$$M_b = \rho_s \forall_s = \rho_s A_s t = \rho_s \pi D^2 t = 1.28 \times 999\,\frac{\text{kg}}{\text{m}^3} \times \pi(116)^2\,\text{m}^2 \times 1.3 \times 10^{-5}\,\text{m}$$

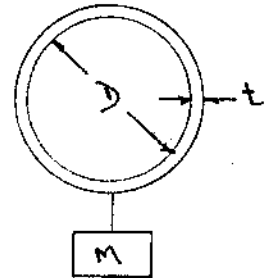
$M_b = 703$ kg ← M_b

Problem *3.98

Given: A pressurized helium balloon is to be designed to lift a payload of mass, M, to an altitude of 40 km, where

$P = 3.0$ mbar and $T = -25°C$.

The balloon skin has a specific gravity, S.G = 1.28 and thickness, $t = 0.015$ mm. The gage pressure of the helium is 0.45 mbar. The allowable tensile stress in the balloon skin is $\sigma = 62\ MN/m^2$

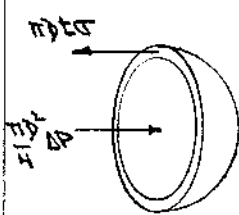
Find: (a) Maximum balloon diameter
(b) Payload, M

Solution:

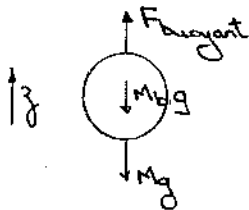
Basic equation: $\Sigma \vec{F} = m\vec{a} = 0$

Assumptions: (1) static equilibrium at altitude.
(2) air and helium exhibit ideal gas behavior.

The balloon diameter is limited by tensile stress

$$\Sigma F = 0 = \frac{\pi D^2}{4}\Delta P - \pi D t \sigma$$

$$D_{max} = \frac{4t\sigma}{\Delta P}$$

$$D_{max} = 4 \times 1.50 \times 10^{-5}\ m \times 62 \times 10^{6}\ \frac{N}{m^2} \times \frac{1}{0.45 \times 10^{-3}\ bar} \times \frac{bar\ m^2}{10^5\ N}$$

$$D_{max} = 82.7\ m$$ ← D_{max}

$$\Sigma F_z = 0 = F_{buoy} - M_{He}g - M_b g - Mg \qquad M_{He} = \rho_{He}\forall$$

$$F_{buoy} - M_{He}g = (\rho_{air} - \rho_{He})g\forall = (\rho_{air} - \rho_{He})g\frac{\pi D^3}{6}$$

$$M_b = \rho_s \forall_s = \rho_s A_s t_s = \rho_s \pi D^2 t$$

$$\therefore M = \frac{F_{buoy}}{g} - M_b = (\rho_{air} - \rho_{He})\frac{\pi D^3}{6} - \rho_s \pi D^2 t$$

$$M = \pi D^2 \left[(\rho_{air} - \rho_{He})\frac{D}{6} - \rho_s t\right]$$

From ideal gas law

$$\rho_{air} = \frac{P}{RT} = 3.0 \times 10^{-3}\ bar \times \frac{kg \cdot K}{287\ J} \times \frac{1}{248\ K} \times \frac{10^5\ Pa}{bar} \times \frac{N}{Pa\ m^2} \times \frac{J}{N \cdot m} = 4.21 \times 10^{-3}\ kg/m^3$$

$$\rho_{He} = \frac{(P + \Delta P)}{RT} = 3.45 \times 10^{-3}\ bar \times \frac{kg \cdot K}{2080\ J} \times \frac{1}{248\ K} \times \frac{10^5\ Pa}{bar} \times \frac{N}{Pa \cdot m^2} \times \frac{J}{N \cdot m} = 6.69 \times 10^{-4}\ kg/m^3$$

Then,

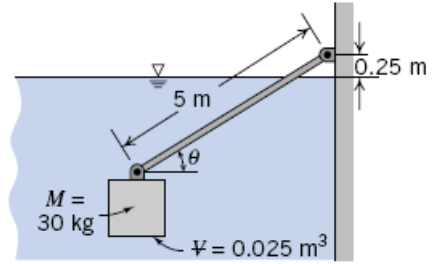
$$M = \pi (82.7)^2\ m^2 \left[(42.1 - 6.69) \times 10^{-4}\ \frac{kg}{m^3} \times \frac{82.7\ m}{6} - 1.28 \times 999\ \frac{kg}{m^3} \times 1.5 \times 10^{-5}\ m\right]$$

$$M = 637\ kg$$ ← M

Problem 3.99

*3.99 A block of mass 30 kg and volume 0.025 m^3 is allowed to sink in water as shown. A circular rod 5 m long and 20 cm^2 in cross section is attached to the weight and also to the wall. If the rod mass is 1.25 kg, what will be the angle, θ, for equilibrium?

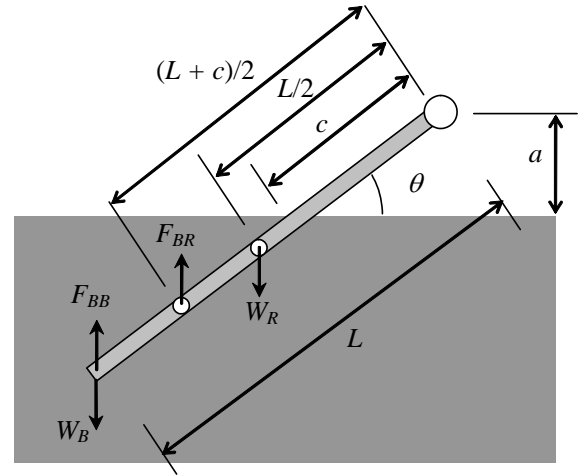
NEW PROBLEM STATEMENT NEEDED

NOTE: Cross section is 25 cm^2

Given: Geometry of block and rod

Find: Angle for equilibrium

Solution:

Basic equations $\quad \Sigma M_{Hinge} = 0 \quad F_B = \rho \cdot g \cdot V$ (Buoyancy)

The free body diagram is as shown. F_{BB} and F_{BR} are the buoyancy of the block and rod, respectively; c is the (unknown) exposed length of the rod

Taking moments about the hinge

$$\left(W_B - F_{BB}\right) \cdot L \cdot \cos(\theta) - F_{BR} \cdot \frac{(L + c)}{2} \cdot \cos(\theta) + W_R \cdot \frac{L}{2} \cdot \cos(\theta) = 0$$

with $\quad W_B = M_B \cdot g \quad F_{BB} = \rho \cdot g \cdot V_B \quad F_{BR} = \rho \cdot g \cdot (L - c) \cdot A \quad W_R = M_R \cdot g$

Combining equations $\quad \left(M_B - \rho \cdot V_B\right) \cdot L - \rho \cdot A \cdot (L - c) \cdot \frac{(L + c)}{2} + M_R \cdot \frac{L}{2} = 0$

We can solve for c $\quad \rho \cdot A \cdot \left(L^2 - c^2\right) = 2 \cdot \left(M_B - \rho \cdot V_B + \frac{1}{2} \cdot M_R\right) \cdot L$

$$c = \sqrt{L^2 - \frac{2 \cdot L}{\rho \cdot A} \cdot \left(M_B - \rho \cdot V_B + \frac{1}{2} \cdot M_R\right)}$$

$$c = \sqrt{(5 \cdot m)^2 - 2 \times 5 \cdot m \times \frac{m^3}{1000 \cdot kg} \times \frac{1}{25} \cdot \frac{1}{cm^2} \times \left(\frac{100 \cdot cm}{1 \cdot m}\right)^2 \times \left[30 \cdot kg - \left(1000 \cdot \frac{kg}{m^3} \times 0.025 \cdot m^3\right) + \frac{1}{2} \times 1.25 \cdot kg\right]}$$

$$c = 1.58 m$$

Then $\quad \sin(\theta) = \frac{a}{c} \quad$ with $\quad a = 0.25 \cdot m \quad \theta = \operatorname{asin}\left(\frac{a}{c}\right) \quad \theta = 9.1 \cdot deg$

Problem *3.100

Given: Glass hydrometer used to measure SG of liquids.

Stem has $D = 6$ mm; distance between marks on stem is $d = 3$ mm per 0.1 SG

Hydrometer floats in ethyl alcohol (assume contact angle is 0°).

Find: Magnitude of error introduced by surface tension.

Solution: Consider a free-body diagram of the floating hydrometer

Surface tension will cause the hydrometer to sink Δh lower into the liquid. Thus for this change,

$$\sum F_z = \Delta F_B - F_\sigma = m a_z = 0$$

Computing equation: $\Delta F_B = \rho g \Delta V$

Assumptions: (1) Static liquid (2) Incompressible liquid (3) $\theta \approx 0$

D = 6 mm; $d = \frac{3 \text{ mm}}{0.1 \text{ SG}}$; z; F_σ; Ethyl alcohol; ΔF_B

Then $\Delta V = \frac{\pi D^2}{4} \Delta h$ and $\Delta F_B = \rho g \frac{\pi D^2}{4} \Delta h$

and $F_\sigma = \pi D \sigma \cos\theta = \pi D \sigma$

Combining $\rho g \frac{\pi D^2}{4} \Delta h = \pi D \sigma$ or $\Delta h = \frac{4\sigma}{\rho g D} = \frac{4\sigma}{SG \rho_{H_2O} g D}$

From Table A.2, SG = 0.789 and from Table A.4, $\sigma = 22.3$ mN/m for ethanol, so

$$\Delta h = \frac{4}{0.789} \times 22.3 \times 10^{-3} \frac{N}{m} \times \frac{m^3}{1000 \text{ kg}} \times \frac{s^2}{9.81 \text{ m}} \times \frac{1}{0.006 \text{ m}} \times \frac{\text{kg} \cdot \text{m}}{N \cdot s^2} = 1.92 \times 10^{-3} \text{ m}$$

Thus the change in SG will be

$$\Delta SG = 1.92 \times 10^{-3} \text{ m} \times \frac{0.1 \text{ SG}}{3 \text{ mm}} \times 1000 \frac{\text{mm}}{\text{m}} = 0.0640$$

ΔS

{ From the diagram, surface tension acts to cause the hydrometer to float lower in the liquid. Therefore surface tension results in an indicated SG <u>smaller</u> than the actual SG. }

Problem 3.101

***3.101** If the mass M in Problem 3.99 is released from the rod, at equilibrium how much of the rod will remain submerged? What will be the minimum required upward force at the tip of the rod to just lift it out of the water?

Given: Geometry of rod

Find: How much of rod is submerged; force to lift rod out of water

Solution:

Basic equations $\qquad \Sigma M_{Hinge} = 0 \qquad F_B = \rho \cdot g \cdot V \quad$ (Buoyancy)

The free body diagram is as shown. F_{BR} is the buoyancy of the rod; c is the (unknown) exposed length of the rod

Taking moments about the hinge

$$-F_{BR} \cdot \frac{(L+c)}{2} \cdot \cos(\theta) + W_R \cdot \frac{L}{2} \cdot \cos(\theta) = 0$$

with $\qquad F_{BR} = \rho \cdot g \cdot (L - c) \cdot A \qquad W_R = M_R \cdot g$

Hence $\qquad -\rho \cdot A \cdot (L - c) \cdot \frac{(L+c)}{2} + M_R \cdot \frac{L}{2} = 0$

We can solve for c $\qquad \rho \cdot A \cdot \left(L^2 - c^2\right) = M_R \cdot L$

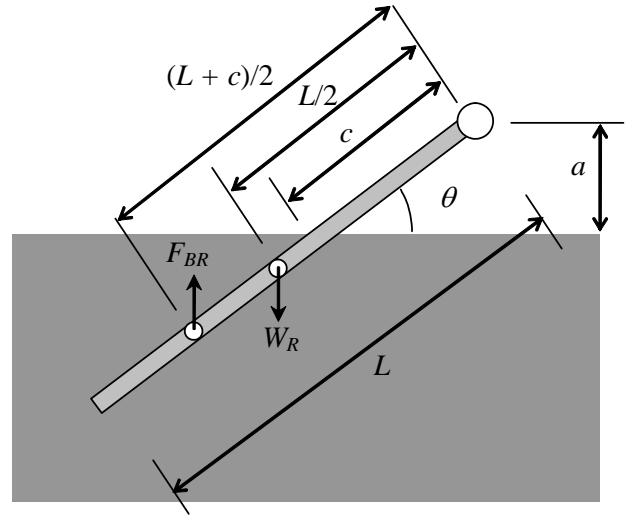
$$c = \sqrt{L^2 - \frac{L \cdot M_R}{\rho \cdot A}}$$

$$c = \sqrt{(5 \cdot m)^2 - 5 \cdot m \times \frac{m^3}{1000 \cdot kg} \times \frac{1}{25} \cdot \frac{1}{cm^2} \times \left(\frac{100 \cdot cm}{1 \cdot m}\right)^2 \times 1.25 \cdot kg}$$

$$c = 4.74\,m$$

Then the submerged length is $\qquad L - c = 0.257\,m$

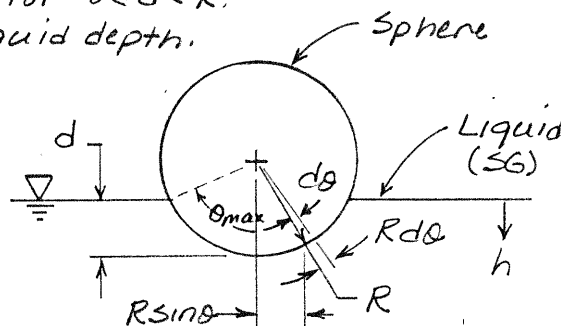
To lift the rod out of the water requires a force equal to half the rod weight (the reaction also takes half the weight)

$$F = \frac{1}{2} \cdot M_R \cdot g = \frac{1}{2} \times 1.25 \cdot kg \times 9.81 \cdot \frac{m}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} \qquad F = 6.1\,N$$

Given: Sphere partially immersed in liquid of specific gravity, SG.

Find: (a) Formula, algebraic expression for buoyancy force, as a function of submersion depth, d, for $0 \le d \le R$.
(b) Plot of results over range of liquid depth.

<u>Solution</u>: Apply fluid statics

Basic equations: $\frac{dp}{dh} = \rho g$

$dF = p\,dA$

Assumptions: (1) Static liquid
(2) Incompressible, so $p = p_0 + \rho g h$
(3) Neglect p_{atm} since it acts everywhere

Then $dF_V = \cos\theta\, p\, dA$; $p = \rho g h$; $d = h + R(1-\cos\theta)$; $h = d - R(1-\cos\theta)$

$$dA = 2\pi(R\sin\theta)\,R\,d\theta = 2\pi R^2 \sin\theta\, d\theta$$

$$dF_V = \cos\theta\, \rho g\left[d - R(1-\cos\theta)\right] 2\pi R^2 \sin\theta\, d\theta = 2\pi R^3\left[\frac{d}{R} - (1-\cos\theta)\right]\sin\theta\cos\theta\, d\theta\, \rho g$$

Now

$$F_V = \int_A dF_V = \int_0^{\theta_{max}} 2\pi R^3\left[\frac{d}{R} - (1-\cos\theta)\right]\sin\theta\cos\theta\, d\theta\, \rho g$$

$$F_V = 2\pi R^3\left[(1-d/R)\frac{\cos^2\theta}{2} - \frac{\cos^3\theta}{3}\right]_0^{\theta_{max}} \rho g \quad ; \quad \rho = SG\,\rho_{H_2O}$$

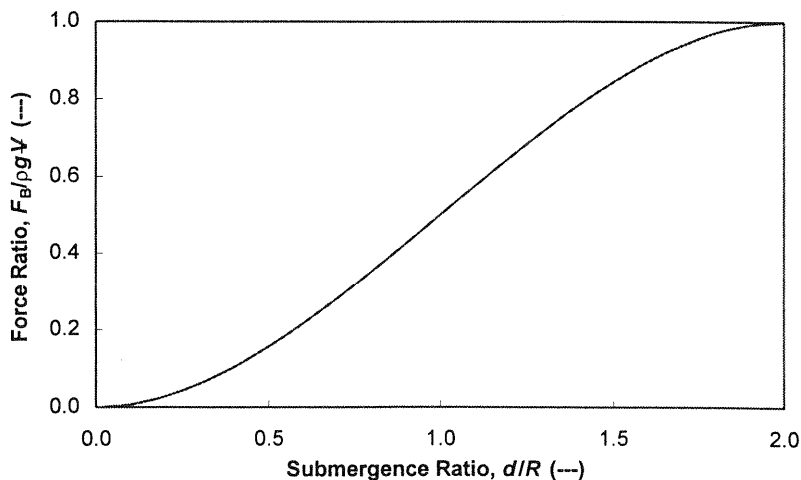
At θ_{max}, $\cos\theta_{max} = \frac{R-d}{R} = 1 - d/R$, so

$$F_V = 2\pi\rho g R^3\left\{(1-d/R)\left[(1-d/R)^2/2 - 1/2\right] - \frac{(1-d/R)^3}{3} + \frac{1}{3}\right\}$$

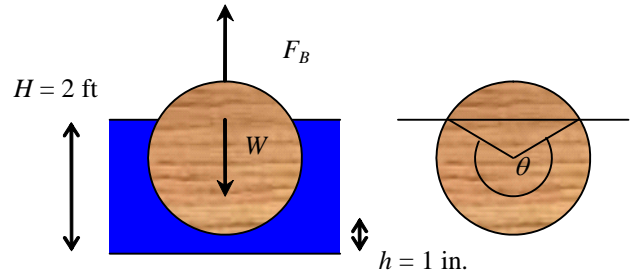
$$F_V = 2\pi\rho g R^3\left[\frac{1}{6}\left(1-\frac{d}{R}\right)^3 - \frac{1}{2}\left(1-\frac{d}{R}\right) + \frac{1}{3}\right] \qquad F_V$$

Dividing both sides by the vertical force on a fully submerged sphere,

$$\frac{F_V}{\rho g \frac{4\pi R^3}{3}} = \frac{3}{2}\left[\frac{1}{6}()^3 - \frac{1}{2}() + \frac{1}{3}\right]$$

where $() = (1 - \frac{d}{R})$.

Problem *3.103

[2]

*3.103 In a logging operation, timber floats downstream to a lumber mill. It is a dry year, and the river is running low, as low as 2 feet in some locations. What is the largest diameter log that may be transported in this fashion (leaving a minimum 1 in. clearance between the log and the bottom of the river)? For the wood, SG = 0.8.

Given: Data on river

Find: Largest diameter of log that will be transported

Solution:

Basic equation $F_B = \rho \cdot g \cdot V_{sub}$ and $\Sigma F_y = 0$ $\Sigma F_y = 0 = F_B - W$

where $F_B = \rho \cdot g \cdot V_{sub} = \rho \cdot g \cdot A_{sub} \cdot L$ $W = SG \cdot \rho \cdot g \cdot V = SG \cdot \rho \cdot g \cdot A \cdot L$

From references (trying Googling "segment of a circle") $A_{sub} = \frac{R^2}{2} \cdot (\theta - \sin(\theta))$ where R is the radius and θ is the included angle

Hence $\rho \cdot g \cdot \frac{R^2}{2} \cdot (\theta - \sin(\theta)) \cdot L = SG \cdot \rho \cdot g \cdot \pi \cdot R^2 \cdot L$

$$\theta - \sin(\theta) = 2 \cdot SG \cdot \pi = 2 \times 0.8 \times \pi$$

This equation can be solved by manually iterating, or by using a good calculator, or by using Excel's Goal Seek

$$\theta = 239 \cdot deg$$

From geometry the submerged amount of a log is $H - h$ and also $R + R \cdot \cos\left(\pi - \frac{\theta}{2}\right)$

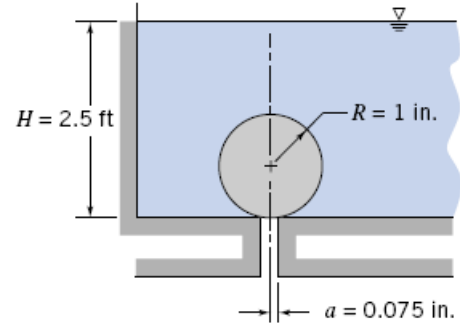
Hence $H - h = R + R \cdot \cos\left(\pi - \frac{\theta}{2}\right)$

Solving for R $R = \frac{H - h}{1 + \cos\left(180deg - \frac{\theta}{2}\right)}$ $R = \frac{\left(2 - \frac{1}{12}\right) \cdot ft}{1 + \cos\left[\left(180 - \frac{239}{2}\right) \cdot deg\right]}$ $R = 1.28 \cdot ft$

$D = 2 \cdot R$ $D = 2.57 \cdot ft$

Problem *3.104

[4]

*3.104 A sphere of radius R, made from material of specific gravity SG, is submerged in a tank of water. The sphere is placed over a hole, of radius a, in the tank bottom. Develop a general expression for the range of specific gravities for which the sphere will float to the surface. For the dimensions given, determine the minimum SG required for the sphere to remain in the position shown.

Given: Data on sphere and tank bottom

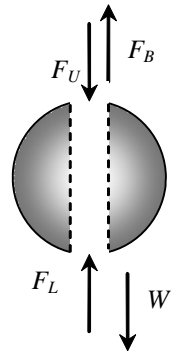
Find: Expression for SG of sphere at which it will float to surface; minimum SG to remain in position

Solution:

Basic equations $\quad F_B = \rho \cdot g \cdot V \quad$ and $\quad \Sigma F_y = 0 \qquad \Sigma F_y = 0 = F_L - F_U + F_B - W$

where $\quad F_L = p_{atm} \cdot \pi \cdot a^2 \qquad F_U = \left[p_{atm} + \rho \cdot g \cdot (H - 2 \cdot R)\right] \cdot \pi \cdot a^2$

$$F_B = \rho \cdot g \cdot V_{net} \qquad V_{net} = \frac{4}{3} \cdot \pi \cdot R^3 - \pi \cdot a^2 \cdot 2 \cdot R$$

$$W = SG \cdot \rho \cdot g \cdot V \quad \text{with} \quad V = \frac{4}{3} \cdot \pi \cdot R^3$$

Note that we treat the sphere as a sphere with SG, and for fluid effects a sphere minus a cylinder (buoyancy) and cylinder with hydrostatic pressures

Hence

$$p_{atm} \cdot \pi \cdot a^2 - \left[p_{atm} + \rho \cdot g \cdot (H - 2 \cdot R)\right] \cdot \pi \cdot a^2 + \rho \cdot g \cdot \left(\frac{4}{3} \cdot \pi \cdot R^3 - 2 \cdot \pi \cdot R \cdot a^2\right) - SG \cdot \rho \cdot g \cdot \frac{4}{3} \cdot \pi \cdot R^3 = 0$$

Solving for SG

$$SG = \frac{3}{4 \cdot \pi \cdot \rho \cdot g \cdot R^3} \cdot \left[-\pi \cdot \rho \cdot g \cdot (H - 2 \cdot R) \cdot a^2 + \rho \cdot g \cdot \left(\frac{4}{3} \cdot \pi \cdot R^3 - 2 \cdot \pi \cdot R \cdot a^2\right)\right]$$

$$SG = 1 - \frac{3}{4} \cdot \frac{H \cdot a^2}{R^3}$$

$$SG = 1 - \frac{3}{4} \times 2.5 \cdot ft \times \left(0.075 \cdot in \times \frac{1 \cdot ft}{12 \cdot in}\right)^2 \times \left(\frac{1}{1 \cdot in} \times \frac{12 \cdot in}{1 \cdot ft}\right)^3 \qquad SG = 0.873$$

This is the minimum SG to remain submerged; any SG above this and the sphere remains on the bottom; any SG less than this and the sphere rises to the surface

Problem *3.105

Given: Cylindrical timber, $D = 0.3$ m and $L = 4$ m is weighted on lower end so it floats vertically with 3m submerged in sea water.
When displaced vertically from equilibrium position the timber oscillates in a vertical direction upon release

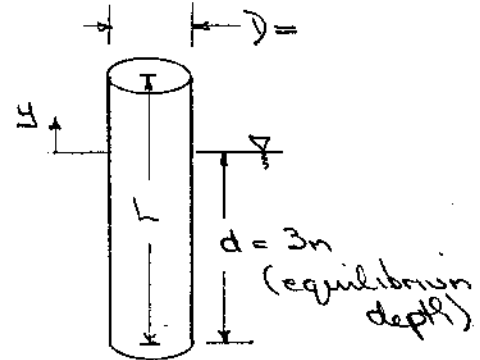
Find: Estimate frequency of oscillation. (Neglect any viscous effects or water motion)

Solution:

At equilibrium

$$\Sigma F_y = 0 = F_b - mg = \rho A d - mg$$

$$\therefore m = \frac{\rho A d}{g}$$

$D =$, y, L, $d = 3$m (equilibrium depth)

For displacement y

$$\Sigma F_y = m\frac{d^2y}{dt^2} = m\ddot{y}$$

$$F_b - mg = m\ddot{y} \quad \text{where } F_b = \rho A(d-y)$$

$$\therefore \rho A(d-y) - mg = m\ddot{y}$$

$$\rho A d - \rho A y - \frac{\rho A d}{g} g = m\ddot{y}$$

or

$$m\ddot{y} + \rho A y = 0$$

$$\ddot{y} + \frac{\rho A}{m} y = 0 = \ddot{y} + \omega^2 y = 0$$

$$\text{where } \omega^2 = \frac{\rho A}{m} = \frac{\rho A g}{\rho A d} = \frac{g}{d}$$

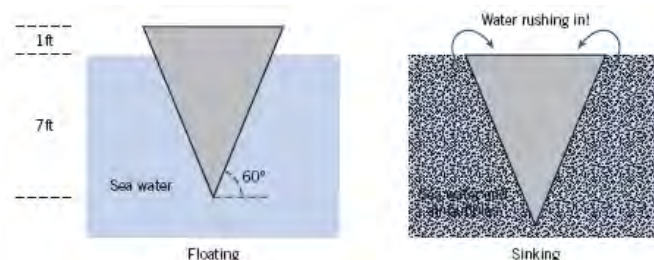
$$\omega = \left(\frac{g}{d}\right)^{1/2} = \left[9.81\frac{m}{s^2} \times \frac{1}{3m}\right]^{1/2} = 1.81 \text{ rad/s}$$

ω

$$f = \frac{\omega}{2\pi} = 1.81\frac{\text{rad}}{s} \times \frac{\text{cycle}}{2\pi \text{ rad}} = 0.288 \text{ cycle/s}$$

$$\tau = \frac{1}{f} = 3.47 \text{ s}$$

Problem *3.106

*3.106 You are in the Bermuda Triangle when you see a bubble plume eruption (a large mass of air bubbles, similar to a foam) off to the side of the boat. Do you want to head toward it and be part of the action? What is the effective density of the water and air bubbles in the drawing on the right that will cause the boat to sink? Your boat is 10 ft long, and weight is the same in both cases.

Given: Data on boat

Find: Effective density of water/air bubble mix if boat sinks

Floating Sinking

$H = 8$ ft

$h = 7$ ft

$\theta = 60^{o}$

Solution:

Basic equations $F_B = \rho \cdot g \cdot V$ and $\Sigma F_y = 0$

We can apply the sum of forces for the "floating" free body

$$\Sigma F_y = 0 = F_B - W \qquad \text{where} \qquad F_B = SG_{sea} \cdot \rho \cdot g \cdot V_{subfloat}$$

$$V_{subfloat} = \frac{1}{2} \cdot h \cdot \left(\frac{2 \cdot h}{\tan \cdot \theta} \right) \cdot L = \frac{L \cdot h^2}{\tan(\theta)} \qquad SG_{sea} = 1.024 \qquad \text{(Table A.2)}$$

Hence $$W = \frac{SG_{sea} \cdot \rho \cdot g \cdot L \cdot h^2}{\tan(\theta)} \qquad (1)$$

We can apply the sum of forces for the "sinking" free body

$$\Sigma F_y = 0 = F_B - W \qquad \text{where} \qquad F_B = SG_{mix} \cdot \rho \cdot g \cdot V_{sub} \qquad V_{subsink} = \frac{1}{2} \cdot H \cdot \left(\frac{2 \cdot H}{\tan \cdot \theta} \right) \cdot L = \frac{L \cdot H^2}{\tan(\theta)}$$

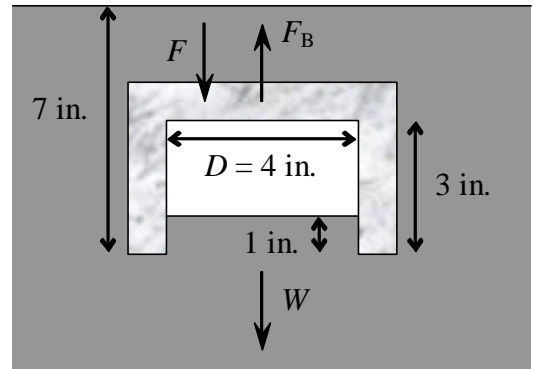
Hence $$W = \frac{SG_{mix} \cdot \rho \cdot g \cdot L \cdot H^2}{\tan(\theta)} \qquad (2)$$

Comparing Eqs. 1 and 2 $$W = \frac{SG_{sea} \cdot \rho \cdot g \cdot L \cdot h^2}{\tan(\theta)} = \frac{SG_{mix} \cdot \rho \cdot g \cdot L \cdot H^2}{\tan(\theta)}$$

$$SG_{mix} = SG_{sea} \cdot \left(\frac{h}{H} \right)^2 \qquad SG_{mix} = 1.024 \times \left(\frac{7}{8} \right)^2 \qquad SG_{mix} = 0.784$$

The density is $$\rho_{mix} = SG_{mix} \cdot \rho \qquad \rho_{mix} = 0.784 \times 1.94 \cdot \frac{slug}{ft^3} \qquad \rho_{mix} = 1.52 \frac{slug}{ft^3}$$

Problem *3.107

*3.107 A bowl is inverted symmetrically and held in BXYB fluid, SG = 15.6, to a depth of 7 in. measured along the centerline of the bowl from the bowl rim. The bowl height is 3 in., and the BXYB fluid rises 1 in. inside the bowl. The bowl is unique: Its base is 4 in. inside diameter, and it is made from an old clay recipe, SG = 5.7. The volume of the bowl is about 56 in.^{3}. What is the force required to hold it in place?

Given: Data on inverted bowl and BXYB fluid

Find: Force to hold in place

Solution:

Basic equation $F_B = \rho \cdot g \cdot V$ and $\Sigma F_y = 0$ $\qquad \Sigma F_y = 0 = F_B - F - W$

Hence $F = F_B - W$

For the buoyancy force $F_B = SG_{BXYB} \cdot \rho \cdot g \cdot V_{sub}$ with $V_{sub} = V_{bowl} + V_{air}$

For the weight $W = SG_{bowl} \cdot \rho \cdot g \cdot V_{bowl}$

Hence $F = SG_{BXYB} \cdot \rho \cdot g \cdot \left(V_{bowl} + V_{air}\right) - SG_{bowl} \cdot \rho \cdot g \cdot V_{bowl}$

$$F = \rho \cdot g \cdot \left[SG_{BXYB} \cdot \left(V_{bowl} + V_{air}\right) - SG_{bowl} \cdot V_{bowl}\right]$$

$$F = 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times \left[15.6 \times \left[56 \cdot in^3 + (3-1) \cdot in \cdot \frac{\pi \cdot (4 \cdot in)^2}{4}\right] - 5.7 \times 56 \cdot in^3\right] \times \left(\frac{1 \cdot ft}{12 \cdot in}\right)^3 \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

$$F = 34.2 \cdot lbf$$

Problem *3.108

[4]

Consider a conical funnel held upside down and submerged slowly in a container of water. Discuss the force needed to submerge the funnel if the spout is open to the atmosphere. Compare with the force needed to submerge the funnel when the spout opening is blocked by a rubber stopper.

Open-Ended Problem Statement: Consider a conical funnel held upside down and submerged slowly in a container of water. Discuss the force needed to submerge the funnel if the spout is open to the atmosphere. Compare with the force needed to submerge the funnel when the spout opening is blocked by a rubber stopper.

Discussion: Let the weight of the funnel in air be W_a. Assume the funnel is held with its spout vertical and the conical section down. Then W_a will also be vertical.

Two possible cases are with the funnel spout open to atmosphere or with the funnel spout sealed.
With the funnel spout open to atmosphere, the pressures inside and outside the funnel are equal, so no net pressure force acts on the funnel. The force needed to support the funnel will remain constant until it first contacts the water. Then a buoyancy force will act vertically upward on every element of volume located beneath the water surface.

The first contact of the funnel with the water will be at the widest part of the conical section. The buoyancy force will be caused by the volume formed by the funnel thickness and diameter as it begins to enter the water. The buoyancy force will reduce the force needed to support the funnel. The buoyancy force will increase as the depth of submergence of the funnel increases until the funnel is fully submerged. At that point the buoyancy force will be constant and equal to the weight of water displaced by the volume of the material from which the funnel is made.

If the funnel material is less dense than water, it would tend to float partially submerged in the water. The force needed to support the funnel would decrease to zero and then become negative (i.e., down) to fully submerge the funnel.

If the funnel material were denser than water it would not tend to float even when fully submerged. The force needed to support the funnel would decrease to a minimum when the funnel became fully submerged, and then would remain constant at deeper submersion depths.
With the funnel spout sealed, air will be trapped inside the funnel. As the funnel is submerged gradually below the water surface, it will displace a volume equal to the volume of the funnel material plus the volume of trapped air. Thus its buoyancy force will be much larger than when the spout is open to atmosphere. Neglecting any change in air volume (pressures caused by submersion should be small compared to atmospheric pressure) the buoyancy force would be from the entire volume encompassed by the outside of the funnel. Finally, when fully submerged, the volume of the rubber stopper (although small) will also contribute to the total buoyancy force acting on the funnel.

Problem *3.109 [4]

In the ‘‘Cartesian diver’’ child’s toy, a miniature ‘‘diver’’ is immersed in a column of liquid. When a diaphragm at the top of the column is pushed down, the diver sinks to the bottom. When the diaphragm is released, the diver again rises. Explain how the toy might work.

Open-Ended Problem Statement: In the “Cartesian diver” child's toy, a miniature “diver” is immersed in a column of liquid. When a diaphragm at the top of the column is pushed down, the diver sinks to the bottom. When the diaphragm is released, the diver again rises. Explain how the toy might work.

Discussion: A possible scenario is for the toy to have a flexible bladder that contains air. Pushing down on the diaphragm at the top of the liquid column would increase the pressure at any point in the liquid. The air in the bladder would be compressed slightly as a result. The volume of the bladder, and therefore its buoyancy, would decrease, causing the diver to sink to the bottom of the liquid column.

Releasing the diaphragm would reduce the pressure in the water column. This would allow the bladder to expand again, increasing its volume and therefore the buoyancy of the diver. The increased buoyancy would permit the diver to rise to the top of the liquid column and float in a stable, partially submerged position, on the surface of the liquid.

Problem *3.110 [4]

A proposed ocean salvage scheme involves pumping air into ‘‘bags’’ placed within and around a wrecked vessel on the sea bottom. Comment on the practicality of this plan, supporting your conclusions with analyses.

Open-Ended Problem Statement: A proposed ocean salvage scheme involves pumping air into “bags” placed within and around a wrecked vessel on the sea bottom. Comment on the practicality of this plan, supporting your conclusions with analyses.

Discussion: This plan has several problems that render it impractical. First, pressures at the sea bottom are very high. For example, *Titanic* was found in about 12,000 ft of seawater. The corresponding pressure is nearly 6,000 psi. Compressing air to this pressure is possible, but would require a multi-stage compressor and very high power.

Second, it would be necessary to manage the buoyancy force after the bag and object are broken loose from the sea bed and begin to rise toward the surface. Ambient pressure would decrease as the bag and artifact rise toward the surface. The air would tend to expand as the pressure decreases, thereby tending to increase the volume of the bag. The buoyancy force acting on the bag is directly proportional to the bag volume, so it would increase as the assembly rises. The bag and artifact thus would tend to accelerate as they approach the sea surface. The assembly could broach the water surface with the possibility of damaging the artifact or the assembly.

If the bag were of constant volume, the pressure inside the bag would remain essentially constant at the pressure of the sea floor, e.g., 6,000 psi for *Titanic*. As the ambient pressure decreases, the pressure differential from inside the bag to the surroundings would increase. Eventually the difference would equal sea floor pressure. This probably would cause the bag to rupture.

If the bag permitted some expansion, a control scheme would be needed to vent air from the bag during the trip to the surface to maintain a constant buoyancy force just slightly larger than the weight of the artifact in water. Then the trip to the surface could be completed at low speed without danger of broaching the surface or damaging the artifact.

Problem *3.111

*3.111 Three steel balls (each about half an inch in diameter) lie at the bottom of a plastic shell floating on the water surface in a partially filled bucket. Someone removes the steel balls from the shell and carefully lets them fall to the bottom of the bucket, leaving the plastic shell to float empty. What happens to the water level in the bucket? Does it rise, go down, or remain unchanged? Explain.

Given: Steel balls resting in floating plastic shell in a bucket of water

Find: What happens to water level when balls are dropped in water

Solution: Basic equation $F_B = \rho \cdot V_{disp} \cdot g = W$ for a floating body weight W

When the balls are in the plastic shell, the shell and balls displace a volume of water equal to their own weight - a large volume because the balls are dense. When the balls are removed from the shell and dropped in the water, the shell now displaces only a small volume of water, and the balls sink, displacing only their own volume. Hence the difference in displaced water before and after moving the balls is the difference between the volume of water that is equal to the weight of the balls, and the volume of the balls themselves. The amount of water displaced is significantly reduced, so the water level in the bucket drops.

Volume displaced before moving balls: $$V_1 = \frac{W_{plastic} + W_{balls}}{\rho \cdot g}$$

Volume displaced after moving balls: $$V_2 = \frac{W_{plastic}}{\rho \cdot g} + V_{balls}$$

Change in volume displaced $$\Delta V = V_2 - V_1 = V_{balls} - \frac{W_{balls}}{\rho \cdot g} = V_{balls} - \frac{SG_{balls} \cdot \rho \cdot g \cdot V_{balls}}{\rho \cdot g}$$

$$\Delta V = V_{balls} \cdot \left(1 - SG_{balls}\right)$$

Hence initially a large volume is displaced; finally a small volume is displaced ($\Delta V < 0$ because $SG_{balls} > 1$)

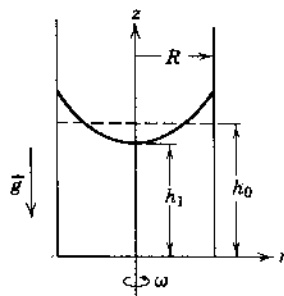
Problem *3.112

Given: Cylindrical container rotating as in Example 3.10

$R = 0.5$ ft

$h_0 = 4$ in.

Determine: (a) value of ω such that $h_1 = 0$

(b) if solution is dependent on ρ

Solution:

In order to obtain the solution we need an expression for the shape of the free surface in terms of ω, r, and h_0

The required expression was derived in Example 3.10. The equation is

$$z = h_0 - \frac{(\omega R)^2}{2g}\left[\frac{1}{2} - \left(\frac{r}{R}\right)^2\right]$$

Since $h_1 = 0$ corresponds to $z = 0$ and $r = 0$ we must determine ω such that

$$0 = h_0 - \frac{(\omega R)^2}{4g}$$

Solving for ω,

$$\omega = \frac{2}{R}\sqrt{g h_0}$$

$$= \frac{2}{0.5\text{ ft}}\left(32.2\frac{\text{ft}}{\text{s}^2} \times 4\text{ in} \times \frac{\text{ft}}{12\text{ in}}\right)^{1/2}$$

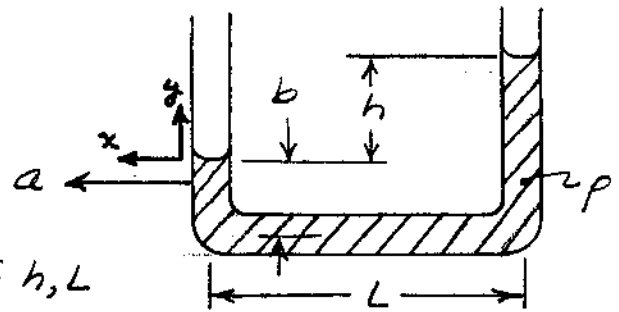
$$= 4 \times 3.28 \ \frac{1}{\text{s}}$$

$$\omega = 13.1 \text{ rad/s}$$

ω

The solution is independent of ρ since the equation of the free surface is independent of ρ.

Problem *3.113

Given: U-tube accelerometer

Find: Acceleration in terms of h, L

Solution: Apply x, y components of hydrostatic equation.

Basic equations:

$$-\frac{\partial p}{\partial x} + \rho g_x = \rho a_x \qquad a_x = a \qquad g_x = 0$$

$$-\frac{\partial p}{\partial y} + \rho g_y = \rho a_y \qquad a_y = 0 \qquad g_y = -g$$

Assumptions: (1) Neglect sloshing
(2) Ignore corners

Then $\frac{\partial p}{\partial x} = -\rho a$, $\frac{\partial p}{\partial y} = -\rho g$. Evaluate Δp from left leg to right:

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy$$

$$\Delta p = \frac{\partial p}{\partial x} \Delta x + \frac{\partial p}{\partial y} \Delta y$$

$$= (-\rho g)(-b) + (-\rho a)(-L) + (-\rho g)(b+h)$$

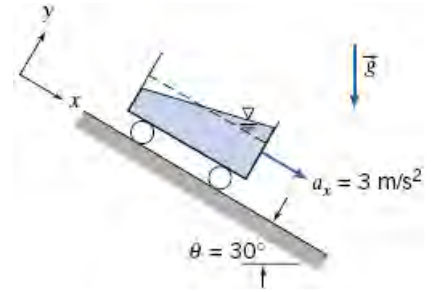
$$\Delta p = \rho a L - \rho g h = 0$$

Solving,

$$a = g\left(\frac{h}{L}\right)$$

a

Problem *3.114

*3.114 A rectangular container of water undergoes constant acceleration down an incline as shown. Determine the slope of the free surface using the coordinate system shown.

Given: Rectangular container with constant acceleration

Find: Slope of free surface

Solution: Basic equation $-\nabla p + \rho\vec{g} = \rho\vec{a}$

In components

$$-\frac{\partial}{\partial x}p + \rho \cdot g_x = \rho \cdot a_x \qquad -\frac{\partial}{\partial y}p + \rho \cdot g_y = \rho \cdot a_y \qquad -\frac{\partial}{\partial z}p + \rho \cdot g_z = \rho \cdot a_z$$

We have

$$a_y = a_z = 0 \qquad g_x = g \cdot \sin(\theta) \qquad g_y = -g \cdot \cos(\theta) \qquad g_z = 0$$

Hence

$$-\frac{\partial}{\partial x}p + \rho \cdot g \cdot \sin(\theta) = \rho \cdot a_x \quad (1) \qquad -\frac{\partial}{\partial y}p - \rho \cdot g \cdot \cos(\theta) = 0 \quad (2) \qquad -\frac{\partial}{\partial z}p = 0 \quad (3)$$

From Eq. 3 we can simplify from $p = p(x,y,z)$ to $p = p(x,y)$

Hence a change in pressure is given by $dp = \frac{\partial}{\partial x}p \cdot dx + \frac{\partial}{\partial y}p \cdot dy$

At the free surface p = const., so $dp = 0 = \frac{\partial}{\partial x}p \cdot dx + \frac{\partial}{\partial y}p \cdot dy$ or $\frac{dy}{dx} = -\frac{\frac{\partial}{\partial x}p}{\frac{\partial}{\partial y}p}$ at the free surface

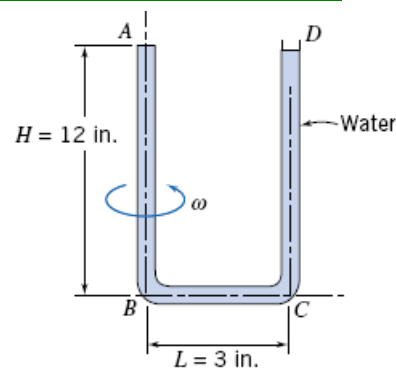
Hence at the free surface, using Eqs 1 and 2

$$\frac{dy}{dx} = -\frac{\frac{\partial}{\partial x}p}{\frac{\partial}{\partial y}p} = \frac{\rho \cdot g \cdot \sin(\theta) - \rho \cdot a_x}{\rho \cdot g \cdot \cos(\theta)} = \frac{g \cdot \sin(\theta) - a_x}{g \cdot \cos(\theta)}$$

$$\frac{dy}{dx} = \frac{9.81 \cdot (0.5) \cdot \frac{m}{s^2} - 3 \cdot \frac{m}{s^2}}{9.81 \cdot (0.866) \cdot \frac{m}{s^2}}$$

At the free surface, the slope is $\frac{dy}{dx} = 0.224$

Problem *3.115

*3.115 The U-tube shown is filled with water at $T = 68°F$. It is sealed at A and open to the atmosphere at D. The tube is rotated about vertical axis AB. For the dimensions shown, compute the maximum angular speed if there is to be no cavitation.

Given: Spinning U-tube sealed at one end

Find: Maximum angular speed for no cavitation

Solution: Basic equation $-\nabla p + \rho\vec{g} = \rho\vec{a}$

In components $-\frac{\partial}{\partial r}p = \rho \cdot a_r = -\rho \cdot \frac{V^2}{r} = -\rho \cdot \omega^2 \cdot r$ $\quad \frac{\partial}{\partial z}p = -\rho \cdot g$

Between D and C, r = constant, so $\frac{\partial}{\partial z}p = -\rho \cdot g$ and so $p_D - p_C = -\rho \cdot g \cdot H$ (1)

Between B and A, r = constant, so $\frac{\partial}{\partial z}p = -\rho \cdot g$ and so $p_A - p_B = -\rho \cdot g \cdot H$ (2)

Between B and C, z = constant, so $\frac{\partial}{\partial r}p = \rho \cdot \omega^2 \cdot r$ and so $\int_{p_B}^{p_C} 1\, dp = \int_0^L \rho \cdot \omega^2 \cdot r\, dr$

Integrating $p_C - p_B = \rho \cdot \omega^2 \cdot \frac{L^2}{2}$ (3)

Since $p_D = p_{atm}$, then from Eq 1 $p_C = p_{atm} + \rho \cdot g \cdot H$

From Eq. 3 $p_B = p_C - \rho \cdot \omega^2 \cdot \frac{L^2}{2}$ so $p_B = p_{atm} + \rho \cdot g \cdot H - \rho \cdot \omega^2 \cdot \frac{L^2}{2}$

From Eq. 2 $p_A = p_B - \rho \cdot g \cdot H$ so $p_A = p_{atm} - \rho \cdot \omega^2 \cdot \frac{L^2}{2}$

Thus the minimum pressure occurs at point A (not B)

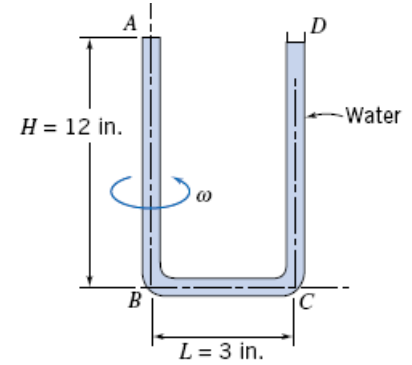
At 68°F from steam tables, the vapor pressure of water is $p_v = 0.339 \cdot psi$

Solving for ω with $p_A = p_v$, we obtain $\omega = \sqrt{\frac{2 \cdot (p_{atm} - p_v)}{\rho \cdot L^2}} = \left[2 \cdot (14.7 - 0.339) \cdot \frac{lbf}{in^2} \times \frac{ft^3}{1.94 \cdot slug} \times \frac{1}{(3 \cdot in)^2} \times \left(\frac{12 \cdot in}{1 \cdot ft}\right)^4 \times \frac{slug\, ft}{s^2 \cdot lbf}\right]^{\frac{1}{2}}$

$$\omega = 185 \cdot \frac{rad}{s} \qquad \omega = 1764 \cdot rpm$$

Problem *3.116

[2]

*3.116 If the U-tube of Problem 3.115 is spun at 200 rpm, what will be the pressure at A? If a small leak appears at A, how much water will be lost at D?

Given: Spinning U-tube sealed at one end

Find: Pressure at A; water loss due to leak

Solution: Basic equation $-\nabla p + \rho\vec{g} = \rho\vec{a}$

From the analysis of Example Problem 3.10, solving the basic equation, the pressure p at any point (r,z) in a continuous rotating fluid is given by

$$p = p_0 + \frac{\rho\cdot\omega^2}{2}\cdot\left(r^2 - r_0^2\right) - \rho\cdot g\cdot\left(z - z_0\right) \qquad (1)$$

where p_0 is a reference pressure at point (r_0,z_0)

In this case $p = p_A$ $\quad p_0 = p_D$ $\quad z = z_A = z_D = z_0 = H$ $\quad r = 0$ $\quad r_0 = r_D = L$

The speed of rotation is $\omega = 200\cdot\text{rpm}$ $\quad \omega = 20.9\cdot\frac{\text{rad}}{\text{s}}$

The pressure at D is $p_D = 0\cdot\text{kPa}$ (gage)

Hence $$p_A = \frac{\rho\cdot\omega^2}{2}\cdot\left(-L^2\right) - \rho\cdot g\cdot(0) = -\frac{\rho\cdot\omega^2\cdot L^2}{2} = -\frac{1}{2}\times 1.94\cdot\frac{\text{slug}}{\text{ft}^3}\times\left(20.9\cdot\frac{\text{rad}}{\text{s}}\right)^2\times(3\cdot\text{in})^2\times\left(\frac{1\cdot\text{ft}}{12\cdot\text{in}}\right)^4\times\frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

$p_A = -0.18\cdot\text{psi}$ (gage)

When the leak appears,the water level at A will fall, forcing water out at point D. Once again, from the analysis of Example Problem 3.10, we can use Eq 1

In this case $p = p_A = 0$ $\quad p_0 = p_D = 0$ $\quad z = z_A$ $\quad z_0 = z_D = H$ $\quad r = 0$ $\quad r_0 = r_D = L$

Hence $$0 = \frac{\rho\cdot\omega^2}{2}\cdot\left(-L^2\right) - \rho\cdot g\cdot\left(z_A - H\right)$$

$$z_A = H - \frac{\omega^2\cdot L^2}{2\cdot g} = 12\text{in} - \frac{1}{2}\times\left(20.9\cdot\frac{\text{rad}}{\text{s}}\right)^2\times(3\cdot\text{in})^2\times\frac{\text{s}^2}{32.2\cdot\text{ft}}\times\frac{1\cdot\text{ft}}{12\cdot\text{in}} \qquad z_A = 6.91\cdot\text{in}$$

The amount of water lost is $\Delta h = H - z_A = 12\cdot\text{in} - 6.91\cdot\text{in}$ $\qquad \Delta h = 5.09\cdot\text{in}$

Problem *3.117

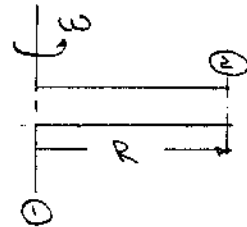
Given: Centrifugal micromanometer consists of pair of parallel disks that rotate to develop a radial pressure difference. There is no flow between the disks.

Find: (a) An expression for the pressure difference, ΔP, as a function of ω, R, and ρ
(b) Find ω if $\Delta P = 8\ \mu m\ H_2O$ and $R = 50\,mm$.

Solution:

ω ② R ①

Basic equation: $-\nabla P + \rho \vec{g} = \rho \vec{a}$

(r component) $-\frac{\partial P}{\partial r} + \rho g_r = \rho a_r$

Assumptions: (1) standard air between disks
(2) r horizontal, so $g_r = 0$
(3) rigid body motion, so $a_r = -\frac{V^2}{r} = -\frac{(r\omega)^2}{r} = -r\omega^2$

Then

$$\frac{\partial P}{\partial r} = \rho r \omega^2 \qquad (\rho \text{ is a constant})$$

Separating variables and integrating, we obtain

$$\int_{P}^{P+\Delta P} dP = \rho\omega^2 \int_0^R r\,dr$$

$$\Delta P = \frac{\rho \omega^2 R^2}{2} \qquad \Delta P$$

Then

$$\omega^2 = \frac{2\Delta P}{\rho R^2}$$

where $\Delta P = \rho_{H_2O}\, g\, \Delta h$ and $\Delta h = 8 \times 10^{-6}\,m$

$$\omega^2 = \frac{2\rho_{H_2O}\, g\, \Delta h}{\rho R^2}$$

$$= 2 \times \frac{999\ kg/m^3}{1.225\ kg/m^3} \times 9.81\,\frac{m}{s^2} \times 8\times10^{-6}\,m \times \frac{1}{(0.05)^2\,m^2}$$

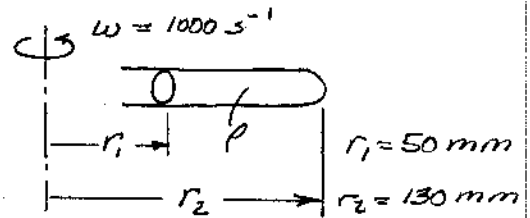
$$\omega^2 = 51.2\ s^{-2}$$

$$\omega = 7.16\ rad/s \qquad \omega$$

Problem *3.118

[2]

Given: Test tube with water

Find: (a) Radial acceleration
(b) Radial pressure gradient, $\partial p/\partial r$
(c) Maximum pressure on bottom.

Solution: Apply equation for rigid-body motion

Basic equation: $-\nabla p + \rho \vec{g} = \rho \vec{a}$

(r component) $-\frac{\partial p}{\partial r} + \rho g_r = \rho a_r$

Assumptions: (1) Rigid-body motion, so $a_r = -\frac{V^2}{r} = -\frac{(r\omega)^2}{r} = -r\omega^2$
(2) r horizontal, so $g_r = 0$

a_r

Then $\frac{\partial p}{\partial r} = -\rho a_r = -\rho(-r\omega^2) = \rho r \omega^2$

$\frac{\partial p}{\partial r}$

Integrating, $p_2 - p_1 = \int_1^2 \frac{\partial p}{\partial r} dr = \int_{r_1}^{r_2} \rho r \omega^2 = \rho \frac{r^2 \omega^2}{2} \Big]_{r_1}^{r_2} = \frac{1}{2}\rho\omega^2(r_2^2 - r_1^2)$

$$p_{max} = p_2 - p_1 = \frac{1}{2} \times 999 \frac{kg}{m^3} \times \frac{(1000)^2}{s^2} \times [(0.130)^2 - (0.050)^2] m^2 \times \frac{N \cdot s^2}{kg \cdot m} = 7.19\ MPa$$

p_{ma}

Problem *3.119

***3.119** A cubical box, 80 cm on a side, half-filled with oil (SG = 0.80), is given a constant horizontal acceleration of 0.25 g parallel to one edge. Determine the slope of the free surface and the pressure along the horizontal bottom of the box.

Given: Cubical box with constant acceleration

Find: Slope of free surface; pressure along bottom of box

Solution: Basic equation $-\nabla p + \rho\vec{g} = \rho\vec{a}$

In components $-\frac{\partial}{\partial x}p + \rho \cdot g_x = \rho \cdot a_x$ $\quad -\frac{\partial}{\partial y}p + \rho \cdot g_y = \rho \cdot a_y$ $\quad -\frac{\partial}{\partial z}p + \rho \cdot g_z = \rho \cdot a_z$

We have $a_x = a_x \quad g_x = 0 \quad a_y = 0 \quad g_y = -g \quad a_z = 0 \quad g_z = 0$

Hence $\frac{\partial}{\partial x}p = -SG \cdot \rho \cdot a_x$ (1) $\quad \frac{\partial}{\partial y}p = -SG \cdot \rho \cdot g$ (2) $\quad \frac{\partial}{\partial z}p = 0$ (3)

From Eq. 3 we can simplify from $p = p(x,y,z)$ to $p = p(x,y)$

Hence a change in pressure is given by $dp = \frac{\partial}{\partial x}p \cdot dx + \frac{\partial}{\partial y}p \cdot dy$ (4)

At the free surface p = const., so $dp = 0 = \frac{\partial}{\partial x}p \cdot dx + \frac{\partial}{\partial y}p \cdot dy$ or $\frac{dy}{dx} = -\frac{\frac{\partial}{\partial x}p}{\frac{\partial}{\partial y}p} = -\frac{a_x}{g} = -\frac{0.25 \cdot g}{g}$

Hence at the free surface $\frac{dy}{dx} = -0.25$

The equation of the free surface is then $y = -\frac{x}{4} + C$ and through volume conservation the fluid rise in the rear balances the fluid fall in the front, so at the midpoint the free surface has not moved from the rest position

For size $L = 80 \cdot cm$ at the midpoint $x = \frac{L}{2}$ $\quad y = \frac{L}{2}$ (box is half filled) $\quad \frac{L}{2} = -\frac{1}{4} \cdot \frac{L}{2} + C \quad C = \frac{5}{8} \cdot L \quad y = \frac{5}{8} \cdot L - \frac{x}{4}$

Combining Eqs 1, 2, and 4 $\quad dp = -SG \cdot \rho \cdot a_x \cdot dx - SG \cdot \rho \cdot g \cdot dy$ or $p = -SG \cdot \rho \cdot a_x \cdot x - SG \cdot \rho \cdot g \cdot y + c$

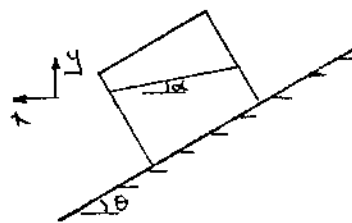
We have $p = p_{atm}$ when $x = 0 \quad y = \frac{5}{8} \cdot L$ so $\quad p_{atm} = -SG \cdot \rho \cdot g \cdot \frac{5}{8} \cdot L + c \quad c = p_{atm} + SG \cdot \rho \cdot g \cdot \frac{5}{8} \cdot L$

$$p(x,y) = p_{atm} + SG \cdot \rho \cdot \left(\frac{5}{8} \cdot g \cdot L - a_x \cdot x - g \cdot y\right) = p_{atm} + SG \cdot \rho \cdot g \cdot \left(\frac{5}{8} \cdot L - \frac{x}{4} - y\right)$$

On the bottom y = 0 so $p(x,0) = p_{atm} + SG \cdot \rho \cdot g \cdot \left(\frac{5}{8} \cdot L - \frac{x}{4}\right) = 101 + 0.8 \times 1000 \cdot \frac{kg}{m^3} \times \frac{N \cdot s^2}{kg \cdot m} \times 9.81 \cdot \frac{m}{s^2} \times \left(\frac{5}{8} \times 0.8 \cdot m - \frac{x}{4}\right) \times \frac{kPa}{10^3 \cdot Pa}$

$p(x,0) = 105 - 1.96 \cdot x$ (p in kPa, x in m)

Problem *3.120

Given: Rectangular container of base dimensions 0.4m × 0.2m and height 0.4m is filled with water to a depth, $d = 0.2m$
Mass of empty container is $M_c = 10$ kg
Container slides down an incline, $\theta = 30°$
Coefficient of sliding friction is 0.30

Find: The angle of the water surface relative to the horizontal

Solution:

Basic equations: $-\nabla P + \rho \vec{g} = \rho \vec{a}$ $\quad \Sigma \vec{F} = M\vec{a}$

Assumptions: (1) fluid moves as solid body, ie no sloshing

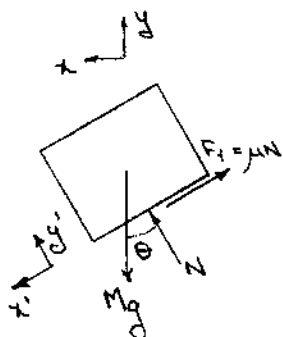
Writing component equations,

$$-\frac{\partial P}{\partial x} = \rho a_x \qquad \frac{\partial P}{\partial x} = -\rho a_x$$

$$-\frac{\partial P}{\partial y} - \rho g = \rho a_y \qquad \frac{\partial P}{\partial y} = -\rho(g + a_y)$$

$P = P(x,y)$ $\quad dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy$. Along the water surface, $dP = 0$

$$\frac{dy}{dx} = -\frac{\partial P/\partial x}{\partial P/\partial y} = -\frac{a_x}{g + a_y}$$

To determine a_x and a_y consider the container and contents

$$M = M_c + M_{H_2O} = M_c + \rho V = 10\,\text{kg} + 999\frac{\text{kg}}{\text{m}^3} \times 0.4\text{m} \times 0.2\text{m} \times 0.2\text{m}$$

$$M = 26\ \text{kg}$$

(Free body diagram: x, y, x', y', $F_f = \mu N$, Mg, N, θ)

$$\Sigma F_{y'} = 0 = N - Mg\cos\theta$$

$$N = Mg\cos\theta = 26\,\text{kg} \times 9.81\frac{\text{m}}{\text{s}^2} \times \cos 30° \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} = 221\ \text{N}$$

$$\Sigma F_{x'} = Ma_{x'} = Mg\sin 30° - F_f = Mg\sin 30° - \mu N$$

$$a_{x'} = g\sin 30° - \mu\frac{N}{M} = 9.81\frac{\text{m}}{\text{s}^2}\sin 30° - 0.3 \times 221\,\text{N} \times \frac{1}{26\,\text{kg}} \times \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}$$

$$a_{x'} = 2.36\ \text{m/sec}^2$$

Then

$$a_x = a_{x'}\cos\theta = 2.36\frac{\text{m}}{\text{sec}^2} \times \cos 30° = 2.04\ \text{m/s}^2$$

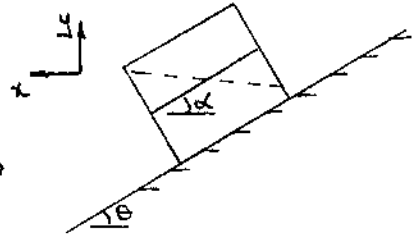
$$a_y = -a_{x'}\sin\theta = -2.36\frac{\text{m}}{\text{sec}^2} \times \sin 30° = -1.18\ \text{m/s}^2$$

and

$$\frac{dy}{dx} = \frac{-a_x}{g + a_y} = -\frac{2.04}{9.81 - 1.18} = -0.236$$

$$\alpha = \tan^{-1} 0.236 = 13.3°$$ ← α

Problem *3.121

Given: Rectangular container of base dimensions 0.4m × 0.2m and height 0.4m is filled with water to a depth, $d = 0.2m$
Mass of empty container is $M_c = 10$ kg
Container slides down an incline, $\theta = 30°$ without friction

Find: (a) The angle of the water surface relative to the horizontal.
(b) Slope of the free surface for the same acceleration up the plane

Solution:

Basic equations: $-\nabla P + \rho\vec{g} = M\vec{a}$ $\quad\quad \Sigma\vec{F} = M\vec{a}$

Assumptions: (1) fluid moves as solid body, ie. no sloshing

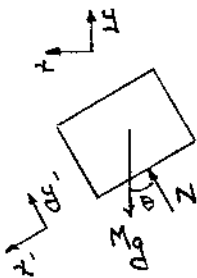
Writing component equations,

$$-\frac{\partial P}{\partial x} = \rho a_x \qquad \frac{\partial P}{\partial x} = -\rho a_x$$

$$-\frac{\partial P}{\partial y} - \rho g = \rho a_y \qquad \frac{\partial P}{\partial y} = -\rho(g + a_y)$$

$P = P(x,y)$ $\quad dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy$ $\quad$ Along the water surface, $dP = 0$

$$\frac{dy}{dx} = \frac{-\partial P/\partial x}{\partial P/\partial y} = -\frac{a_x}{(g + a_y)}$$

For motion without friction

$$\Sigma F_{x'} = Ma_{x'} = Mg\sin\theta \qquad \therefore a_{x'} = g\sin\theta$$

$$a_x = a_{x'}\cos\theta = g\sin\theta\cos\theta$$

$$a_y = -a_{x'}\sin\theta = -g\sin^2\theta$$

$$\frac{dy}{dx} = -\frac{a_x}{(g+a_y)} = -\frac{g\sin\theta\cos\theta}{(g - g\sin^2\theta)} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

$$\frac{dy}{dx} = -\tan 30° = -0.577$$

$$\alpha = \tan^{-1} 0.577 = 30°$$ $\longleftarrow \alpha$

For the same acceleration up the incline, $a_x = -g\sin\theta\cos\theta$ $\quad a_y = g\sin^2\theta$

$$\frac{dy}{dx} = \frac{-a_x}{(g+a_y)} = \frac{g\sin\theta\cos\theta}{(g + g\sin^2\theta)} = \frac{\sin\theta\cos\theta}{1+\sin^2\theta} = \frac{\sin 30\cos 30}{1 + \sin^2 30}$$

$$\frac{dy}{dx} = 0.346$$

Problem *3.122

Given: Gas centrifuge, with maximum peripheral speed, $V_{max} = 300$ m/sec contains uranium hexafluoride gas ($M = 352$ kg/kgmol) at 325C.

Find: (a) Develop an expression for ratio of maximum pressure to pressure at centrifuge axis
(b) Evaluate for given conditions.

Solution:

Basic equation: $-\nabla p + \rho\vec{g} = \rho\vec{a}$ $\qquad p = \rho RT$

(r component) $-\frac{\partial p}{\partial r} + \rho g_r = \rho a_r$

$V_{max} = \omega r_2$

Assumptions: (1) ideal gas behavior, T = constant
(2) r horizontal, so $g_r = 0$
(3) rigid body motion, so

$$a_r = -\frac{V^2}{r} = -\frac{(r\omega)^2}{r} = -r\omega^2$$

Then $\frac{\partial p}{\partial r} = -\rho a_r = \rho r\omega^2 = \frac{p}{RT} r\omega^2$

Separating variables and integrating, we obtain

$$\int_{p_1}^{p_2} \frac{dp}{p} = \frac{\omega^2}{RT}\int_{r_1=0}^{r_2} r\,dr = \frac{\omega^2}{RT}\frac{r_2^2}{2} \qquad V_{max} = \omega r_2$$

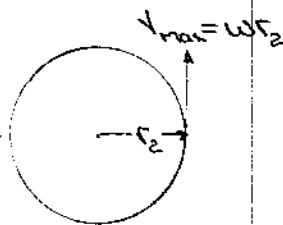
$$\ln\frac{p_2}{p_1} = \frac{V_{max}^2}{2RT}$$

$$\frac{p_2}{p_1} = e^{\frac{V_{max}^2}{2RT}} \qquad \frac{p_2}{p_1}$$

To evaluate, $R = \frac{R_u}{M} = 8314\,\frac{N\cdot m}{kgmol\cdot K} \times \frac{kgmol}{352\,kg} = 23.62\,\frac{N\cdot m}{kg\cdot K}$

$$\frac{V_{max}^2}{2RT} = \frac{(300)^2}{2}\frac{m^2}{s^2} \times \frac{kg\cdot K}{23.62\,N\cdot m} \times \frac{1}{598\,K} \times \frac{N\cdot s^2}{kg\cdot m} = 3.186$$

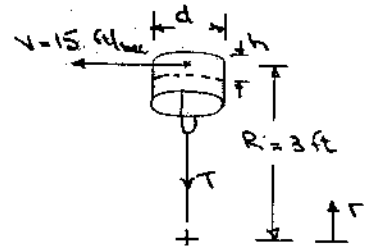
$$\therefore \frac{p_2}{p_1} = e^{3.186} = 24.2$$

Problem *3.123

Given: Pail, 1 ft in diameter and 1 ft deep, weighs 3 lbf and contains 8 in. of water.
Pail is swung in a vertical circle of 3 ft radius and a speed of 15 ft/s.
Water moves as solid body
Point of interest is top of trajectory.

Determine: (a) tension in string
(b) pressure on pail bottom from water

Solution

Assumption: center of mass of bucket and of water are located at $r = 3$ ft where $V = r\omega = 15$ ft/s

$V = 15$ ft/sec, d, h, r, $R = 3$ ft, T, r

Summing forces in radial direction

$$\Sigma F_r \hat{e}_r = M_b a_{b_r} \hat{e}_r + M_w a_{w_r} \hat{e}_r$$

$$-T - (M_b + M_w) g = M_b a_{b_r} + M_w a_{w_r}$$

But $a_{b_r} = a_{w_r} = -\omega^2 r = -\frac{V^2}{r}$

$$\therefore T = \left(\frac{V^2}{r} - g\right)(M_b + M_w)$$

where

$$M_w = \rho_w \forall_w = \rho_w \frac{\pi d^2}{4} h = 1.94 \frac{\text{slug}}{\text{ft}^3} \times \pi \times \frac{1\,\text{ft}^2}{4} \times 8\,\text{in} \times \frac{\text{ft}}{12\,\text{in}} = 1.02\ \text{slug}$$

Then

$$T = \left((15)^2 \frac{\text{ft}^2}{\text{s}^2} \times \frac{1}{3\,\text{ft}} - 32.2 \frac{\text{ft}}{\text{s}^2}\right)\left(3\,\text{lbf} \times \frac{1}{32.2}\frac{\text{s}^2}{\text{ft}} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2} + 1.02\ \text{slug}\right) \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}}$$

$$T = 47.6\ \text{lbf}$$

T

In the water $-\nabla p + \rho \vec{g} = \rho \vec{a}$

Writing the component in the r direction

$$-\frac{\partial p}{\partial r} - \rho g = \rho a_r = -\rho \frac{V^2}{r}$$

$$\frac{\partial p}{\partial r} = \rho\left(\frac{V^2}{r} - g\right) = 1.94 \frac{\text{slug}}{\text{ft}^3}\left((15)^2 \frac{\text{ft}^2}{\text{sec}^2} \times \frac{1}{3\,\text{ft}} - 32.2 \frac{\text{ft}}{\text{sec}^2}\right) \times \frac{\text{lbf}\cdot\text{sec}^2}{\text{ft}\cdot\text{slug}}$$

$$\frac{\partial p}{\partial r} = 83.0\ \text{lbf/ft}^3$$

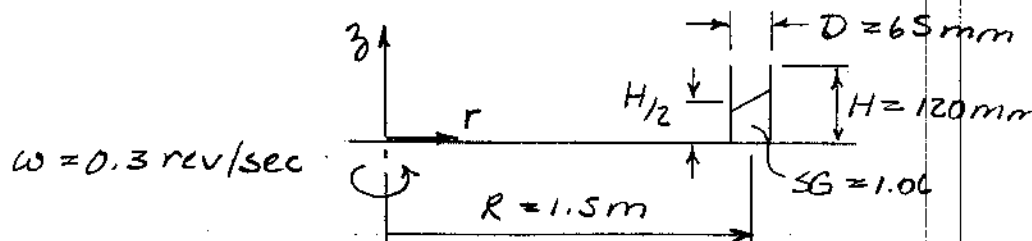
Assuming that $\partial p/\partial r$ is constant throughout the water then

$$p_{bottom} \cong p_{surface} + \frac{\partial p}{\partial r}\Delta r$$

$$p_{bottom} = p_{atm} + 83.0 \frac{\text{lbf}}{\text{ft}^3} \times 8\,\text{in} \times \frac{\text{ft}}{12\,\text{in}} = p_{atm} + 55.3 \frac{\text{lbf}}{\text{ft}^2}$$

$$p_{bottom} - p_{atm} = 55.3\ \text{lbf/ft}^2\ \text{(gage)}$$

Problem *3.124

Given: Soft drink can at outer edge of merry-go-round.

Find: (a) Slope of free surface
(b) Spin rate to spill
(c) Likelihood of spilling vs. slipping

Solution: Assume rigid-body motion

Basic equation: $-\nabla p + \rho\vec{g} = \rho\vec{a}_r \quad a_r = -\frac{V^2}{r} = -\frac{(r\omega)^2}{r} = -r\omega^2$

$$\left.\begin{aligned} -\frac{\partial p}{\partial r} + \rho g_r &= \rho a_r \quad (g_r = 0\ (2)) \\ -\frac{\partial p}{\partial z} + \rho g_z &= \rho a_z \quad (a_z = 0\ (3)) \end{aligned}\right\} \quad \begin{aligned} \frac{\partial p}{\partial r} &= -\rho a_r = \rho r \omega^2 \\ \frac{\partial p}{\partial z} &= +\rho g_z = -\rho g \end{aligned}$$

Assumptions: (1) Rigid-body motion, (2) $g_r = 0$, (3) $a_z = 0$, (4) $g_z = -g$

Then $p = p(r, z)$ so $dp = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial z}dz$

$dp = 0$ along free surface, so $\frac{dz}{dr} = -\frac{\partial p/\partial r}{\partial p/\partial z} = -\frac{\rho r\omega^2}{-\rho g} = \frac{r\omega^2}{g}$

$$\omega = 0.3\ \frac{\text{rev}}{\text{sec}} \times 2\pi\ \frac{\text{rad}}{\text{rev}} = 1.88\ \text{rad/s}$$

$$\left.\frac{dz}{dr}\right)_{surface} = 1.5\ \text{m} \times (1.88)^2\ \frac{\text{rad}^2}{\text{s}^2} \times \frac{\text{s}^2}{9.81\ \text{m}} = 0.540$$

Slope

To spill, slope must be [diagonal H/D] $H/D = 120/65 = 1.85$

$$\text{Thus} \quad \omega = \left[\frac{g}{r}\frac{dz}{dr}\right]^{1/2} = \left[9.81\ \frac{\text{m}}{\text{s}^2} \times 1.85 \times \frac{1}{1.5\ \text{m}}\right]^{1/2} = 3.48\ \text{rad/s}$$

Spill

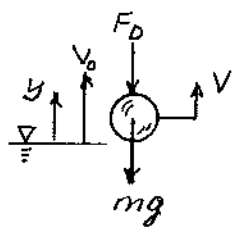
This is nearly double the speed.

The coefficient of static friction between the can and surface is probably $\mu_s \leq 0.5$.

Thus the can would likely not spill or tip: it would slide off!

Problem *3.125

Open-Ended Problem Statement: When a water polo ball is submerged below the surface in a swimming pool and released from rest, it is observed to pop out of the water. How would you expect the height to which it rises above the water to vary with depth of submersion below the surface? Would you expect the same results for a beach ball? For a table-tennis ball?

Discussion: Separate the problem into two parts: (1) motion of the ball in water below the pool surface, and (2) motion of the ball in air above the pool surface.

Below the pool water surface the motion of each ball is controlled by buoyancy force and inertia. For small depths of submersion ball speed upon reaching the pool surface will be small. As depth is increased, ball speed will increase until terminal speed in water is approached. For large depths, the actual depth will be irrelevant because the ball will reach terminal speed before reaching the pool water surface. All three balls are relatively light for their diameters, so terminal speed in water should be reached quickly. The depth of submersion needed to reach terminal speed should be fairly small, perhaps 1 meter or less.[1]

Buoyancy is proportional to volume and inertia is proportional to mass. The ball with the largest volume per unit mass should accelerate most quickly to terminal speed. This probably will be the beach ball, followed by the table-tennis ball and the water polo ball.

The ball with the largest diameter has the smallest frontal area per unit volume; the terminal speed should be highest for this ball. The beach ball should have the highest terminal speed, followed by the water polo ball and the table-tennis ball.

Above the pool water surface the motion of each ball is controlled by aerodynamic drag force, gravity force, and inertia (see equation below). Without aerodynamic drag, the height above the pool water surface reached by each ball would depend only its initial speed.[2] Aerodynamic drag reduces the height reached by each ball.

Aerodynamic drag force is proportional to frontal area. The heaviest ball per unit frontal area (probably the water polo ball) should reach the maximum height and the lightest ball per unit frontal area (probably the beach ball) should reach the minimum height.

F_D, V_0, y, V, mg

$$\Sigma F_y = -F_D - mg = ma_y = m\frac{dV}{dt}$$

$$-C_D A \tfrac{1}{2}\rho V^2 - mg = m\frac{dV}{dt}, \quad \text{since } F_D = C_D A \tfrac{1}{2}\rho V^2$$

$$-\frac{C_D A \frac{1}{2}\rho V^2}{m} - g = \frac{dV}{dt} = V\frac{dV}{dy} \qquad (1)$$

Separating variables
$$\frac{V\,dV}{1+\frac{C_D A \rho}{mg}\frac{V^2}{2}} = -g\,dy$$

Integrating,
$$\frac{mg}{\rho C_D A}\ln\left[1+\frac{\rho C_D A}{mg}\frac{V^2}{2}\right]_{V_0}^{0} = -\frac{mg}{\rho C_D A}\ln\left[1+\frac{\rho C_D A}{mg}\frac{V_0^2}{2}\right] = -g y_{max}$$

---

[1] The initial water depth required to reach terminal speed may be calculated using the methods of Chapter 9.

[2] The maximum height reached by a ball in air with aerodynamic drag may be calculated using the methods of Chapter 9.

Thus $y_{max} = \frac{m}{\rho C_D A} \ln\left[1 + \frac{\rho C_D A}{mg}\frac{V_0^2}{2}\right] = \frac{m}{\rho C_D A} \ln\left[1 + \frac{F_{D_0}}{mg}\right]$ (2) y_{max}

With no aerodynamic drag, Eq. 1 reduces to

$$-mg = mV\frac{dV}{dy} \quad \text{or} \quad V\,dV = -g\,dy$$

Integrating from V_0 to 0, $\left.\frac{V^2}{2}\right]_{V_0}^{0} = -g\,y_{max}$

$$y_{max} = \frac{V_0^2}{2g} \qquad (3)$$

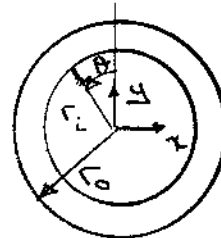
y_{max} ($C_D = 0$)

Check the limiting value predicted by Eq. 2 as $C_D \to 0$:

$$\lim_{C_D \to 0} y_{max} = \lim_{C_D \to 0} \frac{m}{\rho C_D A}\frac{\rho C_D A}{mg}\frac{V_0^2}{2} = \frac{V_0^2}{2g} \quad \checkmark\checkmark$$

Problem *3.126

Given: A steel liner of length $L = 2m$, outer radius $r_o = 0.15\,m$, and inner radius $r_i = 0.10\,m$ is to be formed in a spinning horizontal mold. To insure uniform thickness the minimum radial acceleration should be $10g$. For steel, S.G = 7.8.

Find: (a) The required angular velocity
(b) the maximum and minimum pressures on the surface of the mold.

Solution:

Basic equation: $\nabla p + \rho\vec{g} = \rho\vec{a}$

Writing component equations,

$$-\frac{\partial p}{\partial r} + \rho g_r = \rho a_r \quad \text{and} \quad \frac{\partial p}{\partial r} = \rho g_r - \rho a_r = \rho(-g\cos\theta) - \rho(-r\omega^2) = \rho r\omega^2 - \rho g\cos\theta$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho g_\theta = 0 \quad \text{and} \quad \frac{\partial p}{\partial \theta} = \rho g_\theta r = \rho g \sin\theta\, r$$

Then, $dp = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial \theta}d\theta = (\rho r\omega^2 - \rho g\cos\theta)dr + \rho g r\sin\theta\, d\theta$

$\left.\frac{\partial p}{\partial r}\right)_\theta = \text{const} = \rho r\omega^2 - \rho g\cos\theta$. Since $p = p_{atm}$ at $r = r_i$, then

$p - p_{atm} = \int_{r_i}^{r}(\rho r\omega^2 - \rho g\cos\theta)\,dr + f(\theta)$ where, $f(\theta)$ is an arbitrary function

$\therefore\ p = p_{atm} + \rho\omega^2\frac{(r^2 - r_i^2)}{2} - \rho g\cos\theta(r - r_i) + f(\theta)$. Then,

$\frac{\partial p}{\partial \theta} = \rho g \sin\theta(r - r_i) + \frac{df}{d\theta} = \rho g\sin\theta\, r$

Hence, $\frac{df}{d\theta} = \rho g \sin\theta\, r_i$ and $f = -\rho g r_i\cos\theta + c$

$\therefore\ p = p_{atm} + \rho\omega^2\frac{(r^2 - r_i^2)}{2} - \rho g\cos\theta(r - r_i) - \rho g r_i\cos\theta + c$

At $r = r_i$, $p = p_{atm}$ for any value of θ. Hence, $c = \rho g r_i \cos\theta$ and

$$p = p_{atm} + \rho\omega^2\frac{(r^2 - r_i^2)}{2} - \rho g\cos\theta(r - r_i)$$

Minimum value of $a_r = 10g = r\omega^2$ occurs at r_i for given ω. Hence,

$$\omega_{min} = \left[\frac{10g}{r_i}\right]^{1/2} = \left[10 \times 9.81\frac{m}{s^2} \times \frac{1}{0.10m}\right]^{1/2} = 31.3\ \text{rad/s}$$ ← ω

p_{max} on the surface of the mold ($r = r_o$) occurs at $\theta = \pi$

$p_{max} - p_{atm} = \frac{\rho\omega^2}{2}(r_o^2 - r_i^2) - \rho g\cos\theta(r - r_i)$.

$$p_{max} - p_{atm} = \frac{1}{2}\times 7.8 \times 999\frac{kg}{m^3}\times\frac{(31.3)^2}{s^2}\times[(0.15)^2 - (0.10)^2]m^2 \times \frac{N\cdot s^2}{kg\cdot m} - 7.8\times 999\frac{kg}{m^3}\times 9.81\frac{m}{s^2}(-1)[0.15 - 0.10]\frac{N\cdot s^2}{kg\cdot m}$$

$p_{max} = 51.5$ kPa (gage) ← p_{max}

p_{min} on the surface of the mold ($r = r_o$) occurs at $\theta = 0$

$p_{min} - p_{atm} = \frac{\rho\omega^2}{2}(r_o^2 - r_i^2) - \rho g\cos\theta(r - r_i)$

$$p_{min} - p_{atm} = \frac{1}{2}\times 7.8\times 999\frac{kg}{m^3}\times\frac{(31.3)^2}{s^2}[(0.15)^2 - (0.10)^2]m^2\times\frac{N\cdot s^2}{kg\cdot m} - 7.8\times 999\frac{kg}{m^3}\times 9.81\frac{m}{s^2}(+1)[0.15 - 0.10]\frac{N\cdot s^2}{kg\cdot m}$$

$p_{min} = 43.9$ kPa (gage) ← p_{min}

Problem *3.127

Open-Ended Problem Statement: The analysis of problem 3.120 suggests that it maybe possible to determine the coefficient of sliding friction between two surfaces by measuring the slope of the free surface in a liquid-filled container sliding down an inclined surface. Investigate the feasibility of this idea.

Discussion: A certain minimum angle of inclination would be needed to overcome static friction and start the container into motion down the incline. Once the container is in motion, the retarding force would be provided by sliding (dynamic) friction. The coefficient of dynamic friction usually is smaller than the static friction coefficient. Thus the container would continue to accelerate as it moved down the incline. This acceleration would provide a non-zero slope to the free surface of the liquid in the container.

In principle the slope could be measured and the coefficient of dynamic friction calculated. In practice several problems would arise.

To calculate dynamic friction coefficient one must assume the liquid moves as a solid body (i.e., that there is no sloshing). This condition could only be achieved if there were minimum initial disturbance and the sliding distance were long.

It would be difficult to measure the slope of the free surface of liquid in the moving container. Images made with a video camera or digital still camera might be processed to obtain the required slope information.

$$\Sigma F_y = N - mg\cos\theta \; ; \; N = mg\cos\theta$$

$$\Sigma F_x = mg\sin\theta - F_f = ma_x \; ; \; F_f = \mu_k N = \mu_k mg\cos\theta$$

$$a_x = g\sin\theta - \mu_k g\cos\theta = g(\sin\theta - \mu_k\cos\theta)$$

For static liquid $-\nabla p + \rho\vec{g} = \rho\vec{a}$

$$-\frac{\partial p}{\partial x} + \rho g\sin\theta = \rho a_x = \rho g(\sin\theta - \mu_k\cos\theta) \; ; \; \frac{\partial p}{\partial x} = \rho g\mu_k\cos\theta$$

$$-\frac{\partial p}{\partial y} - \rho g\cos\theta = \rho a_y^{\,0} \; ; \; \frac{\partial p}{\partial y} = -\rho g\cos\theta$$

For the free surface, $dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = 0$, so $\frac{dy}{dx} = -\frac{\partial p/\partial x}{\partial p/\partial y}$

Thus $\frac{dy}{dx} = -\frac{\rho g\mu_k\cos\theta}{-\rho g\cos\theta} = \mu_k \; ; \; \alpha = \tan^{-1}(\mu_k)$

Since it was necessary to make the container slip on the surface,

$$\theta > \tan^{-1}(\mu_s) > \tan^{-1}(\mu_k) = \alpha$$

α

Thus $\alpha < \theta$, as shown in the sketch above.

Problem 4.1

4.1 A mass of 3 kg falls freely a distance of 5 m before contacting a spring attached to the ground. If the spring stiffness is 400 N/m, what is the maximum spring compression?

Given: Data on mass and spring

Find: Maximum spring compression

Solution:

The given data is $M = 3\cdot kg$ $\quad h = 5\cdot m$ $\quad k = 400\cdot\frac{N}{m}$

Apply the First Law of Thermodynamics: for the system consisting of the mass and the spring (the spring has gravitional potential energy and the spring elastic potential energy)

Total mechanical energy at initial state $\quad E_1 = M\cdot g\cdot h$

Total mechanical energy at instant of maximum compression *x* $\quad E_2 = M\cdot g\cdot(-x) + \frac{1}{2}\cdot k\cdot x^2$

Note: The datum for zero potential is the top of the uncompressed spring

But $\quad E_1 = E_2$

so $\quad M\cdot g\cdot h = M\cdot g\cdot(-x) + \frac{1}{2}\cdot k\cdot x^2$

Solving for x $\quad x^2 - \frac{2\cdot M\cdot g}{k}\cdot x - \frac{2\cdot M\cdot g\cdot h}{k} = 0$

$$x = \frac{M\cdot g}{k} + \sqrt{\left(\frac{M\cdot g}{k}\right)^2 + \frac{2\cdot M\cdot g\cdot h}{k}}$$

$$x = 3\cdot kg\times 9.81\cdot\frac{m}{s^2}\times\frac{m}{400\cdot N} + \sqrt{\left(3\cdot kg\times 9.81\cdot\frac{m}{s^2}\times\frac{m}{400\cdot N}\right)^2 + 2\times 3\cdot kg\times 9.81\cdot\frac{m}{s^2}\times 5\cdot m\times\frac{m}{400\cdot N}}$$

$$x = 0.934\,m$$

Note that ignoring the loss of potential of the mass due to spring compression x gives

$$x = \sqrt{\frac{2\cdot M\cdot g\cdot h}{k}} \qquad x = 0.858\,m$$

Note that the deflection if the mass is dropped from immediately above the spring is

$$x = \frac{2\cdot M\cdot g}{k} \qquad x = 0.147\,m$$

Problem 4.2

Given: Six-pack cooled from 25°C to 5°C in freezer.

Find: Change in specific entropy.

Solution: Apply the Tds equation.

Basic equation: $T ds = du + p dv$ ($\simeq 0(1)$)

Assumptions: (1) Neglect volume change
(2) Liquid properties are similar to water

Then

$$T ds = du = c_v \, dT$$

or

$$ds = c_v \frac{dT}{T}$$

Integrating,

$$s_2 - s_1 = c_v \ln\left(\frac{T_2}{T_1}\right)$$

$$= 1 \frac{\text{kcal}}{\text{kg} \cdot \text{K}} \times \ln\left(\frac{273+5}{273+25}\right) \times 4190 \frac{\text{J}}{\text{kcal}}$$

$$s_2 - s_1 = -0.291 \text{ kJ/kg} \cdot \text{K}$$

Δs

Problem 4.3

[2]

4.3 A fully loaded Boeing 777-200 jet transport aircraft weighs 325,000 kg. The pilot brings the 2 engines to full takeoff thrust of 450 kN each before releasing the brakes. Neglecting aerodynamic and rolling resistance, estimate the minimum runway length and time needed to reach a takeoff speed of 225 kph. Assume engine thrust remains constant during ground roll.

Given: Data on Boeing 777-200 jet

Find: Minimum runway length for takeoff

Solution:

Basic equation $\Sigma F_x = M \cdot \frac{dV}{dt} = M \cdot V \cdot \frac{dV}{dx} = F_t = \text{constant}$ Note that the "weight" is already in mass units!

Separating variables $M \cdot V \cdot dV = F_t \cdot dx$

Integrating $x = \frac{M \cdot V^2}{2 \cdot F_t}$

$$x = \frac{1}{2} \times 325 \times 10^3 \text{kg} \times \left(225 \frac{\text{km}}{\text{hr}} \times \frac{1 \cdot \text{km}}{1000 \cdot \text{m}} \times \frac{1 \cdot \text{hr}}{3600 \cdot \text{s}}\right)^2 \times \frac{1}{2 \times 425 \times 10^3} \cdot \frac{1}{\text{N}} \times \frac{\text{N} \cdot \text{s}^2}{\text{kg} \cdot \text{m}} \qquad x = 747\,\text{m}$$

For time calculation $M \cdot \frac{dV}{dt} = F_t$ $\qquad dV = \frac{F_t}{M} \cdot dt$

Integrating $t = \frac{M \cdot V}{F_t}$

$$t = 325 \times 10^3 \text{kg} \times 225 \frac{\text{km}}{\text{hr}} \times \frac{1 \cdot \text{km}}{1000 \cdot \text{m}} \times \frac{1 \cdot \text{hr}}{3600 \cdot \text{s}} \times \frac{1}{2 \times 425 \times 10^3} \cdot \frac{1}{\text{N}} \times \frac{\text{N} \cdot \text{s}^2}{\text{kg} \cdot \text{m}} \qquad t = 23.9\,\text{s}$$

Aerodynamic and rolling resistances would significantly increase both these results

Problem 4.4

Given: Small steel ball of radius, r, atop large sphere of radius, R, begins to roll. Neglect rolling and air resistance.

Find: Location where ball loses contact and becomes a projectile.

<u>Solution</u>: Sum forces in n direction

$$\Sigma F_n = F_n - mg\cos\theta = ma_n$$

$$a_n = -\frac{V^2}{(R+r)}$$

Contact is lost when $F_n \to 0$, or

$$-mg\cos\theta = -m\frac{V^2}{(R+r)}$$

or

$$V^2 = (R+r)g\cos\theta \qquad (1)$$

Energy must be conserved if there is no resistance. Thus

$$E = mgz + m\frac{V^2}{2} = mg(R+r)\cos\theta + m\frac{V^2}{2} = E_0 = mg(R+r)$$

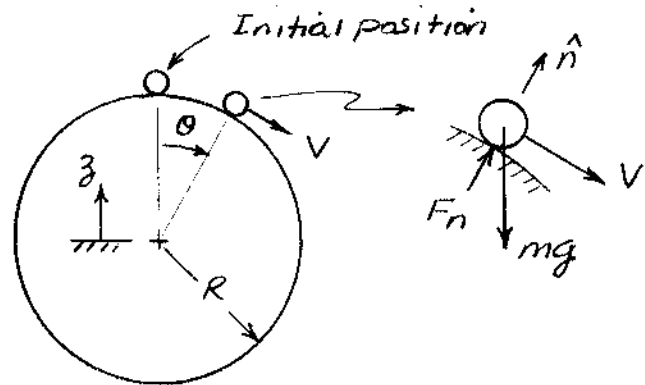
Thus from energy considerations

$$V^2 = 2g(R+r)(1-\cos\theta) \qquad (2)$$

Combining Eqs. 1 and 2,

$$V^2 = 2g(R+r)(1-\cos\theta) = (R+r)g\cos\theta$$

or $\quad 2(1-\cos\theta) = 2 - 2\cos\theta = \cos\theta$

Thus $\cos\theta = \frac{2}{3}$ and $\theta = \cos^{-1}(\frac{2}{3}) = 48.2$ degrees $\quad \theta$

Problem 4.5

Given: Auto skids to stop in 50 meters on level road with $\mu = 0.6$.

Find: Initial speed.

Solution: Apply Newton's second law to a system (auto).

Basic equations: $\Sigma F_x = ma_x = \frac{W}{g}\frac{d^2x}{dt^2}$

Assumptions: (1) $F_f = \mu W$
(2) Neglect air resistance

Then $\Sigma F_x = -F_f = -\mu W = \frac{W}{g}\frac{d^2x}{dt^2}$

or $\frac{d^2x}{dt^2} = -\mu g$

Integrating,

$$\frac{dx}{dt} = -\mu g t + C_1 = -\mu g t + V_0 \qquad (1)$$

since $V = V_0$ at $t = 0$. Integrating again,

$$x = -\frac{1}{2}\mu g t^2 + V_0 t + C_2 = -\frac{1}{2}\mu g t^2 + V_0 t \qquad (2)$$

since $x = 0$ at $t = 0$.

Now at $x = L$, $\frac{dx}{dt} = 0$, and $t = t_f$. From Eq. 1,

$$0 = -\mu g t_f + V_0 \quad \text{or} \quad t_f = \frac{V_0}{\mu g}$$

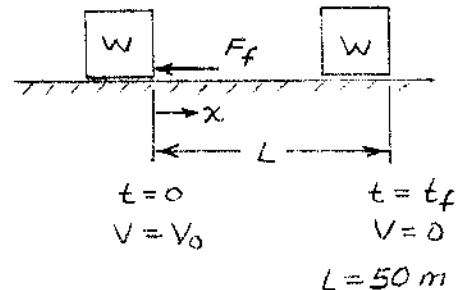
Substituting into Eq. 2, evaluated at $t = t_f$,

$$L = -\frac{1}{2}\mu g t_f^2 + V_0 t_f = -\frac{1}{2}\mu g \frac{V_0^2}{(\mu g)^2} + V_0 \frac{V_0}{\mu g}$$

$$L = -\frac{1}{2}\frac{V_0^2}{\mu g} + \frac{V_0^2}{\mu g} = \frac{1}{2}\frac{V_0^2}{\mu g}$$

Solving, $V_0 = \sqrt{2\mu g L} = \sqrt{2(0.6)\, 9.81\,\frac{m}{s^2} \times 50\ m} = 24.3\ m/s$

or

$$V_0 = 24.3\,\frac{m}{s} \times \frac{km}{1000\ m} \times 3600\,\frac{s}{hr} = 87.5\ km/hr$$

V_0

Problem 4.6

[2]

4.6 Air at 68°F and an absolute pressure of 1 atm is compressed adiabatically, without friction, to an absolute pressure of 3 atm. Determine the internal energy change.

Given: Data on air compression process

Find: Internal energy change

Solution:

Basic equation $\delta Q - \delta W = dE$

Assumptions: 1) Adiabatic so $\delta Q = 0$ 2) Stationary system $dE = dU$ 3) Frictionless process $\delta W = pdV = Mpdv$

Then $dU = -\delta W = -M \cdot p \cdot dv$

Before integrating we need to relate p and v. An adiabatic frictionless (reversible) process is isentropic, which for an ideal gas gives

$$p \cdot v^k = C \qquad \text{where} \qquad k = \frac{c_p}{c_v}$$

Hence $$v = C^{\frac{1}{k}} \cdot p^{-\frac{1}{k}} \qquad \text{and} \qquad dv = C^{\frac{1}{k}} \cdot \frac{1}{k} \cdot p^{-\frac{1}{k}-1} \cdot dp$$

Substituting $$du = \frac{dU}{M} = -p \cdot dv = -p \cdot C^{\frac{1}{k}} \cdot \frac{1}{k} \cdot p^{-\frac{1}{k}-1} \cdot dp = \frac{-C^{\frac{1}{k}}}{k} \cdot p^{-\frac{1}{k}} \cdot dp$$

Integrating between states $$\Delta u = \frac{C^{\frac{1}{k}}}{k-1} \cdot \left(p_2^{\frac{k-1}{k}} - p_1^{\frac{k-1}{k}} \right) = \frac{C^{\frac{1}{k}} \cdot p_1^{\frac{k-1}{k}}}{k-1} \cdot \left[\left(\frac{p_2}{p_1} \right)^{\frac{k-1}{k}} - 1 \right]$$

But $$C^{\frac{1}{k}} \cdot p^{\frac{k-1}{k}} = C^{\frac{1}{k}} \cdot p^{-\frac{1}{k}} \cdot p = p \cdot v = R_{air} \cdot T$$

Hence $$\Delta u = \frac{R_{air} \cdot T_1}{k-1} \cdot \left[\left(\frac{p_2}{p_1} \right)^{\frac{k-1}{k}} - 1 \right]$$

From Table A.6 $$R_{air} = 53.33 \cdot \frac{ft \cdot lbf}{lbm \cdot R} \qquad \text{and} \qquad k = 1.4$$

$$\Delta u = \frac{1}{0.4} \times 53.33 \cdot \frac{ft \cdot lbf}{lbm \cdot R} \times (68 + 460) R \times \left[\left(\frac{3}{1} \right)^{\frac{1.4-1}{1.4}} - 1 \right] \qquad \Delta u = 2.6 \times 10^4 \cdot \frac{ft \cdot lbf}{lbm}$$

$$\Delta u = 33.4 \cdot \frac{Btu}{lbm} \qquad \Delta u = 1073 \cdot \frac{Btu}{slug}$$ (Using conversions from Table G.2)

Problem 4.7

4.7 In an experiment with a can of soda, it took 3 hr to cool from an initial temperature of 75°F to 50°F in a 40°F refrigerator. If the can is now taken from the refrigerator and placed in a room at 68°F, how long will the can take to reach 60°F? You may assume that for both processes the heat transfer is modeled by $\dot{Q} \approx k(T - T_{amb})$, where T is the can temperature, T_{amb} is the ambient temperature, and k is a heat transfer coefficient.

Given:

Data on cooling of a can of soda in a refrigerator

Find:

How long it takes to warm up in a room

Solution:

The First Law of Thermodynamics for the can (either warming or cooling) is

$$M \cdot c \cdot \frac{dT}{dt} = -k \cdot (T - T_{amb}) \qquad \text{or} \qquad \frac{dT}{dt} = -A \cdot (T - T_{amb}) \qquad \text{where} \qquad A = \frac{k}{M \cdot c}$$

where M is the can mass, c is the average specific heat of the can and its contents, T is the temperature, and T_{amb} is the ambient temperature

Separating variables $\quad \frac{dT}{T - T_{amb}} = -A \cdot dt$

Integrating $\quad T(t) = T_{amb} + (T_{init} - T_{amb}) \cdot e^{-A\,t}$

where T_{init} is the initial temperature. The available data from the coolling can now be used to obtain a value for constant A

Given data for cooling $\quad T_{init} = (25 + 273) \cdot K \quad T_{init} = 298\,K \quad T_{amb} = (5 + 273) \cdot K \quad T_{amb} = 278\,K$

$T = (10 + 273) \cdot K \quad T = 283\,K \quad$ when $\quad t = \tau = 10 \cdot hr$

Hence $\quad A = \frac{1}{\tau} \cdot \ln\left(\frac{T_{init} - T_{amb}}{T - T_{amb}}\right) = \frac{1}{3 \cdot hr} \times \frac{1 \cdot hr}{3600 \cdot s} \times \ln\left(\frac{298 - 278}{283 - 278}\right) \qquad A = 1.284 \times 10^{-4}\,s^{-1}$

Then, for the warming up process

$T_{init} = (10 + 273) \cdot K \quad T_{init} = 283\,K \quad T_{amb} = (20 + 273) \cdot K \quad T_{amb} = 293\,K$

$T_{end} = (15 + 273) \cdot K \quad T_{end} = 288\,K$

with $\quad T_{end} = T_{amb} + (T_{init} - T_{amb}) \cdot e^{-A\,\tau}$

Hence the time τ is $\quad \tau = \frac{1}{A} \cdot \ln\left(\frac{T_{init} - T_{amb}}{T_{end} - T_{amb}}\right) = \frac{s}{1.284 \cdot 10^{-4}} \cdot \ln\left(\frac{283 - 293}{288 - 293}\right) \qquad \tau = 5.40 \times 10^{3}\,s \qquad \tau = 1.50\,hr$

Problem 4.8

4.8 The average rate of heat loss from a person to the surroundings when not actively working is about 85W. Suppose that in an auditorium with volume of approximately $3.5 \times 10^5\ m^3$, containing 6000 people, the ventilation system fails. How much does the internal energy of the air in the auditorium increase during the first 15 min after the ventilation system fails? Considering the auditorium and people as a system, and assuming no heat transfer to the surroundings, how much does the internal energy of the system change? How do you account for the fact that the temperature of the air increases? Estimate the rate of temperature rise under these conditions.

Given:

Data on heat loss from persons, and people-filled auditorium

Find:

Internal energy change of air and of system; air temperature rise

Solution:

Basic equation $\quad Q - W = \Delta E$

Assumptions: 1) Stationary system dE =dU 2) No work W = 0

Then for the air $\quad \Delta U = Q = 85 \cdot \frac{W}{person} \times 6000 \cdot people \times 15 \cdot min \times \frac{60 \cdot s}{min} \qquad \Delta U = 459\,MJ$

For the air and people $\quad \Delta U = Q_{surroundings} = 0$

The increase in air energy is equal and opposite to the loss in people energy

For the air $\quad \Delta U = Q \quad$ but for air (an ideal gas) $\quad \Delta U = M \cdot c_V \cdot \Delta T \quad$ with $\quad M = \rho \cdot V = \frac{p \cdot V}{R_{air} \cdot T}$

Hence $\quad \Delta T = \frac{Q}{M \cdot c_V} = \frac{R_{air} \cdot Q \cdot T}{c_V \cdot p \cdot V}$

From Table A.6 $\quad R_{air} = 286.9 \cdot \frac{J}{kg \cdot K} \quad$ and $\quad c_V = 717.4 \cdot \frac{J}{kg \cdot K}$

$$\Delta T = \frac{286.9}{717.4} \times 459 \times 10^6 \cdot J \times (20 + 273) K \times \frac{1}{101 \times 10^3} \cdot \frac{m^2}{N} \times \frac{1}{3.5 \times 10^5} \cdot \frac{1}{m^3} \qquad \Delta T = 1.521\,K$$

This is the temperature change in 15 min. The rate of change is then $\quad \frac{\Delta T}{15 \cdot min} = 6.09 \frac{K}{hr}$

Problem 4.9

Given: Aluminum beverage can, $m_c = 20\,g$, $D = 65$ mm, $H = 120$ mm.

Maximum contents level is h_{max}, when $\forall_b = 354$ mL of beverage.

SG of beverage is 1.05.

Find: (a) Center of mass, y_c, vs. level, h.
(b) Level for least tendency to tip.
(c) Minimum coefficient of friction, μ_s, for full can to tip, not slide.
(d) Plot $\mu_s)_{minimum}$ for can to tip (not slide) as a function of beverage level in can.

Solution: $M_b = SG\rho\forall_b = 1.05 \times 1.0\frac{g}{cm^3} \times 354\,mL \times \frac{cm^3}{mL} = 372\,g$ (max)

$$h_{max} = \frac{\forall_b}{A} = \frac{4\forall_b}{\pi D^2} = \frac{4}{\pi} \times 354\,mL \times \frac{1}{(6.5)^2\,cm^2} \times \frac{cm^3}{mL} \times \frac{10\,mm}{cm} = 107\,mm$$

At any level, $m_b = \frac{h}{h_{max}} M_b$; $m_b(g) = \frac{h(mm)}{107\,mm} \times 372\,g = 3.47\,h\,(mm)$

From moment considerations,

$$y_c M = \frac{h}{2} m_b + \frac{H}{2} m_c = \frac{1}{2}\left[h(3.47h) + 120(20)\right] = \frac{1}{2}(3.47h^2 + 2400)$$

$$M = m_b + m_c = 3.47h + 20$$

$$y_c = \frac{3.47\,h^2 + 2400}{6.94\,h + 40} \quad (h \text{ in mm})$$

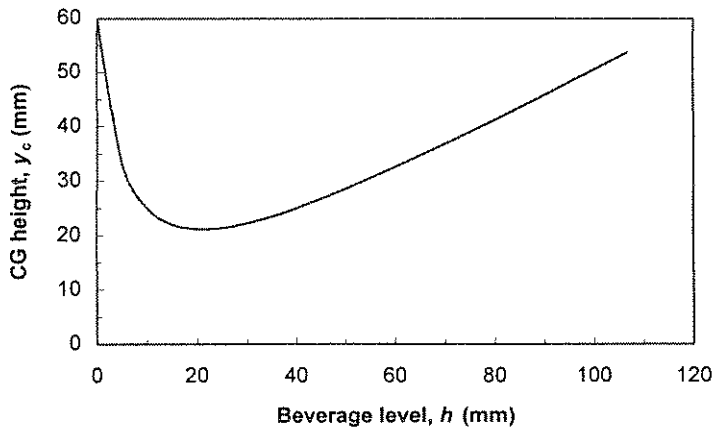
y_c

Tendency to tip will be least when y_c is a minimum. Thus

$$\frac{dy_c}{dh} = \frac{2(3.47h)}{6.94h + 40} + (-1)(6.95)\frac{3.47h^2 + 2400}{(6.94\,h + 40)^2} = \frac{24.1h^2 + 278\,h - 16{,}700}{(6.94\,h + 40)^2} = 0$$

Using the quadratic formula,

$$h\,(\text{at } y_c \text{ min}) = \frac{-278 \pm \sqrt{(278)^2 + 4(24.1)16{,}700}}{2(24.1)} = 21.2\,mm$$

h (y_c min)

Plotting,

Next page

Draw a free-body diagram of the can at tipping:

$$\Sigma F_x = F_f = ma_x$$

$$\Sigma F_y = F_n - mg = ma_y = 0$$

$$F_n = mg$$

Since $F_f \leq \mu_s F_n$, then $\mu_s F_n \geq ma_x$

Summing moments about point O:

$$\Sigma \overset{+}{M_O} = y_c\, ma_x - \frac{D}{2} F_n = 0$$

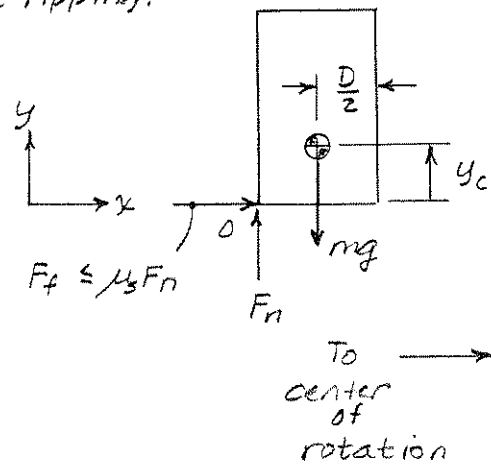
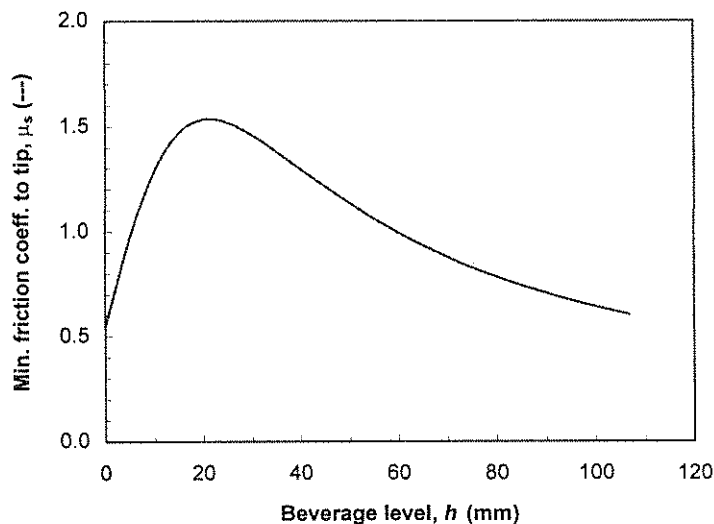
or $\quad y_c\, ma_x = \frac{D}{2} F_n$

But $ma_x \leq \mu_s F_n$, so

$$y_c \mu_s F_n \geq \frac{D}{2} F_n$$

Thus to tip

$$\mu_s \geq \frac{D}{2y_c}$$

μ_s

Plotting,

For the full can with $y_c = 53.8$ mm,

$$\mu_s \geq \frac{1}{2} \times 65\,mm \times \frac{1}{53.8\,mm} = 0.604$$

μ_s

This value is much higher than the can could develop. Therefore the can <u>will</u> <u>not</u> tip; it will slide.

The corresponding acceleration is $a_x = \mu_s g = 0.593\ m/s^2$

Problem 4.10

[3]

4.10 The velocity field in the region shown is given by $\vec{V} = az\hat{j} + b\hat{k}$, where $a = 10\ \text{s}^{-1}$ and $b = 5$ m/s. For the 1 m × 1 m triangular control volume (depth $w = 1$ m perpendicular to the diagram), an element of area ① may be represented by $w(-dz\hat{j} + dy\hat{k})$ and an element of area ② by $wdz\hat{j}$.

a. Find an expression for $\vec{V} \cdot d\vec{A}_1$.
b. Evaluate $\int_{A_1} \vec{V} \cdot d\vec{A}_1$.
c. Find an expression for $\vec{V} \cdot d\vec{A}_2$.
d. Find an expression for $\vec{V}(\vec{V} \cdot d\vec{A}_2)$.
e. Evaluate $\int_{A_2} \vec{V}(\vec{V} \cdot d\vec{A}_2)$.

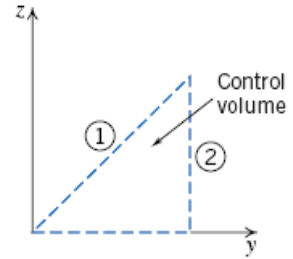
Given: Data on velocity field and control volume geometry

Find: Several surface integrals

Solution:

$$d\vec{A}_1 = -wdz\hat{j} + wdy\hat{k} \qquad d\vec{A}_1 = -dz\hat{j} + dy\hat{k}$$

$$d\vec{A}_2 = wdz\hat{j} \qquad d\vec{A}_2 = dz\hat{j}$$

$$\vec{V} = \left(az\hat{j} + b\hat{k}\right) \qquad \vec{V} = \left(10z\hat{j} + 5\hat{k}\right)$$

(a) $$\vec{V} \cdot dA_1 = \left(10z\hat{j} + 5\hat{k}\right) \cdot \left(-dz\hat{j} + dy\hat{k}\right) = -10zdz + 5dy$$

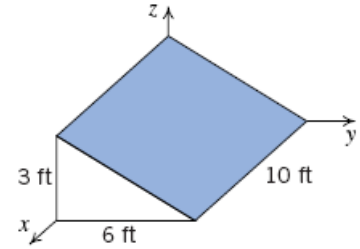
(b) $$\int_{A_1} \vec{V} \cdot dA_1 = -\int_0^1 10zdz + \int_0^1 5dy = -5z^2\Big|_0^1 + 5y\Big|_0^1 = 0$$

(c) $$\vec{V} \cdot dA_2 = \left(10z\hat{j} + 5\hat{k}\right) \cdot \left(dz\hat{j}\right) = 10zdz$$

(d) $$\vec{V}\left(\vec{V} \cdot dA_2\right) = \left(10z\hat{j} + 5\hat{k}\right)10zdz$$

(e) $$\int_{A_2} \vec{V}\left(\vec{V} \cdot dA_2\right) = \int_0^1 \left(10z\hat{j} + 5\hat{k}\right)10zdz = \frac{100}{3}z^3\hat{j}\Big|_0^1 + 25z^2\hat{k}\Big|_0^1 = 33.3\hat{j} + 25\hat{k}$$

Problem 4.11

4.11 The shaded area shown is in a flow where the velocity field is given by $\vec{V} = ax\hat{i} - by\hat{j}$; $a = b = 1\ \text{s}^{-1}$, and the coordinates are measured in meters. Evaluate the volume flow rate and the momentum flux through the shaded area.

Given: Geometry of 3D surface

Find: Volume flow rate and momentum flux through area

Solution:

$$d\vec{A} = dxdz\hat{j} + dxdy\hat{k}$$

$$\vec{V} = ax\hat{i} - by\hat{j} \qquad \vec{V} = x\hat{i} - y\hat{j}$$

We will need the equation of the surface: $z = 3 - \frac{1}{2}y$ or $y = 6 - 2z$

a) Volume flow rate

$$Q = \int_A \vec{V} \cdot dA = \int_A \left(x\hat{i} - y\hat{j}\right)\cdot\left(dxdz\hat{j} + dxdy\hat{k}\right)$$

$$= \int_0^{10}\int_0^3 -ydzdx = \int_0^3 -10ydz = \int_0^3 -10(6-2z)dz = -60z + 10z^2\Big|_0^3$$

$$Q = (-180 + 90)\frac{\text{ft}^3}{\text{s}}$$

$$Q = -90\frac{\text{ft}^3}{\text{s}}$$

b) Momentum flux

$$\rho\int_A \vec{V}\left(\vec{V}\cdot d\vec{A}\right) = \rho\int_A \left(x\hat{i} - y\hat{j}\right)(-ydxdz)$$

$$= \rho\int_0^{10}\int_0^3 (-xy)dzdx\hat{i} + \rho\int_0^3 10y^2dz\,\hat{j}$$

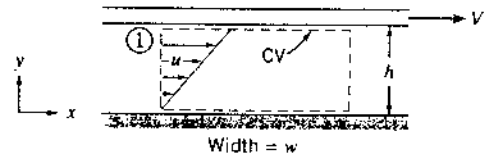
$$= -\rho\int_0^{10} xdx\int_0^3 (6-2z)dz\hat{i} + \rho\int_0^3 10(6-2z)^2dz\hat{j}$$

$$= \rho\left(-\frac{x^2}{2}\Big|_0^{10}\right)\left(6z - z^2\Big|_0^3\right)\hat{i} + \rho\left(10\left(36z - 12z^2 + \frac{4}{3}z^3\right)\Big|_0^3\right)\hat{j}$$

$$= \rho(-50)(18-9)\hat{i} + \rho(10(108 - 108 + 36))\hat{j}$$

$$= -450\rho\hat{i} + 360\rho\hat{j} \qquad \left(\frac{\text{slug}\cdot{}^{\text{ft}}/_{\text{s}}}{\text{s}} \text{ if } \rho \text{ is in } \frac{\text{slug}}{\text{ft}^3}\right)$$

Problem 4.12

Given: Control volume with linear velocity distribution across surface ① as shown; width = w.

Find: (a) Volume flow rate, and (b) Momentum flux, through surface ①

Solution:

The volume flow rate is $Q = \int \vec{V} \cdot d\vec{A}$

At surface ①, $\vec{V} = \frac{V}{h} y \hat{i}$ and $dA = -w\,dy\,\hat{i}$

Thus

$$Q = \int_{y=0}^{h} \frac{V}{h} y \hat{i} \cdot (-w\,dy\,\hat{i}) = -\frac{Vw}{h} \int_{y=0}^{h} y\,dy = -\frac{Vw}{h} \frac{y^2}{2} \Big]_0^h$$

$$Q = -\frac{1}{2} Vhw \qquad \text{Volume flow rate}$$

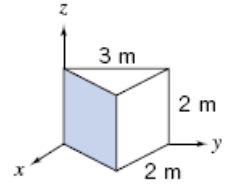
The momentum flux is given by $m.f. = \int \vec{V} (\rho \vec{V} \cdot d\vec{A})$

Thus,

$$m.f. = \int_0^h \frac{V}{h} y \hat{i} \left(-\rho \frac{Vw}{h} y\,dy\right) = -\rho \frac{V^2 w}{h^2} \hat{i} \int_0^h y^2\,dy = -\rho \frac{V^2 w}{h^2} \hat{i} \frac{y^3}{3} \Big]_0^h$$

$$m.f. = -\frac{1}{3} \rho V^2 w h\,\hat{i} \qquad \text{Momentum flux}$$

Problem 4.13

4.13 The area shown shaded is in a flow where the velocity field is given by $\vec{V} = -ax\hat{i} + by\hat{j} + c\hat{k}$; $a = b = 2\ \text{s}^{-1}$ and $c = 2.5$ m/s. Write a vector expression for an element of the shaded area. Evaluate the integrals $\int \vec{V} \cdot d\vec{A}$ and $\int \vec{V}(\vec{V} \cdot d\vec{A})$ over the shaded area.

Given: Geometry of 3D surface

Find: Surface integrals

Solution:

$$d\vec{A} = dydz\hat{i} - dxdz\hat{j}$$

$$\vec{V} = -ax\hat{i} + by\hat{j} + c\hat{k} \qquad \vec{V} = -2x\hat{i} + 2y\hat{j} + 2.5\hat{k}$$

We will need the equation of the surface: $y = \frac{3}{2}x$ or $x = \frac{2}{3}y$

$$\int_A \vec{V} \cdot dA = \int_A \left(-ax\hat{i} + by\hat{j} + c\hat{k}\right) \cdot \left(dydz\hat{i} - dxdz\hat{j}\right)$$

$$= \int_0^2 \int_0^3 -axdydz - \int_0^2 \int_0^2 bydxdz = -a\int_0^2 dz \int_0^3 \frac{2}{3}ydy - b\int_0^2 dz \int_0^2 \frac{3}{2}xdx = -2a\frac{1}{3}y^2\Big|_0^3 - 2b\frac{3}{4}x^2\Big|_0^2$$

$$Q = (-6a - 6b)$$

$$Q = -24\frac{\text{m}^3}{\text{s}}$$

We will again need the equation of the surface: $y = \frac{3}{2}x$ or $x = \frac{2}{3}y$, and also $dy = \frac{3}{2}dx$ and $a = b$

$$\int_A \vec{V}\left(\vec{V} \cdot d\vec{A}\right) = \int_A \left(-ax\hat{i} + by\hat{j} + c\hat{k}\right)\left(-ax\hat{i} + by\hat{j} + c\hat{k}\right) \cdot \left(dydz\hat{i} - dxdz\hat{j}\right)$$

$$= \int_A \left(-ax\hat{i} + by\hat{j} + c\hat{k}\right)\left(-axdydz - bydxdz\right)$$

$$= \int_A \left(-ax\hat{i} + \frac{3}{2}ax\hat{j} + c\hat{k}\right)\left(-ax\frac{3}{2}dxdz - a\frac{3}{2}xdxdz\right)$$

$$= \int_A \left(-ax\hat{i} + \frac{3}{2}ax\hat{j} + c\hat{k}\right)\left(-3axdxdz\right)$$

$$= 3\int_0^2 \int_0^2 a^2x^2dxdz\hat{i} - \frac{9}{2}\int_0^2 \int_0^2 a^2x^2dxdz\hat{j} - 3\int_0^2 \int_0^2 acxdxdz\hat{k}$$

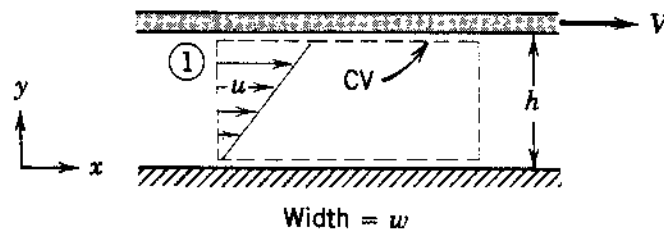
$$= (6)\left(a^2\frac{x^3}{3}\Big|_0^2\right)\hat{i} - (9)\left(a^2\frac{x^3}{3}\Big|_0^2\right)\hat{j} - (6)\left(ac\frac{x^2}{2}\Big|_0^2\right)$$

$$= 16a^2\hat{i} - 24a^2\hat{j} - 12ac\hat{k}$$

$$= 64\hat{i} - 96\hat{j} - 60\hat{k} \quad \frac{\text{m}^4}{\text{s}^2}$$

Problem 4.14

[2]____

Given: Flow and CV of Problem 4.12, as shown

Find: Expression for kinetic energy flux through cross-section ① of CV.

Solution: Kinetic energy flux is defined as $kef = \int_A \frac{V^2}{2} \rho \vec{V} \cdot d\vec{A}$

Model the velocity profile as $u = V\frac{y}{h}$. Then

$$\vec{V} = u\hat{i} = V\frac{y}{h}\hat{i}\ ;\ V^2 = V^2\left(\frac{y}{h}\right)^2$$

Since flow is into the CV, $\vec{V} \cdot d\vec{A} = -u\,dA = -V\frac{y}{h}w\,dy$

Substituting,

$$kef = \int_A \frac{V^2}{2}\left(\frac{y}{h}\right)^2 \left\{-\rho V \frac{y}{h} w\,dy\right\} = -\frac{\rho V^3 w}{2h^3}\int_0^h y^3\,dy$$

$$= -\frac{\rho V^3 w}{2h^3}\left[\frac{y^4}{4}\right]_0^h$$

$$kef = -\frac{\rho V^3 w h}{8}$$

kef

Check dimensions:

$$[kef] = \frac{M}{L^3}\left(\frac{L}{t}\right)^3 LL = \frac{ML^2}{t^3} \times \frac{Ft^2}{ML} = \frac{FL}{t} = \frac{Energy}{Time}$$ ✓✓

Problem 4.15

Given: Velocity distribution for laminar flow in a long circular tube

$$\vec{V} = u\hat{i} = u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]\hat{i}$$

where R is the tube radius.

Evaluate: (a) The volume flow rate, and (b) the momentum flux, through a section normal to the pipe axis.

Solution: The volume flow rate is given by

$$\int_{A_{tube}} \vec{V}\cdot d\vec{A} = \int_0^R u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]\hat{i}\cdot 2\pi r\,dr\,\hat{i} \qquad \{A = \pi r^2,\ dA = 2\pi r\,dr\}$$

$$= u_{max}2\pi\int_0^R\left[1 - \left(\frac{r}{R}\right)^2\right]r\,dr = u_{max}2\pi\int_0^R\left[r - \frac{r^3}{R^2}\right]dr$$

$$= u_{max}2\pi\left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R = u_{max}2\pi\left[\frac{R^2}{2} - \frac{R^2}{4}\right]$$

$$\int_{A_{tube}} \vec{V}\cdot d\vec{A} = \frac{1}{2}u_{max}\pi R^2 \longleftarrow \text{volume flow rate}$$

The momentum flux is given by

$$\int_{A_{tube}} \vec{V}(\vec{V}\cdot d\vec{A}) = \int_0^R u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]\hat{i}\left\{u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]\hat{i}\cdot 2\pi r\,dr\,\hat{i}\right\}$$

$$= \int_0^R u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]\hat{i}\left\{u_{max}2\pi\left[r - \frac{r^3}{R^2}\right]dr\right\}$$

$$= u_{max}^2 2\pi\int_0^R\left(r - \frac{2r^3}{R^2} + \frac{r^5}{R^4}\right)dr\,\hat{i}$$

$$= u_{max}^2 2\pi\left[\frac{r^2}{2} - \frac{r^4}{2R^2} + \frac{r^6}{6R^4}\right]_0^R\hat{i}$$

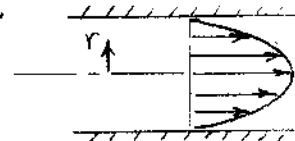
$$= u_{max}^2 2\pi R^2\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right]\hat{i}$$

$$\int_{A_{tube}} \vec{V}(\vec{V}\cdot d\vec{A}) = \frac{1}{3}u_{max}^2\pi R^2\hat{i} \longleftarrow \text{momentum flux}$$

Problem 4.16

[2]

Given: Velocity profile in a circular tube,

$$\vec{V} = u\hat{i} = u_{max}\left[1-\left(\frac{r}{R}\right)^2\right]\hat{i}$$

Find: Expression for kinetic energy flux, $kef = \int \frac{V^2}{2}\rho\vec{V}\cdot d\vec{A}$

Solution: $V^2 = \vec{V}\cdot\vec{V} = u_{max}^2\left[1-\left(\frac{r}{R}\right)^2\right]^2 = u_{max}^2\left[1-2\left(\frac{r}{R}\right)^2+\left(\frac{r}{R}\right)^4\right]$

$$d\vec{A} = 2\pi r\,dr\,\hat{i}$$

$$\vec{V}\cdot d\vec{A} = 2\pi r\,u_{max}\left[1-\left(\frac{r}{R}\right)^2\right]$$

Then

$$kef = \int_0^R \frac{u_{max}^2}{2}\left[1-2\left(\frac{r}{R}\right)^2+\left(\frac{r}{R}\right)^4\right]\rho 2\pi r u_{max}\left[1-\left(\frac{r}{R}\right)^2\right]dr$$

$$= \pi\rho u_{max}^3\int_0^R\left[1-3\left(\frac{r}{R}\right)^2+3\left(\frac{r}{R}\right)^4-\left(\frac{r}{R}\right)^6\right]r\,dr$$

$$= \pi\rho u_{max}^3 R^2\int_0^1\left[1-3\left(\frac{r}{R}\right)^2+3\left(\frac{r}{R}\right)^4-\left(\frac{r}{R}\right)^6\right]\frac{r}{R}\,d\left(\frac{r}{R}\right)$$

$$= \pi\rho u_{max}^3 R^2\left[\frac{1}{2}\left(\frac{r}{R}\right)^2-\frac{3}{4}\left(\frac{r}{R}\right)^4+\frac{1}{2}\left(\frac{r}{R}\right)^6-\frac{1}{8}\left(\frac{r}{R}\right)^8\right]_0^1$$

$$= \pi R^2\rho u_{max}^3\left[\frac{1}{2}-\frac{3}{4}+\frac{1}{2}-\frac{1}{8}\right]$$

$$kef = \frac{\pi R^2\rho u_{max}^3}{8}$$

kef

Problem 4.17

4.17 A farmer is spraying a liquid through 10 nozzles, ⅛th in. ID, at an average exit velocity of 10 ft/s. What is the average velocity in the 1-in. ID head feeder? What is the system flow rate, in gpm?

Given: Data on flow through nozzles

Find: Average velocity in head feeder; flow rate

Solution:

Basic equation $$\sum_{CS} \left(\vec{V} \cdot \vec{A}\right) = 0$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Uniform flow

Then for the nozzle flow $$\sum_{CS} \left(\vec{V} \cdot \vec{A}\right) = -V_{feeder} \cdot A_{feeder} + 10 \cdot V_{nozzle} \cdot A_{nozzle} = 0$$

Hence $$V_{feeder} = V_{nozzle} \cdot \frac{10 \cdot A_{nozzle}}{A_{feeder}} = V_{nozzle} \cdot 10 \cdot \left(\frac{D_{nozzle}}{D_{feeder}}\right)^2$$

$$V_{feeder} = 10 \cdot \frac{ft}{s} \times 10 \times \left(\frac{\frac{1}{8}}{1}\right)^2 \qquad V_{feeder} = 1.56 \cdot \frac{ft}{s}$$

The flow rate is $$Q = V_{feeder} \cdot A_{feeder} = V_{feeder} \cdot \frac{\pi \cdot D_{feeder}^2}{4}$$

$$Q = 1.56 \cdot \frac{ft}{s} \times \frac{\pi}{4} \times \left(1 \cdot in \times \frac{1 \cdot ft}{12 \cdot in}\right)^2 \times \frac{7.48 \cdot gal}{1 \cdot ft^3} \times \frac{60 \cdot s}{1 \cdot min} \qquad Q = 3.82 \cdot gpm$$

Problem 4.18

[3]

4.18 A cylindrical holding water tank has a 3 m ID, and a height of 3 m. There is one inlet of diameter 10 cm, an exit of diameter 8 cm, and a drain. The tank is initially empty when the inlet pump is turned on, producing an average inlet velocity of 5 m/s. When the level in the tank reaches 0.7 m, the exit pump turns on, causing flow out of the exit; the exit average velocity is 3 m/s. When the water level reaches 2 m the drain is opened such that the level remains at 2 m. Find (a) the time at which the exit pump is switched on, (b) the time at which the drain is opened, and (c) the flow rate into the drain (m^3/min).

Given: Data on flow into and out of tank

Find: Time at which exit pump is switched on; time at which drain is opened; flow rate into drain

Solution:

Basic equation $$\frac{\partial}{\partial t}M_{CV} + \sum_{CS}\left(\rho\cdot\vec{V}\cdot\vec{A}\right) = 0$$

Assumptions: 1) Uniform flow 2) Incompressible flow

After inlet pump is on $$\frac{\partial}{\partial t}M_{CV} + \sum_{CS}\left(\rho\cdot\vec{V}\cdot\vec{A}\right) = \frac{\partial}{\partial t}M_{tank} - \rho\cdot V_{in}\cdot A_{in} = 0 \qquad \frac{\partial}{\partial t}M_{tank} = \rho\cdot A_{tank}\cdot\frac{dh}{dt} = \rho\cdot V_{in}\cdot A_{in}$$ where h is the level of water in the tank

$$\frac{dh}{dt} = V_{in}\cdot\frac{A_{in}}{A_{tank}} = V_{in}\cdot\left(\frac{D_{in}}{D_{tank}}\right)^2$$

Hence the time to reach h_{exit} = 0.7 m is $$t_{exit} = \frac{h_{exit}}{\frac{dh}{dt}} = \frac{h_{exit}}{V_{in}}\cdot\left(\frac{D_{tank}}{D_{in}}\right)^2 \qquad t_{exit} = 0.7\cdot m\times\frac{1}{5}\cdot\frac{s}{m}\times\left(\frac{3\cdot m}{0.1\cdot m}\right)^2 \qquad t_{exit} = 126\,s$$

After exit pump is on $$\frac{\partial}{\partial t}M_{CV} + \sum_{CS}\left(\rho\cdot\vec{V}\cdot\vec{A}\right) = \frac{\partial}{\partial t}M_{tank} - \rho\cdot V_{in}\cdot A_{in} + \rho\cdot V_{exit}\cdot A_{exit} = 0 \qquad A_{tank}\cdot\frac{dh}{dt} = V_{in}\cdot A_{in} - V_{exit}\cdot A_{exit}$$

$$\frac{dh}{dt} = V_{in}\cdot\frac{A_{in}}{A_{tank}} - V_{exit}\cdot\frac{A_{exit}}{A_{tank}} = V_{in}\cdot\left(\frac{D_{in}}{D_{tank}}\right)^2 - V_{exit}\cdot\left(\frac{D_{exit}}{D_{tank}}\right)^2$$

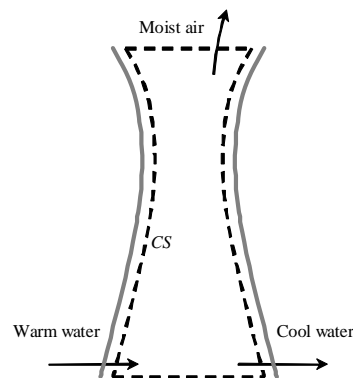
Hence the time to reach h_{drain} = 2 m is $$t_{drain} = t_{exit} + \frac{\left(h_{drain} - h_{exit}\right)}{\frac{dh}{dt}} = \frac{\left(h_{drain} - h_{exit}\right)}{V_{in}\cdot\left(\frac{D_{in}}{D_{tank}}\right)^2 - V_{exit}\cdot\left(\frac{D_{exit}}{D_{tank}}\right)^2}$$

$$t_{drain} = 126\cdot s + (2 - 0.7)\cdot m\times\frac{1}{5\cdot\frac{m}{s}\times\left(\frac{0.1\cdot m}{3\cdot m}\right)^2 - 3\cdot\frac{m}{s}\times\left(\frac{0.08\cdot m}{3\cdot m}\right)^2} \qquad t_{drain} = 506\,s$$

The flow rate into the drain is equal to the net inflow (the level in the tank is now constant)

$$Q_{drain} = V_{in}\cdot\frac{\pi\cdot D_{in}^2}{4} - V_{exit}\cdot\frac{\pi\cdot D_{exit}^2}{4} \qquad Q_{drain} = 5\cdot\frac{m}{s}\times\frac{\pi}{4}\times(0.1\cdot m)^2 - 3\cdot\frac{m}{s}\times\frac{\pi}{4}\times(0.08\cdot m)^2 \qquad Q_{drain} = 0.0242\,\frac{m^3}{s}$$

Problem 4.19

4.19 A wet cooling tower cools warm water by spraying it into a forced dry-air flow. Some of the water evaporates in this air and is carried out of the tower into the atmosphere; the evaporation cools the remaining water droplets, which are collected at the exit pipe (6 in. ID) of the tower. Measurements indicate the warm water mass flow rate is 250,000 lb/hr, and the cool water (70°F) flows at an average speed of 5.55 ft/s in the exit pipe. The flow rate of the moist air is to be obtained from measurements of the velocity at four points, each representing 1/4 of the air stream cross-sectional area of 13.2 ft^2. The moist air density is 0.066 lb/ft^3. Find (a) the volume and mass flow rates of the cool water, (b) the mass flow rate of the moist air, and (c) the mass flow rate of the dry air.

Given: Data on flow into and out of cooling tower

Find: Volume and mass flow rate of cool water; mass flow rate of moist and dry air

Solution:

Basic equation $\sum_{CS} \left(\rho \cdot \vec{V} \cdot \vec{A}\right) = 0$ and at each inlet/exit $Q = V \cdot A$

Assumptions: 1) Uniform flow 2) Incompressible flow

At the cool water exit $Q_{cool} = V \cdot A$ $\quad Q_{cool} = 5.55 \cdot \frac{ft}{s} \times \frac{\pi}{4} \times (0.5 \cdot ft)^2$ $\quad Q_{cool} = 1.09 \frac{ft^3}{s}$ $\quad Q_{cool} = 489\,gpm$

The mass flow rate is $m_{cool} = \rho \cdot Q_{cool}$ $\quad m_{cool} = 1.94 \cdot \frac{slug}{ft^3} \times 1.09 \cdot \frac{ft^3}{s}$ $\quad m_{cool} = 2.11 \frac{slug}{s}$ $\quad m_{cool} = 2.45 \times 10^5 \frac{lb}{hr}$

NOTE: Software does not allow dots over terms, so m represents mass flow rate, not mass!

For the air flow we need to use $\sum_{CS} \left(\rho \cdot \vec{V} \cdot \vec{A}\right) = 0$ to balance the water flow

We have $-m_{warm} + m_{cool} + m_v = 0$ $\quad m_v = m_{warm} - m_{cool}$ $\quad m_v = 5073 \frac{lb}{hr}$

This is the mass flow rate of water vapor. We need to use this to obtain air flow rates. From psychrometrics $x = \frac{m_v}{m_{air}}$

where x is the relative humidity. It is also known (try Googling "density of moist air") that $\frac{\rho_{moist}}{\rho_{dry}} = \frac{1 + x}{1 + x \cdot \frac{R_{H2O}}{R_{air}}}$

We are given $\rho_{moist} = 0.066 \cdot \frac{lb}{ft^3}$

For dry air we could use the ideal gas equation $\rho_{dry} = \frac{p}{R \cdot T}$ but here we use atmospheric air density (Table A.3)

$\rho_{dry} = 0.002377 \cdot \frac{slug}{ft^3}$ $\quad \rho_{dry} = 0.002377 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{lb}{slug}$ $\quad \rho_{dry} = 0.0765 \frac{lb}{ft^3}$

Note that moist air is less dense than dry air!

Hence $$\frac{0.066}{0.0765} = \frac{1 + x}{1 + x \cdot \frac{85.78}{53.33}}$$ using data from Table A.6

$$x = \frac{0.0765 - 0.066}{0.066 \cdot \frac{85.78}{53.33} - .0765}$$ $x = 0.354$

Hence $$\frac{m_v}{m_{air}} = x$$ leads to $$m_{air} = \frac{m_v}{x}$$ $$m_{air} = 5073 \cdot \frac{lb}{hr} \times \frac{1}{0.354}$$ $$m_{air} = 14331 \frac{lb}{hr}$$

Finally, the mass flow rate of moist air is $$m_{moist} = m_v + m_{air}$$ $$m_{moist} = 19404 \frac{lb}{hr}$$

Problem 4.20

[1]

4.20 A university laboratory wishes to build a wind tunnel with variable speeds. Rather than use a variable speed fan, it is proposed to build the tunnel with a sequence of three circular test sections: Section 1 will have a diameter of 5 ft, Section 2 a diameter of 3 ft, and Section 3 a diameter of 2 ft. If the average speed in Section 1 is 20 mph, what will be the speeds in the other two sections? What will be the flow rate (ft^3/min)?

Given: Data on wind tunnel geometry

Find: Average speeds in wind tunnel

Solution:

Basic equation $Q = V \cdot A$

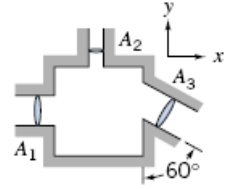
Assumptions: 1) Steady flow 2) Incompressible flow 3) Uniform flow

Between sections 1 and 2 $$Q = V_1 \cdot A_1 = V_1 \cdot \frac{\pi \cdot D_1^{\ 2}}{4} = V_2 \cdot A_2 = V_2 \cdot \frac{\pi \cdot D_2^{\ 2}}{4}$$

Hence $$V_2 = V_1 \cdot \left(\frac{D_1}{D_2}\right)^2 \qquad V_2 = 20 \cdot mph \cdot \left(\frac{5}{3}\right)^2 \qquad V_2 = 55.6\,mph$$

Similarly $$V_3 = V_1 \cdot \left(\frac{D_1}{D_3}\right)^2 \qquad V_3 = 20 \cdot mph \cdot \left(\frac{5}{2}\right)^2 \qquad V_3 = 125\,mph$$

Problem 4.21

4.21 Fluid with 65 lbm/ft^3 density is flowing steadily through the rectangular box shown. Given $A_1 = 0.5$ ft^2, $A_2 = 0.1$ ft^2, $A_3 = 0.6$ ft^2, $\vec{V}_1 = 10\hat{i}$ ft/s, and $\vec{V}_2 = 20\hat{j}$ ft/s, determine velocity $\vec{V}_3$.

Given: Data on flow through box

Find: Velocity at station 3

Solution:

Basic equation $$\sum_{CS} \left(\vec{V}\cdot\vec{A}\right) = 0$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Uniform flow

Then for the box $$\sum_{CS} \left(\vec{V}\cdot\vec{A}\right) = -V_1\cdot A_1 + V_2\cdot A_2 + V_3\cdot A_3 = 0$$

Note that the vectors indicate that flow is in at location 1 and out at location 2; we assume outflow at location 3

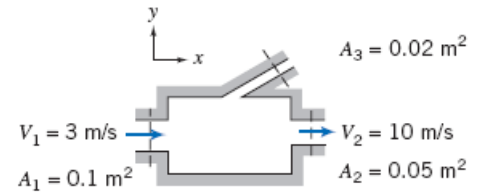
Hence $$V_3 = V_1\cdot\frac{A_1}{A_3} - V_2\cdot\frac{A_2}{A_3} \qquad V_3 = 10\cdot\frac{ft}{s}\times\frac{0.5}{0.6} - 20\cdot\frac{ft}{s}\times\frac{0.1}{0.6} \qquad V_3 = 5\frac{ft}{s}$$

Based on geometry $$V_x = V_3\cdot\sin(60\cdot deg) \qquad V_x = 4.33\frac{ft}{s}$$

$$V_y = -V_3\cdot\cos(60\cdot deg) \qquad V_y = -2.5\frac{ft}{s}$$

$$\vec{V_3} = \left(4.33\cdot\frac{ft}{s}, -2.5\cdot\frac{ft}{s}\right)$$

Problem 4.22

4.22 Consider steady, incompressible flow through the device shown. Determine the magnitude and direction of the volume flow rate through port 3.

Given: Data on flow through device

Find: Volume flow rate at port 3

Solution:

Basic equation $\displaystyle\sum_{CS} \left(\vec{V}\cdot\vec{A}\right) = 0$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Uniform flow

Then for the box $\displaystyle\sum_{CS} \left(\vec{V}\cdot\vec{A}\right) = -V_1 \cdot A_1 + V_2 \cdot A_2 + V_3 \cdot A_3 = -V_1 \cdot A_1 + V_2 \cdot A_2 + Q_3$

Note we assume outflow at port 3

Hence $Q_3 = V_1 \cdot A_1 - V_2 \cdot A_2$ $\qquad Q_3 = 3 \cdot \frac{m}{s} \times 0.1 \cdot m^2 - 10 \cdot \frac{m}{s} \times 0.05 \cdot m^2$ $\qquad Q_3 = -0.2 \cdot \frac{m^3}{s}$

The negative sign indicates the flow at port 3 is inwards. Flow rate at port 3 is 0.2 m^3/s inwards

Problem 4.23

4.23 A rice farmer needs to fill her 5 acre field with water to a depth of 3 in. in 1 hr. How many 6 in. diameter supply pipes are needed if the average velocity in each must be less than 10 ft/s?

Given: Water needs of farmer

Find: Number of 6 in. pipes needed

Solution:

Basic equation $Q = V \cdot A$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Uniform flow

Then $Q = n \cdot V \cdot \frac{\pi \cdot D^2}{4}$ where n is the number of pipes, V is the average velocity in the pipes, and D is the pipe diameter

The flow rate is given by $Q = \frac{5 \cdot acre \cdot 0.25 \cdot ft}{1 \cdot hr} = \frac{5 \cdot acre \cdot 0.25 \cdot ft}{1 \cdot hr} \times \frac{43560 \cdot ft^2}{1 \cdot acre} \times \frac{1 \cdot hr}{3600 \cdot s}$ Data on acres from Googling!

$$Q = 15.1 \cdot \frac{ft^3}{s}$$

Hence $n = \frac{4 \cdot Q}{\pi \cdot V \cdot D^2}$ $n = \frac{4}{\pi} \times \frac{s}{10 \cdot ft} \times \left(\frac{1}{0.5 \cdot ft}\right)^2 \times 15.1 \cdot \frac{ft^3}{s}$ $n = 7.69$

Hence we need at least eight pipes

Problem 4.24

4.24 You are filling your car with gasoline at a rate of 5.3 gals/min. Although you can't see it, the gasoline is rising in the tank at a rate of 4.3 in. per minute. What is the horizontal cross-sectional area of your gas tank? Is this a realistic answer?

Given: Data on filling of gas tank

Find: Cross-section area of tank

Solution:

We can treat this as a steady state problem if we choose a CS as the original volume of gas in the tank, so that additional gas "leaves" the gas as the gas level in the tank rises, OR as an unsteady problem if we choose the CS as the entire gas tank. We choose the latter

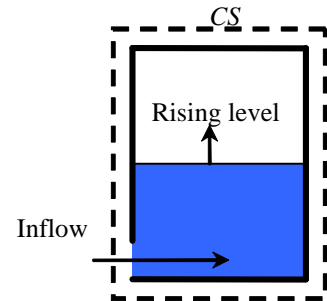
Basic equation $$\frac{\partial}{\partial t}M_{CV} + \sum_{CS}\left(\rho \cdot \vec{V} \cdot \vec{A}\right) = 0$$

Assumptions: 1) Incompressible flow 2) Uniform flow

Hence $$\frac{\partial}{\partial t}M_{CV} = \rho \cdot A \cdot \frac{dh}{dt} = -\sum_{CS}\left(\rho \cdot \vec{V} \cdot \vec{A}\right) = \rho \cdot Q$$

where Q is the gas fill rate, A is the tank cross-section area, and h is the rate of rise in the gas tank

Hence $$A = \frac{Q}{\frac{dh}{dt}} \qquad A = 5.3 \cdot \frac{gal}{min} \times \frac{1 \cdot ft^3}{7.48 \cdot gal} \times \frac{1}{4.3} \cdot \frac{min}{in} \times \frac{12 \cdot in}{1 \cdot ft}$$ Data on gals from Table G.2

$$A = 1.98\,ft^2 \qquad A = 285\,in^2$$ This seems like a reasonable area e.g., 1 ft x 2 ft

Problem 4.25

[1]

4.25 For your sink at home, the flow rate in is 5000 units/hr. Accumulation is 2500 units. What is the accumulation rate if the outflow is 60 units/min? Suddenly, the outflow becomes 13 units/min: What is the accumulation rate? At another time, the flow rate in is 5 units/sec. The accumulation is 50 units. The accumulation rate is −4 units/sec. What is the flow rate out?

Given: Data on filling of a sink

Find: Accumulation rate under various circumstances

Solution:

This is an unsteady problem if we choose the CS as the entire sink

Basic equation $$\frac{\partial}{\partial t}M_{CV} + \sum_{CS}\left(\rho\cdot\vec{V}\cdot\vec{A}\right) = 0$$

Assumptions: 1) Incompressible flow

Hence $$\frac{\partial}{\partial t}M_{CV} = \text{Accumulationrate} = -\sum_{CS}\left(\rho\cdot\vec{V}\cdot\vec{A}\right) = \text{Inflow} - \text{Outflow}$$

$$\text{Accumulationrate} = \text{Inflow} - \text{Outflow}$$

For the first case $$\text{Accumulationrate} = 5000\cdot\frac{\text{units}}{\text{hr}} - 60\cdot\frac{\text{units}}{\text{min}}\times\frac{60\cdot\text{min}}{\text{hr}} \qquad \text{Accumulationrate} = 1400\cdot\frac{\text{units}}{\text{hr}}$$

For the second case $$\text{Accumulationrate} = 5000\cdot\frac{\text{units}}{\text{hr}} - 13\cdot\frac{\text{units}}{\text{min}}\times\frac{60\cdot\text{min}}{\text{hr}} \qquad \text{Accumulationrate} = 4220\cdot\frac{\text{units}}{\text{hr}}$$

For the third case $$\text{Outflow} = \text{Inflow} - \text{Accumulationrate}$$

$$\text{Outflow} = 5\cdot\frac{\text{units}}{\text{s}} - (-4)\cdot\frac{\text{units}}{\text{s}} \qquad \text{Outflow} = 9\cdot\frac{\text{units}}{\text{s}}$$

Problem 4.26

[1]

4.26 You are trying to pump storm water out of your basement during a storm. The pump can extract 10 gpm. The water level in the basement is now sinking about 1 in./hr. What is the flow rate (gpm) from the storm into the basement? The basement is 25 ft by 20 ft.

Given: Data on filling of a basement during a storm

Find: Flow rate of storm into basement

Solution:

This is an unsteady problem if we choose the CS as the entire basement

Basic equation
$$\frac{\partial}{\partial t}M_{CV} + \sum_{CS}\left(\rho\cdot\vec{V}\cdot\vec{A}\right) = 0$$

Assumptions: 1) Incompressible flow

Hence
$$\frac{\partial}{\partial t}M_{CV} = \rho\cdot A\cdot\frac{dh}{dt} = -\sum_{CS}\left(\rho\cdot\vec{V}\cdot\vec{A}\right) = \rho\cdot Q_{storm} - \rho\cdot Q_{pump}$$

where A is the basement area and dh/dt is the rate at which the height of water in the basement changes.

or
$$Q_{storm} = Q_{pump} - A\cdot\frac{dh}{dt}$$

$$Q_{storm} = 10\cdot\frac{gal}{min} - 25\cdot ft \times 20\cdot ft \times \left(-\frac{1}{12}\cdot\frac{ft}{hr}\right) \times \frac{7.48\cdot gal}{ft^3} \times \frac{1\cdot hr}{60\cdot min}$$

Data on gals from Table G.2

$$Q_{storm} = 15.2\,gpm$$

Problem 4.27

[1]

4.27 In steady-state flow downstream, the density is 4 lb/ft^3, the velocity is 10 ft/sec, and the area is 1 ft^2. Upstream, the velocity is 15 ft/sec, and the area is 0.25 ft^2. What is the density upstream?

Given: Data on flow through device

Find: Volume flow rate at port 3

Solution:

Basic equation $\sum_{CS} \left(\rho \cdot \vec{V} \cdot \vec{A}\right) = 0$

Assumptions: 1) Steady flow 2) Uniform flow

Then for the box $\sum_{CS} \left(\rho \cdot \vec{V} \cdot \vec{A}\right) = -\rho_u \cdot V_u \cdot A_u + \rho_d \cdot V_d \cdot A_d = 0$

Hence $\rho_u = \rho_d \cdot \frac{V_d \cdot A_d}{V_u \cdot A_u}$ $\qquad \rho_u = 4 \cdot \frac{lb}{ft^3} \times \frac{10}{15} \times \frac{1}{0.25}$ $\qquad \rho_u = 10.7 \frac{lb}{ft^3}$

Problem 4.28

[2]

4.28 In the incompressible flow through the device shown, velocities may be considered uniform over the inlet and outlet sections. The following conditions are known: $A_1 = 0.1\ \text{m}^2$, $A_2 = 0.2\ \text{m}^2$, $A_3 = 0.15\ \text{m}^2$, $V_1 = 10e^{-t/2}$ m/s, and $V_2 = 2\cos(2\pi t)$ m/s (t in seconds). Obtain an expression for the velocity at section ③, and plot V_3 as a function of time. At what instant does V_3 first become zero? What is the total mean volumetric flow at section ③?

Given: Data on flow through device

Find: Velocity V_3; plot V_3 against time; find when V_3 is zero; total mean flow

Solution:

Governing equation: For incompressible flow (Eq. 4.13) and uniform flow $\int \vec{V}\,d\vec{A} = \sum \vec{V}\cdot\vec{A} = 0$

Applying to the device (assuming V_3 is out) $-V_1\cdot A_1 - V_2\cdot A_2 + V_3\cdot A_3 = 0$

$$V_3 = \frac{V_1\cdot A_1 + V_2\cdot A_2}{A_3} = \frac{10\cdot e^{-\frac{t}{2}}\cdot\frac{m}{s}\times 0.1\cdot m^2 + 2\cdot\cos(2\cdot\pi\cdot t)\cdot\frac{m}{s}\times 0.2\cdot m^2}{0.15\cdot m^2}$$

The velocity at A_3 is $V_3 = 6.67\cdot e^{-\frac{t}{2}} + 2.67\cdot\cos(2\cdot\pi\cdot t)$

The total mean volumetric flow at A_3 is

$$Q = \int_0^{\infty} V_3\cdot A_3\,dt = \int_0^{\infty}\left(6.67\cdot e^{-\frac{t}{2}} + 2.67\cdot\cos(2\cdot\pi\cdot t)\right)\cdot 0.15\,dt\cdot\left(\frac{m}{s}\cdot m^2\right)$$

$$Q = \lim_{t\to\infty}\left(-2\cdot e^{-\frac{t}{2}} + \frac{1}{5\cdot\pi}\cdot\sin(2\cdot\pi\cdot t)\right) - (-2) = 2\cdot m^3 \qquad Q = 2\cdot m^3$$

The time at which V_3 first is zero, and the plot of V_3 is shown in the corresponding *Excel* workbook $t = 2.39\cdot s$

Problem 4.28

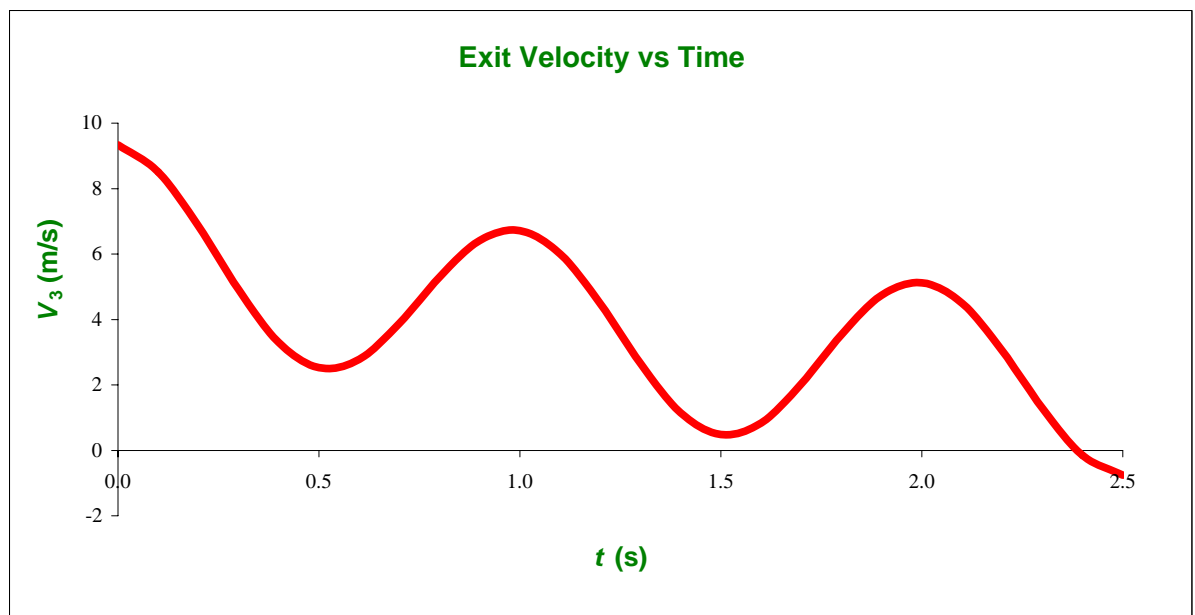
4.28 In the incompressible flow through the device shown, velocities may be considered uniform over the inlet and outlet sections. The following conditions are known: $A_1 = 0.1\ \text{m}^2$, $A_2 = 0.2\ \text{m}^2$, $A_3 = 0.15\ \text{m}^2$, $V_1 = 10e^{-t/2}$ m/s, and $V_2 = 2\cos(2\pi t)$ m/s (t in seconds). Obtain an expression for the velocity at section ③, and plot V_3 as a function of time. At what instant does V_3 first become zero? What is the total mean volumetric flow at section ③?

Given: Data on flow through device

Find: Velocity V_3; plot V_3 against time; find when V_3 is zero; total mean flow

Solution:

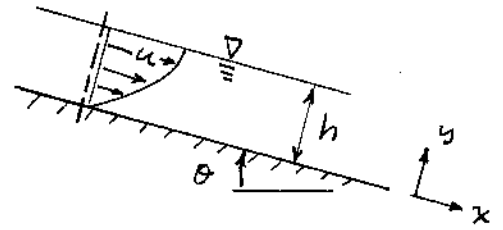
The velocity at A_3 is $$V_3 = 6.67 \cdot e^{-\frac{t}{2}} + 2.67 \cdot \cos(2 \cdot \pi \cdot t)$$

t (s)	V_3 (m/s)
0.00	9.33
0.10	8.50
0.20	6.86
0.30	4.91
0.40	3.30
0.50	2.53
0.60	2.78
0.70	3.87
0.80	5.29
0.90	6.41
1.00	6.71
1.10	6.00
1.20	4.48
1.30	2.66
1.40	1.15
1.50	0.48
1.60	0.84
1.70	2.03
1.80	3.53
1.90	4.74
2.00	5.12
2.10	4.49
2.20	3.04
2.30	1.29
2.40	-0.15
2.50	-0.76

The time at which V_3 first becomes zero can be found using *Goal Seek*

t (s)	V_3 (m/s)
2.39	0.00

Problem 4.29

Given: Oil flow down inclined plane.

$$u = \frac{\rho g \sin\theta}{\mu}\left(hy - \frac{y^2}{2}\right)$$

Find: Mass flow rate per unit width.

Solution: At the dashed cross-section, $\dot{m} = \int \rho u \, dA$

$dA = w\,dy$, where w = width

$$\dot{m} = \int_0^h \rho \frac{\rho g \sin\theta}{\mu}\left(hy - \frac{y^2}{2}\right) w\,dy = \frac{\rho^2 g \sin\theta}{\mu}\int_0^h \left(hy - \frac{y^2}{2}\right) w\,dy$$

$$\dot{m} = \frac{\rho^2 g \sin\theta}{\mu} w \left[\frac{hy^2}{2} - \frac{y^3}{6}\right]_0^h = \frac{\rho^2 g \sin\theta\, w}{\mu}\,\frac{h^3}{3} = \frac{\rho^2 g \sin\theta\, w h^3}{3\mu}$$

Thus

$$\dot{m}/w = \frac{\rho^2 g \sin\theta\, h^3}{3\mu}$$

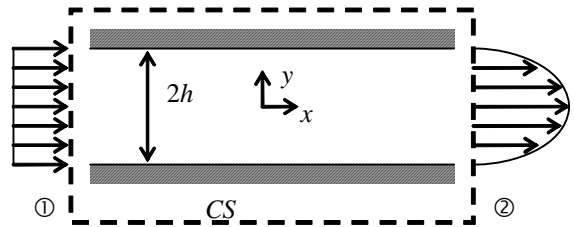
$\dot{m}/w$

Problem 4.30

[2]

4.30 Water enters a wide, flat channel of height $2h$ with a uniform velocity of 2.5 m/s. At the channel outlet the velocity distribution is given by

$$\frac{u}{u_{max}} = 1 - \left(\frac{y}{h}\right)^2$$

where y is measured from the centerline of the channel. Determine the exit centerline velocity, u_{max}.

Given: Data on flow at inlet and outlet of channel

Find: Find u_{max}

Solution:

Basic equation $\int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$

Assumptions: 1) Steady flow 2) Incompressible flow

Evaluating at 1 and 2

$$-\rho \cdot U \cdot 2 \cdot h \cdot w + \int_{-h}^{h} \rho \cdot u(y)\, dy = 0$$

$$\int_{-h}^{h} u_{max} \cdot \left[1 - \left(\frac{y}{h}\right)^2\right] dy = 2 \cdot h \cdot U$$

$$u_{max} \cdot \left[[h - (-h)] - \left[\frac{h^3}{3 \cdot h^2} - \left(-\frac{h^3}{3 \cdot h^2}\right)\right]\right] = 2 \cdot h \cdot U$$

$$u_{max} \cdot \frac{4}{3} \cdot h = 2 \cdot h \cdot U$$

Hence

$$u_{max} = \frac{3}{2} \cdot U = \frac{3}{2} \times 2.5 \cdot \frac{m}{s}$$

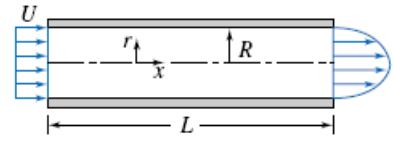
$$u_{max} = 3.75 \cdot \frac{m}{s}$$

Problem 4.31

[2]

4.31 Water flows steadily through a pipe of length L and radius R = 75 mm. Calculate the uniform inlet velocity, U, if the velocity distribution across the outlet is given by

$$u = u_{max}\left[1 - \frac{r^2}{R^2}\right]$$

and u_{max} = 3 m/s.

Given: Data on flow at inlet and outlet of pipe

Find: Find U

Solution:

Basic equation $\int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$

Assumptions: 1) Steady flow 2) Incompressible flow

Evaluating at inlet and exit $-\rho \cdot U \cdot \pi \cdot R^2 + \int_0^R \rho \cdot u(r) \cdot 2 \cdot \pi \cdot r\, dr = 0$

$$\int_0^R u_{max} \cdot \left[1 - \left(\frac{r}{R}\right)^2\right] \cdot 2 \cdot r\, dr = R^2 \cdot U$$

$$u_{max} \cdot \left(R^2 - \frac{1}{2} \cdot R^2\right) = R^2 \cdot U$$

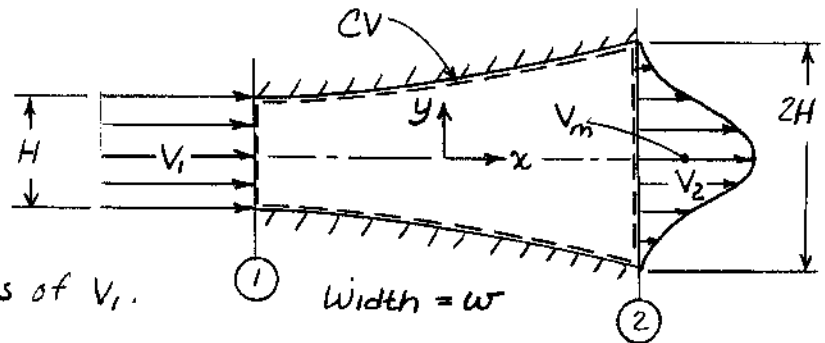
$$U = \frac{1}{2} \cdot u_{max}$$

Hence $U = \frac{1}{2} \times 3 \cdot \frac{m}{s}$ $\quad U = 1.5 \cdot \frac{m}{s}$

Problem 4.32

Given: Incompressible flow in a diverging channel, as shown.

V_1 = constant

$V_2 = V_m \cos\left(\frac{\pi y}{2H}\right)$

Find: Express V_m in terms of V_1.

Solution: Apply conservation of mass using the CV shown.

Basic equation: $$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, dV + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$ (first term $= 0$ (1))

Assumptions: (1) Steady flow
(2) Uniform flow at section 1
(3) Incompressible flow

Then $$0 = \left\{-|\rho V_1 A_1|\right\} + \int_{-H}^{H} \rho V_2 w \, dy$$

Since $A_1 = wH$, then $$V_1 w H = \int_{-H}^{H} V_m \cos\left(\frac{\pi}{2}\frac{y}{H}\right) w \, dy = 2\int_0^H V_m \cos\left(\frac{\pi}{2}\frac{y}{H}\right) w \, dy$$

So $$V_1 H = 2V_m\left(\frac{2H}{\pi}\right)\int_0^H \cos\left(\frac{\pi}{2}\frac{y}{H}\right) d\left(\frac{\pi}{2}\frac{y}{H}\right) = \frac{4V_m H}{\pi}\left[\sin\left(\frac{\pi}{2}\frac{y}{H}\right)\right]_0^H = \frac{4V_m H}{\pi}$$

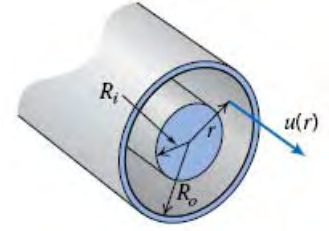
Thus $$V_m = \frac{\pi}{4} V_1$$

V_m

Problem 4.33

4.33 The velocity profile for laminar flow in an annulus is given by

$$u(r) = -\frac{\Delta p}{4\mu L}\left[R_o^2 - r^2 + \frac{R_o^2 - R_i^2}{\ln(R_i/R_o)}\ln\frac{R_o}{r}\right]$$

where $\Delta p/L = -10$ kPa/m is the pressure gradient, μ is the viscosity (SAE 10 oil at 20°C), and $R_o = 5$ mm and $R_i = 1$ mm are the outer and inner radii. Find the volume flow rate, the average velocity, and the maximum velocity. Plot the velocity distribution.

Given: Velocity distribution in annulus

Find: Volume flow rate; average velocity; maximum velocity; plot velocity distribution

Solution:

Governing equation: For the flow rate (Eq. 4.14a) and average velocity (Eq. 4.14b)

$$Q = \int \vec{V}\,d\vec{A} \qquad V_{av} = \frac{Q}{A}$$

The given data is $R_o = 5\cdot mm$ $\quad R_i = 1\cdot mm$ $\quad \frac{\Delta p}{L} = -10\cdot\frac{kPa}{m}$ $\quad \mu = 0.1\cdot\frac{N\cdot s}{m^2}$ (From Fig. A.2)

$$u(r) = \frac{-\Delta p}{4\cdot\mu\cdot L}\cdot\left(R_o^2 - r^2 + \frac{R_o^2 - R_i^2}{\ln\left(\frac{R_i}{R_o}\right)}\cdot\ln\left(\frac{R_o}{r}\right)\right)$$

The flow rate is

$$Q = \int_{R_i}^{R_o} u(r)\cdot 2\cdot\pi\cdot r\,dr$$

Considerable mathematical manipulation leads to

$$Q = \frac{\Delta p\cdot\pi}{8\cdot\mu\cdot L}\cdot\left(R_o^2 - R_i^2\right)\cdot\left[\frac{\left(R_o^2 - R_i^2\right)}{\ln\left(\frac{R_o}{R_i}\right)} - \left(R_i^2 + R_o^2\right)\right]$$

Substituting values

$$Q = \frac{\pi}{8}\cdot\left(-10\cdot 10^3\right)\cdot\frac{N}{m^2\cdot m}\cdot\frac{m^2}{0.1\cdot N\cdot s}\cdot\left(5^2 - 1^2\right)\cdot\left(\frac{m}{1000}\right)^2\cdot\left[\frac{5^2 - 1^2}{\ln\left(\frac{5}{1}\right)} - \left(5^2 + 1^2\right)\right]\cdot\left(\frac{m}{1000}\right)^2$$

$$Q = 1.045\times 10^{-5}\,\frac{m^3}{s} \qquad Q = 10.45\cdot\frac{mL}{s}$$

The average velocity is

$$V_{av} = \frac{Q}{A} = \frac{Q}{\pi\cdot\left(R_o^2 - R_i^2\right)} \qquad V_{av} = \frac{1}{\pi}\times 1.045\times 10^{-5}\cdot\frac{m^3}{s}\times\frac{1}{5^2 - 1^2}\cdot\left(\frac{1000}{m}\right)^2 \qquad V_{av} = 0.139\,\frac{m}{s}$$

The maximum velocity occurs when

$$\frac{du}{dr} = 0 = \frac{d}{dx}\left[\frac{-\Delta p}{4\cdot\mu\cdot L}\cdot\left(R_o^2 - r^2 + \frac{R_o^2 - R_i^2}{\ln\left(\frac{R_i}{R_o}\right)}\cdot\ln\left(\frac{R_o}{r}\right)\right)\right] = -\frac{\Delta p}{4\cdot\mu\cdot L}\cdot\left[-2\cdot r - \frac{\left(R_o^2 - R_i^2\right)}{\ln\left(\frac{R_i}{R_o}\right)\cdot r}\right]$$

$$r = \sqrt{\frac{R_i^2 - R_o^2}{2\cdot\ln\left(\frac{R_i}{R_o}\right)}} \qquad r = 2.73\cdot mm$$

Substituting in u(r)

$$u_{max} = u(2.73\cdot mm) = 0.213\cdot\frac{m}{s}$$

The maximum velocity using Solver instead, and the plot, are also shown in the corresponding *Excel* workbook

Problem 4.33

4.33 The velocity profile for laminar flow in an annulus is given by

$$u(r) = -\frac{\Delta p}{4\mu L}\left[R_o^2 - r^2 + \frac{R_o^2 - R_i^2}{\ln(R_i/R_o)}\ln\frac{R_o}{r}\right]$$

where $\Delta p/L = -10$ kPa/m is the pressure gradient, μ is the viscosity (SAE 10 oil at 20°C), and $R_o = 5$ mm and $R_i = 1$ mm are the outer and inner radii. Find the volume flow rate, the average velocity, and the maximum velocity. Plot the velocity distribution.

Given: Velocity distribution in annulus

Find: Volume flow rate; average velocity; maximum velocity; plot velocity distribution

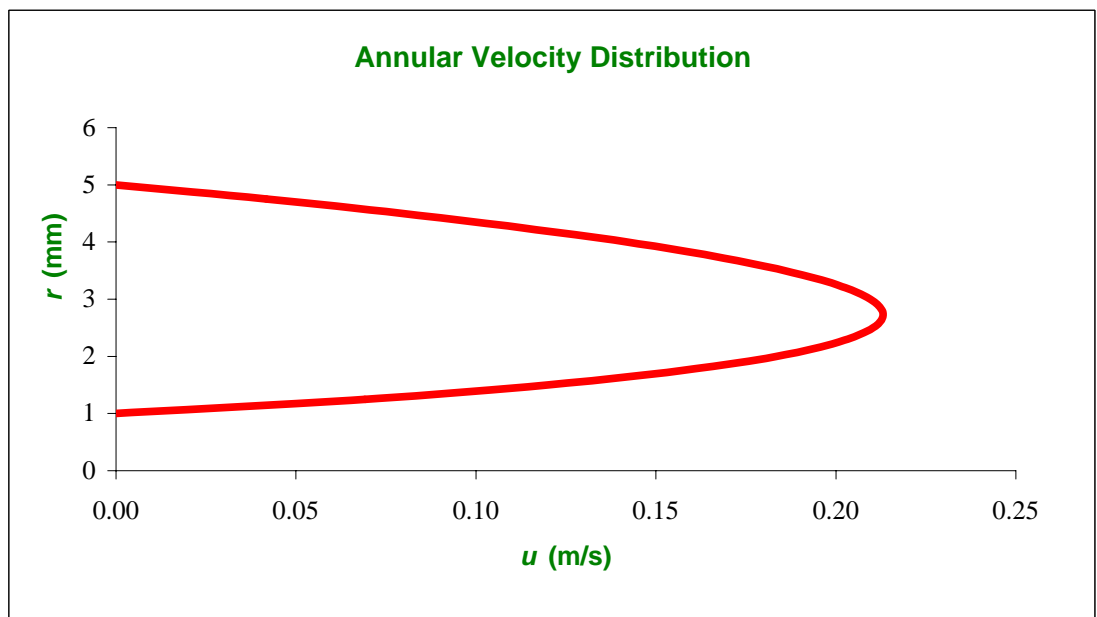
Solution:

R_o = 5 mm

R_i = 1 mm

$\Delta p/L$ = -10 kPa/m

μ = 0.1 N.s/m^2

r (mm)	*u* (m/s)
1.00	0.000
1.25	0.069
1.50	0.120
1.75	0.157
2.00	0.183
2.25	0.201
2.50	0.210
2.75	0.213
3.00	0.210
3.25	0.200
3.50	0.186
3.75	0.166
4.00	0.142
4.25	0.113
4.50	0.079
4.75	0.042
5.00	0.000

The maximum velocity can be found using *Solver*

r (mm)	*u* (m/s)
2.73	0.213

Problem 4.34

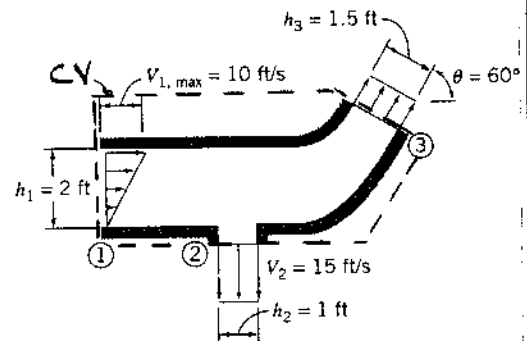
Given: Two-dimensional reducing bend as shown.

Find: Magnitude and direction of uniform velocity at section ③.

Solution: Apply conservation of mass using CV shown.

Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, dV + \int_{CS} \rho \vec{V} \cdot d\vec{A} \quad \left(\frac{\partial}{\partial t} \text{ term} = 0(1)\right)$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at ② and ③

Then

$$0 = \int_{CS} \vec{V} \cdot d\vec{A} = \int_{A_1} \vec{V}_1 \cdot d\vec{A}_1 + \vec{V}_2 \cdot \vec{A}_2 + \vec{V}_3 \cdot \vec{A}_3$$

or

$$\vec{V}_3 \cdot \vec{A}_3 = -\int_{A_1} \vec{V}_1 \cdot d\vec{A}_1 - \vec{V}_2 \cdot \vec{A}_2 = +\int_0^{h_1} V_{1,max} \frac{y}{h_1} w \, dy - V_2 w h_2$$

$$\vec{V}_3 \cdot \vec{A}_3 = V_{1,max} w \left[\frac{y^2}{2h_1}\right]_0^{h_1} - V_2 w h_2 = \frac{V_{1,max} w h_1}{2} - V_2 w h_2$$

So

$$\frac{\vec{V}_3 \cdot \vec{A}_3}{w} = \frac{1}{2} \times 10 \frac{ft}{s} \times 2 ft - 15 \frac{ft}{s} \times 1 ft = -5 \, ft^2/s$$

Since $\vec{V}_3 \cdot \vec{A}_3 < 0$, flow at ③ is into the CV ← Direction

Thus

$$\frac{\vec{V}_3 \cdot \vec{A}_3}{w} = -\frac{V_3 A_3}{w} = -\frac{V_3 w h_3}{w} = -V_3 h_3 = -5 \, ft^2/s$$

$$V_3 = \frac{1}{h_3} \times 5 \frac{ft^2}{s} = \frac{1}{1.5 ft} \times 5 \frac{ft^2}{s} = 3.33 \, ft/s \quad \text{(into CV)}$$ ← V_3

Problem 4.35

Given: Water flow in the two-dimensional square channel shown.

$v_{max} = 2v_{min}$, $U = 7.5$ m/s, $h = 75.5$ mm

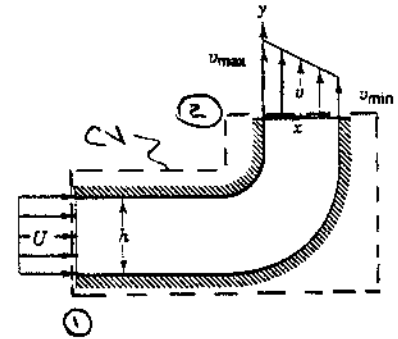
Find: v_{min}

Solution: Apply conservation of mass to the CV shown.

Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

(the time-derivative term is cancelled, marked = 0(1))

Assumptions: (1) steady flow
(2) incompressible flow
(3) uniform flow at section ①

Then

$$0 = \vec{V}_1 \cdot \vec{A}_1 + \int \vec{V}_2 \cdot dA_2$$

$$0 = -Uwh + \int_0^h v\, w\, dx$$

The velocity distribution across the exit at ② is linear

$$v_2 = v_{max} - (v_{max} - v_{min})\frac{x}{h} = 2v_{min} - v_{min}\frac{x}{h} = v_{min}\left(2 - \frac{x}{h}\right)$$

$$\therefore Uwh = \int_0^h v_{min}\left(2 - \frac{x}{h}\right) w\, dx = v_{min}\, w\left[2x - \frac{x^2}{2h}\right]_0^h$$

$$U\cancel{w}\cancel{h} = v_{min}\, w\left[2h - \frac{h}{2}\right] = \frac{3}{2} v_{min}\cancel{w}\cancel{h}$$

$$\therefore v_{min} = \frac{2}{3}U = \frac{2}{3} \times 7.5\,\frac{m}{s} = 5.0 \text{ m/s}$$ ← v_{min}

Problem 4.36

Given: Water flows in a porous round tube of diameter $D = 60\ mm$. At the pipe inlet the flow is uniform with $V_1 = 7.0\ m/sec$. Flow out through the porous wall is radial and axisymmetric with velocity distribution

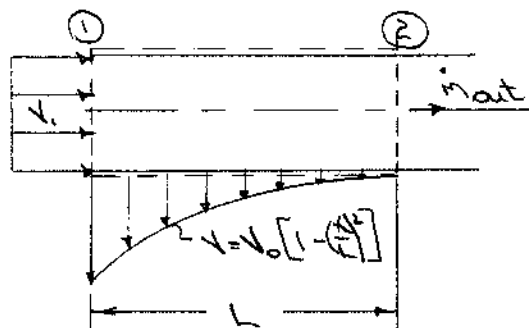
$$v = V_0\left[1 - \left(\frac{x}{L}\right)^2\right]$$

where $V_0 = 0.03\ m/s$ and $L = 0.950\ m$.

Find: the mass flow rate, $\dot{m}_2$, inside the tube at $x = L$

Solution:

Basic equation: $$0 = \cancel{\frac{\partial}{\partial t}\int_{CV} \rho\, dV}^{\,0(1)} + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) steady flow
(2) ρ = constant

Then

$$0 = \int_{A_1} \rho\vec{V}\cdot d\vec{A} + \int_{A_2} \rho\vec{V}\cdot d\vec{A} + \int_{A_{circ}} \rho\vec{V}\cdot d\vec{A}$$

$$= -|\rho V_1 A_1| + \dot{m}_2 + \int_0^L \rho\, V_0\left[1 - \left(\frac{x}{L}\right)^2\right] 2\pi R\, dx$$

$$\dot{m}_2 = \rho V_1 A_1 - 2\pi R \rho V_0 \int_0^L \left[1 - \frac{x^2}{L^2}\right] dx$$

$$= \rho V_1 \frac{\pi D_1^2}{4} - 2\pi R \rho V_0 \left[x - \frac{x^3}{3L^2}\right]_0^L$$

$$= \frac{\pi}{4}\rho V_1 D_1^2 - \frac{4}{3}\pi R \rho V_0 L$$

$$\dot{m}_2 = \frac{\pi}{4} \times 999\frac{kg}{m^3} \times 7.0\frac{m}{s} \times (0.06)^2 m^2 - \frac{4}{3}\pi \times 0.03 m \times 999 \frac{kg}{m^3} \times 0.03\frac{m}{s} \times 0.95 m$$

$$\dot{m}_2 = 19.8\ \frac{kg}{s} - 3.6\ \frac{kg}{s} = 16.2\ kg/s$$ ← $\dot{m}_{out}$

Problem 4.37

Given: A hydraulic accumulator, designed to reduce pressure pulsations in a hydraulic system, is operating under conditions shown, at a given instant.

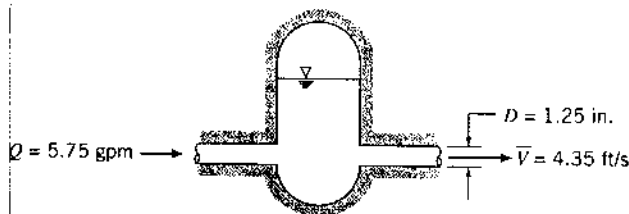
Find: Rate at which accumulator gains or loses hydraulic oil.

Solution:

Use the control volume shown

Q = 5.75 gpm → ; D = 1.25 in. ; $\bar{V}$ = 4.35 ft/s

Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) uniform flow at section ②
(2) ρ = constant

Then,

$$0 = \frac{\partial}{\partial t}(M_{cv}) + \int_{A_1} \{-|\rho V_1 dA_1|\} + \int_{A_2} \{|\rho V_2 dA_2|\}$$

But $\int_{A_1} \rho V_1 dA_1 = \rho Q_1$ where Q = volume flowrate and $\rho = SG\, \rho_{H_2O}$

So $0 = \frac{\partial}{\partial t} M_{cv} - \rho Q_1 + \rho V_2 A_2$

or

$$\frac{\partial M_{cv}}{\partial t} = \rho (Q_1 - V_2 A_2)$$

$$= SG\, \rho_{H_2O} \left(Q_1 - V_2 \pi \frac{D_2^2}{4}\right) \quad \text{where } S.G = 0.88 \text{ (Table A.2)}$$

$$= 0.88 \times 1.94 \frac{slug}{ft^3}\left[5.75 \frac{gal}{min} \times \frac{ft^3}{7.48 gal} \times \frac{min}{60 s} - 4.35 \frac{ft}{s} \times \frac{\pi}{4} \times (1.25)^2 in^2 \times \frac{ft^2}{144 in^2}\right]$$

$$\frac{\partial M_{cv}}{\partial t} = -4.14 \times 10^{-2} \frac{slug}{s} \quad \text{or} \quad -1.33 \, lbm/s$$ ← $\partial M_{cv}/\partial t$

(mass is decreasing in the CV)

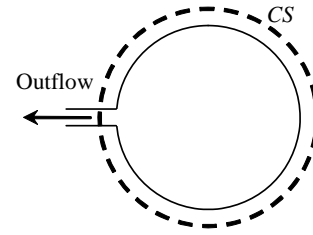
Since $M_{cv} = \rho_{oil} \forall_{oil}$

$$\frac{\partial M_{cv}}{\partial t} = \frac{\partial}{\partial t}(\rho_{oil} \forall_{oil}) = \rho_{oil} \frac{\partial \forall_{oil}}{\partial t} = SG_{oil}\, \rho_{H_2O} \frac{\partial \forall_{oil}}{\partial t}$$

$$\frac{\partial \forall_{oil}}{\partial t} = \frac{1}{SG_{oil}\, \rho_{H_2O}} \frac{\partial M_{cv}}{\partial t} = \frac{1}{0.88} \times \frac{ft^3}{1.94 \, slugs} \times (-4.14) \times 10^{-2} \frac{slug}{s}$$

$$\frac{\partial \forall_{oil}}{\partial t} = -2.43 \times 10^{-2} \frac{ft^3}{s} \quad \text{or} \quad 0.181 \, gal/s$$ ← $\frac{\partial \forall_{oil}}{\partial t}$

Problem 4.38

4.38 A tank of 0.4 m^3 volume contains compressed air. A valve is opened and air escapes with a velocity of 250 m/s through an opening of 100 mm^2 area. Air temperature passing through the opening is −20°C and the absolute pressure is 300 kPa. Find the rate of change of density of the air in the tank at this moment.

Given: Data on airflow out of tank

Find: Find rate of change of density of air in tank

Solution:

Basic equation
$$\frac{\partial}{\partial t}\int_{CV}\rho\, dV + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0$$

Assumptions: 1) Density in tank is uniform 2) Uniform flow 3) Air is an ideal gas

Hence
$$V_{tank}\cdot\frac{d\rho_{tank}}{dt} + \rho_{exit}\cdot V\cdot A = 0 \qquad \frac{d\rho_{tank}}{dt} = -\frac{\rho_{exit}\cdot V\cdot A}{V_{tank}} = -\frac{p_{exit}\cdot V\cdot A}{R_{air}\cdot T_{exit}\cdot V_{tank}}$$

$$\frac{d\rho_{tank}}{dt} = -300\times 10^3\cdot\frac{N}{m^2}\times 250\cdot\frac{m}{s}\times 100\cdot mm^2\times\left(\frac{1\cdot m}{1000\cdot mm}\right)^2\times\frac{1}{286.9}\cdot\frac{kg\cdot K}{N\cdot m}\times\frac{1}{(-20+273)\cdot K}\times\frac{1}{0.4\cdot m^3}$$

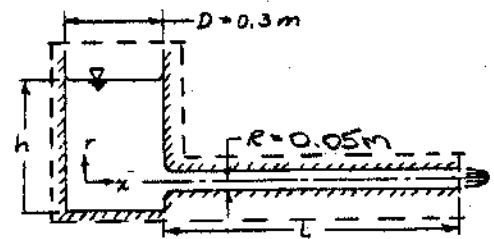
Hence
$$\frac{d\rho_{tank}}{dt} = -0.258\cdot\frac{\frac{kg}{m^3}}{s}$$

The mass in the tank is decreasing, as expected

Problem 4.39

Given: Liquid drains from a tank through a long circular tube. Flow is laminar; velocity profile at tube discharge is given by

$$u = u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

Find: (a) Show that $\bar{V} = 0.5\, u_{max}$ at any instant
(b) rate of change of liquid level in tank when $u_{max} = 0.155$ m/s

Solution:

(a) The average velocity $\bar{V}$ is defined as Q/A.

Since $Q = \int u\,dA$, $dA = 2\pi r\,dr$ and $A = \pi R^2$, then

$$\bar{V} = \frac{Q}{A} = \frac{1}{\pi R^2}\int_0^R u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right] 2\pi r\,dr = \frac{2u_{max}}{R^2}\int_0^R \left[1 - \left(\frac{r}{R}\right)^2\right] r\,dr$$

$$\bar{V} = \frac{2u_{max}}{R^2} R^2 \int_0^1 \left[1 - \left(\frac{r}{R}\right)^2\right]\left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) = 2u_{max}\left[\frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4\right]_0^1$$

$$\bar{V} = \frac{1}{2} u_{max}$$ $\bar{V}$

(b) Apply conservation of mass to the CV shown

Basic equation: $$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) neglect air entering the CV
(2) incompressible flow

Then

$$0 = \rho_L \frac{\partial}{\partial t}\forall_{CV} + \left\{\left|\rho_L \bar{V} A_e\right|\right\} = \rho_L \frac{\partial}{\partial t}\left[\frac{\pi D^2}{4} h + L\pi R^2\right] + \rho \bar{V}\pi R^2$$

$$0 = \frac{\pi D^2}{4}\frac{dh}{dt} + \bar{V}\pi R^2 \qquad \left(\text{note } \frac{dL}{dt} = 0\right)$$

$$\therefore\ \frac{dh}{dt} = -4\bar{V}\left(\frac{R}{D}\right)^2 \qquad \text{But } \bar{V} = \frac{1}{2}u_{max} \text{ and hence}$$

$$\frac{dh}{dt} = -2u_{max}\left(\frac{R}{D}\right)^2 = -2\times 0.155\frac{m}{s}\times\left(\frac{0.05m}{0.30m}\right)^2 \times 1000\frac{mm}{m}$$

$$\frac{dh}{dt} = -8.61 \text{ mm/s} \quad \text{(level is falling)}$$ $\frac{dh}{dt}$

Problem 4.40

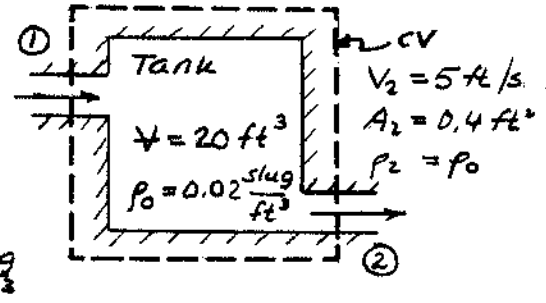
Given: Air flow through tank with conditions shown at time, t_o.

$V_1 = 15$ ft/s
$A_1 = 0.2$ ft²
$\rho_1 = 0.03 \frac{slug}{ft^3}$

Tank
$\forall = 20$ ft³
$\rho_o = 0.02 \frac{slug}{ft^3}$

$V_2 = 5$ ft/s
$A_2 = 0.4$ ft²
$\rho_2 = \rho_o$

CV

① ②

Find: $\frac{\partial \rho}{\partial t}$ in tank at time, t_o.

Solution: Apply conservation of mass, using CV shown.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) Density is uniform in tank, so $\frac{\partial}{\partial t}\int_{CV} \rho d\forall = \frac{\partial}{\partial t}(\rho_o \forall)$

(2) Flow is uniform at inlet and outlet sections.

Then

$$0 = \frac{\partial}{\partial t}(\rho_o \forall) + \rho_1 \vec{V}_1 \cdot \vec{A}_1 + \rho_o \vec{V}_2 \cdot \vec{A}_2$$

$$0 = \rho_o \overset{=0}{\frac{\partial \forall}{\partial t}} + \forall \frac{\partial \rho_o}{\partial t} - |\rho_1 V_1 A_1| + |\rho_o V_2 A_2|$$

or

$$\frac{\partial \rho_o}{\partial t} = \frac{|\rho_1 V_1 A_1| - |\rho_o V_2 A_2|}{\forall}$$

Substituting magnitudes

$$\frac{\partial \rho_o}{\partial t} = \frac{1}{20\,ft^3}\left[\frac{0.03\,slug}{ft^3} \times \frac{15\,ft}{s} \times 0.2\,ft^2 - \frac{0.02\,slug}{ft^3} \times \frac{5\,ft}{s} \times 0.4\,ft^2\right]$$

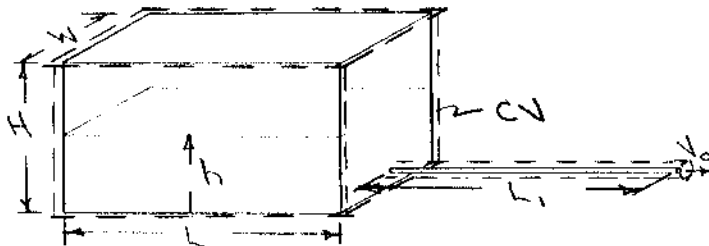
$$\frac{\partial \rho_o}{\partial t} = 2.50 \times 10^{-3}\ slug/(ft^3 \cdot s)$$

$\frac{\partial \rho_o}{\partial t}$

{Note since $\frac{\partial \rho_o}{\partial t} > 0$, mass in tank increases.}

Problem 4.41

Given: Rectangular tank with dimensions H = 230 mm, W = 150 mm, L = 230 mm, supplies water to an outlet tube of diameter, D = 6.35 mm. When the tank is half full the flow in the tube is at Reynolds number Re = 2000. At this instant there is no water flow into the tank.

Find: the rate of change of water level in the tank at this instant.

Solution:

Apply conservation of mass to CV which includes tank and tube.

Basic equation: $$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

Definition: $$Re = \frac{\rho D \bar{V}}{\mu} = \frac{D\bar{V}}{\nu}$$

Assumptions: (1) uniform flow at exit of tube
(2) incompressible flow
(3) neglect air entering the control volume

Then,
$$0 = \frac{\partial}{\partial t}\left[\rho W L h + \rho \frac{\pi D^2}{4} L_1\right] + \left\{+\left|\rho \bar{V}_0 \frac{\pi D^2}{4}\right|\right\}$$

$$0 = WL\frac{dh}{dt} + \bar{V}_0 \frac{\pi D^2}{4} \qquad (\text{note } L_1 = \text{constant})$$

$$\therefore \quad \frac{dh}{dt} = -\bar{V}_0 \frac{\pi D^2}{4WL}$$

To find $\bar{V}$ use the definition of Re

$$\bar{V}_0 = \frac{Re\,\nu}{D}$$

For water at 20C $\nu = 1\times10^{-6}\ m^2/sec$ (Table A.8)

$$V_0 = 2000 \times 1\times10^{-6}\frac{m^2}{sec}\times\frac{1}{6.35\times10^{-3}\ m} = 0.315\ m/sec$$

$$\frac{dh}{dt} = -\bar{V}_0\frac{\pi D^2}{4WL} = -\frac{0.315}{4}\frac{m}{sec}\times\frac{\pi(6.35)^2\ mm^2}{150\ mm\times 230\ mm}\times 10^3\ \frac{mm}{m}$$

$$\frac{dh}{dt} = -0.289\ mm/sec \quad (\text{falling})$$

$\frac{dh}{dt}$

Problem 4.42

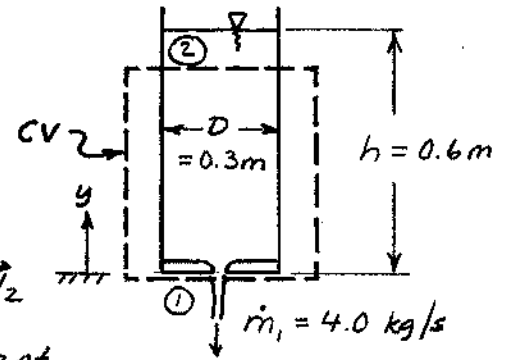
[2]

Given: Circular tank, with $D = 1$ ft draining through a hole in its bottom. Fluid is water

Find: Rate of change of water level at the instant shown.

Solution: Apply conservation of mass to CV shown. Note section ② cuts below free surface, so $\vec{V}_2$ corresponds to free surface velocity; volume of CV is constant.

Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} \qquad \left(\frac{\partial}{\partial t}\text{ term } = 0\ (1)\right)$$

Assumptions: (1) Incompressible flow, so unsteady term is zero, since volume of CV is fixed
(2) Uniform flow at each section

Then

$$0 = \rho \vec{V}_1 \cdot \vec{A}_1 + \rho \vec{V}_2 \cdot \vec{A}_2 = \dot{m}_1 + \rho \vec{V}_2 \cdot \vec{A}_2$$

and

$$\vec{V}_2 \cdot \vec{A}_2 = -\frac{\dot{m}_1}{\rho} = -4.0 \frac{kg}{s} \times \frac{m^3}{999\ kg} = -0.004\ m^3/s$$

Since $\vec{V}_2 \cdot \vec{A}_2 < 0$, flow at section ② is into CV. Therefore

$$V_2 = \frac{|V_2 A_2|}{A_2} = 0.004 \frac{m^3}{s} \times \frac{4}{\pi} \times \frac{1}{(0.3)^2 m^2} = 0.0566\ m/s$$

The water level is falling at 56.6 mm/s.

$$\vec{V}_s = -V_2 \hat{j} = -56.6 \hat{j}\ mm/s$$

$\vec{V}_s$

Problem 4.43

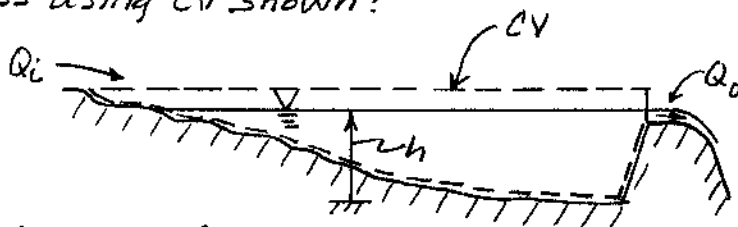
Given: Lake being drained at 2,000 cubic feet per second (cfs). Level falls at 1 ft per 8 hr. Normal flow rate is 290 cfs.

Find: (a) Actual flow rate during draining (gal/s).
(b) Estimate surface area of lake.

Solution: Convert units

$$Q = 2000\,\frac{ft^3}{s} = 2000\,\frac{ft^3}{s} \times 7.48\,\frac{gal}{ft^3} = 1.50 \times 10^4\ gal/s$$

Q

Apply conservation of mass using CV shown:

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumption: (1) ρ = constant

Then $\frac{d\forall}{dt} = A\frac{dh}{dt} = -\int_{CS} \vec{V} \cdot d\vec{A} = -Q_0 + Q_i$

$$A = -\frac{Q_0 - Q_i}{dh/dt} = -\frac{\Delta Q}{dh/dt}\ ;\ \Delta Q = Q_0 - Q_i$$

But $\Delta Q = 1{,}710\ ft^3/s$ and $dh/dt = -1\ ft/8\ hr$, since decreasing.

Thus

$$A = -1{,}710\,\frac{ft^3}{s} \times \frac{8\,hr}{-1\,ft} \times 3600\,\frac{s}{hr} = 4.92 \times 10^7\ ft^2$$

A

Since 1 acre = 43,600 ft^2,

$$A = 4.92 \times 10^7\ ft^2 \times \frac{acre}{43{,}600\ ft^2} \approx 1{,}130\ acres$$

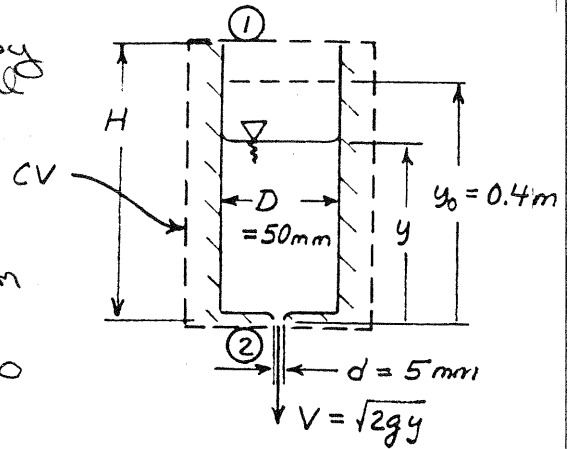
Since 1 square mile = 640 acres, the lake surface area is slightly less than 2 square miles!

Problem 4.44

Given: Cylindrical tank, draining by gravity as shown; initial depth is y_0

Find: Water depth at $t = 12$ s

Plot: (a) y/y_0 vs t for $0.1 \le y_0 \le 1$ m and $D/d = 10$

(b) y/y_0 vs t for $2 \le D/d \le 10$ and $y_0 = 0.4$ m

Solution:

Apply conservation of mass using CV shown

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) incompressible flow
(2) uniform flow at each section
(3) neglect ρ_{air} compared to ρ_{H_2O}

For the CV, $d\forall = A_t\, dy$, so

$$0 = \frac{\partial}{\partial t}\int_0^y \rho_{H_2O} A_t\, dy + \frac{\partial}{\partial t}\int_y^H \rho_{air} A_t\, dy + \{-|\rho_{air} V_1 A_t|\} + \{|\rho_{H_2O} V_2 A_2|\}$$

(the ρ_{air} terms ≈ 0 (3))

or

$$0 = \rho A_t \frac{dy}{dt} + \rho A_2 V_2 = A_t \frac{dy}{dt} + A_2\sqrt{2gy}$$

Separating variables, $\frac{dy}{y^{1/2}} = -\sqrt{2g}\,\frac{A_2}{A_t}\,dt$

Integrating from y_0 at $t=0$ to y at t

$$\int_{y_0}^{y} y^{-1/2}\, dy = 2\left[y^{1/2} - y_0^{1/2}\right] = -\sqrt{2g}\,\frac{A_2}{A_t}\,t$$

$$\frac{y^{1/2}}{y_0^{1/2}} = 1 - \sqrt{\frac{g}{2y_0}}\,\frac{A_2}{A_t}\,t \quad \text{or} \quad y = y_0\left[1 - \sqrt{\frac{g}{2y_0}}\left(\frac{d}{D}\right)^2 t\right]^2 \quad \text{--- (1)}$$

At $t = 12$ sec

$$y = 0.4\,\text{m}\left[1 - \left(\frac{9.81}{2}\frac{\text{m}}{\text{s}^2}\times\frac{1}{0.4\,\text{m}}\right)^{1/2}\left(\frac{5\,\text{mm}}{50\,\text{mm}}\right)^2 12\,\text{s}\right]^2 = 0.134\,\text{m} \quad \leftarrow y_{t=12s}$$

For $D/d = 10$, Eq. 1 gives

$$\frac{y}{y_0} = \left[1 - 2.215\times10^{-2}\, y_0^{-1/2}\, t\right]^2$$

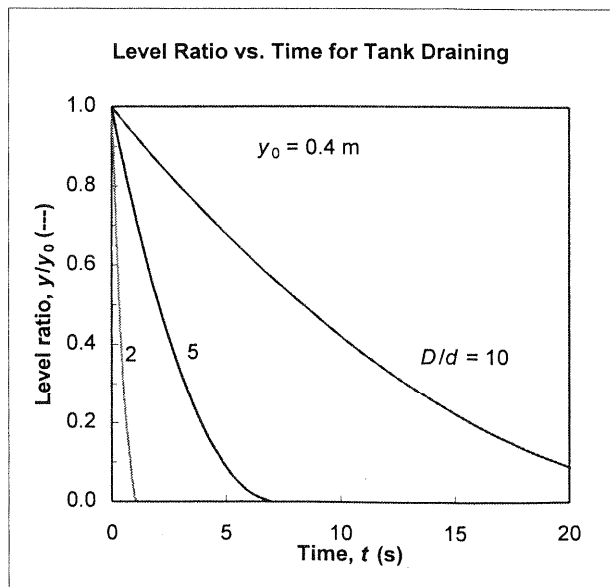
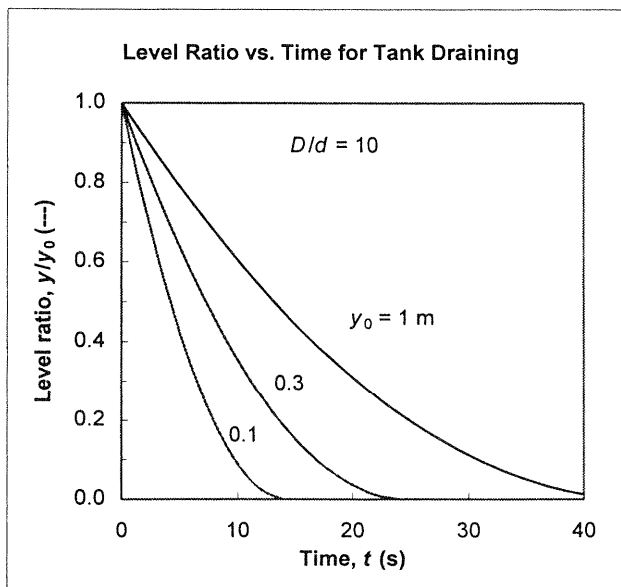
For $y_0 = 0.4\,m$, Eq. 1 gives

$$\frac{y}{y_0} = \left[1 - \frac{3.502}{(D/d)^2}\, t\right]^2$$

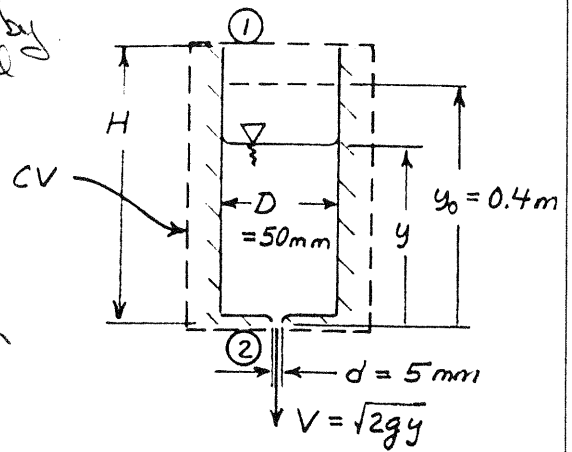
The variation of y/y_0 with t is plotted below for:

- $D/d = 10$ and $0.1 < y_0 \le 1.0\,m$
- $y_0 = 0.4\,m$ and $2 \le D/d \le 10$

y_0 (m) =	0.1	0.3	1	D/d (---) =	2	5	10
Time, t (s)	y/y_0 (---)	y/y_0 (---)	y/y_0 (---)	Time, t (s)	y/y_0 (---)	y/y_0 (---)	y/y_0 (---)
0	1.000	1.000	1.000	0	1.000	1.000	1.000
2	0.739	0.845	0.913	0.5	0.316	0.865	0.965
4	0.518	0.703	0.831	1	0.016	0.739	0.931
6	0.336	0.574	0.752	1.1	0.001	0.716	0.924
8	0.193	0.458	0.677	2		0.518	0.865
10	0.090	0.355	0.606	3		0.336	0.801
12	0.025	0.265	0.539	4		0.193	0.739
14	0.000	0.188	0.476	5		0.090	0.680
16		0.125	0.417	6		0.025	0.624
18		0.074	0.362	7		0.000	0.570
20		0.037	0.310	10			0.422
22		0.012	0.263	12			0.336
24		0.001	0.219	14			0.260
26			0.180	16			0.193
28			0.144	18			0.137
30			0.113	20			0.090
32			0.085	22			0.053
34			0.061	24			0.025
36			0.041	26			0.008
38			0.025	28			0.000
40			0.013				
45			0.000				

Problem 4.45

Given: Cylindrical tank, draining by gravity as shown; initial depth is y_0.

Find: Time to drain tank to depth $y = 20$ mm

Plot: Time t to drain the tank (to $y = 20$ mm) as a function of y/y_0 for $0.1 \le y_0 \le 1$ m with d/D as a parameter for $0.1 \le d/D \le 0.5$.

Solution:

Apply conservation of mass using CV shown.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) incompressible flow
(2) uniform flow at each section.
(3) neglect ρ_{air} compared to ρ_{H_2O}

For the CV, $d\forall = A_t \, dy$, so

$$0 = \frac{\partial}{\partial t}\int_0^y \rho_{H_2O} A_t \, dy + \frac{\partial}{\partial t}\int_y^H \rho_{air} A_t \, dy + \{-|\rho_{air} V_1 A_t|\} + \{|\rho_{H_2O} V_2 A_2|\}$$

(the ρ_{air} terms ≈ 0 (3))

or $$0 = \frac{\partial}{\partial t}\int_0^y \rho_{H_2O} A_t \, dy + \rho_{H_2O} V_2 A_2 = A_t \frac{dy}{dt} + A_2\sqrt{2gy}$$

Separating variables, $$\frac{dy}{y^{1/2}} = -\sqrt{2g}\,\frac{A_2}{A_t}\,dt$$

Integrating from y_0 at $t=0$ to y at t

$$\int_{y_0}^{y} \frac{dy}{y^{1/2}} = 2\left[y^{1/2} - y_0^{1/2}\right] = -\sqrt{2g}\,\frac{A_2}{A_t}\,t$$

$$-\sqrt{2g}\,\frac{A_2}{A_t}\,t = 2y_0^{1/2}\left[\left(\frac{y}{y_0}\right)^{1/2} - 1\right] \quad \text{or} \quad t = \sqrt{\frac{2y_0}{g}}\left(\frac{D}{d}\right)^2\left[1 - \left(\frac{y}{y_0}\right)^{1/2}\right] \quad (1)$$

Evaluating at $y = 20$ mm

$$t = \left[2 \times 0.4\,\text{m} \times \frac{s^2}{9.81\,\text{m}}\right]^{1/2}\left[\frac{50\,\text{mm}}{5\,\text{mm}}\right]^2\left[1 - \left(\frac{0.02\,\text{m}}{0.40\,\text{m}}\right)^{1/2}\right] = 22.2\,\text{s}$$

$t_{y=20\,mm}$

Time t is plotted as a function of y/y_0 ($y = 20$ mm) with d/D as a parameter.

Problem 4.45

Draining of a cylindrical liquid tank:

Input Data:

Initial height:	y_0	0.4	m		
Diameter ratio:	D/d	20	10	5	---

Calculated Results:

Level, *y* (mm)	*D/d* = 20	10	5
	Time, *t* (s)		
400	0	0	0
380	2.89	0.723	0.181
360	5.86	1.47	0.366
340	8.91	2.23	0.557
320	12.1	3.01	0.754
300	15.3	3.83	0.96
280	18.7	4.66	1.17
260	22.1	5.53	1.38
240	25.7	6.44	1.61
220	29.5	7.38	1.84
200	33.5	8.36	2.09
180	37.6	9.40	2.35
160	42.0	10.5	2.62
140	46.6	11.7	2.92
120	51.7	12.9	3.23
100	57.1	14.3	3.57
80	63.1	15.8	3.95
60	70.0	17.5	4.37
40	78.1	19.5	4.88
30	82.9	20.7	5.18
20	88.7	22.2	5.54
10	96.2	24.0	6.01
0	114	28.6	7.14

Problem 4.46

Given: Water flows into the top of a conical flask at a constant rate of $Q = 3.75 \times 10^{-7}\ m^3/hr$. Water drains out through the round opening of diameter $d = 7.35$ mm at the apex of the cone; the flow speed at the exit is $V = (2gy)^{1/2}$ where y is the water depth above the exit plane. At the instant of interest, the water depth $H = 36.8$ mm and the corresponding diameter $D = 29.4$ mm

Find: At the instant of interest:
(a) find the volume flow rate from the bottom of the flask
(b) evaluate the direction and rate of change of water surface level

Solution: Apply continuity to the CV shown.

D = 2R, Q_i, H, y, θ, d, V_o

Basic eq.: $0 = \frac{\partial}{\partial t}\int_{cv} \rho\, d\forall + \int_{cs} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) uniform flow at each section
(2) neglect mass of air.

Then

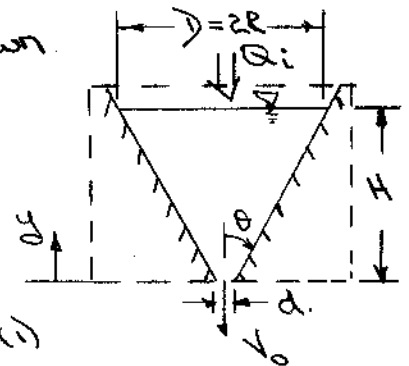
$$0 = \rho \left.\frac{d\forall}{dt}\right)_{water} + \rho Q_{out} - \rho Q_{in} \quad \text{.....(1)}$$

$$Q_{out} = V_o A_o = (2gH)^{1/2} \frac{\pi d^2}{4}$$

$$Q_{out} = \left[2 \times 9.81 \frac{m}{s^2} \times 0.0368\, m\right]^{1/2} \frac{\pi}{4} \times (0.00735)^2\, m^2$$

$$Q_{out} = 3.61 \times 10^{-5}\ m^3/s \quad (0.130\ m^3/hr) \longleftarrow \quad Q_{out}$$

From eq.(1)

$$\left.\frac{d\forall}{dt}\right)_{water} = Q_{in} - Q_{out}$$

$$\forall = \tfrac{1}{3}\ \text{area of base} \times \text{altitude} = \tfrac{1}{3}\pi R^2 y$$

Since $R = y \tan\theta$, $\forall = \frac{1}{3}\pi y^3 \tan^2\theta$

$$\frac{d\forall}{dt} = \frac{1}{3}\pi \tan^2\theta \times 3y^2 \frac{dy}{dt} = \pi y^2 \tan^2\theta \frac{dy}{dt} = \pi R^2 \frac{dy}{dt}$$

$$\therefore \frac{dy}{dt} = \frac{Q_{in} - Q_{out}}{\pi R^2} = \frac{4}{\pi D^2}(Q_{in} - Q_{out})$$

$$= \frac{4}{\pi} \times \frac{1}{(0.0294)^2\, m^2}\left(3.75 \times 10^{-7} - 0.130\right)\frac{m^3}{hr} \times \frac{hr}{3600\, s}$$

$$\frac{dy}{dt} = -0.0532\ m/s \quad \text{(surface moves downward)} \longleftarrow \quad \frac{dy}{dt}$$

Problem 4.47

Given: Conical funnel draining through small hole.

$$V_e = \sqrt{2gy}$$

Find: Rate of change of surface level when $y = H/2$.

Solution: Apply conservation of mass.

(1) Choose CV with top just below surface level.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$ (first term $= 0(1)$)

Assumptions: (1) ρ = constant, $\forall$ = const, so $\partial/\partial t = 0$
(2) Uniform flow at each section.

For CV(1): $0 = \{-|\rho V_s A_s|\} + \{+|\rho V_e A_e|\}$ or $V_s = V_e \frac{A_e}{A_s}$ (in, out)

Thus

$$V_s = V_e\left(\frac{d}{D/2}\right)^2 = \sqrt{2g\frac{H}{2}}\,4\left(\frac{d}{D}\right)^2 = 4\sqrt{gH}\left(\frac{d}{D}\right)^2 = -\frac{dy}{dt} \quad \text{(since y decreases)}$$

But $\tan\theta = \frac{D/2}{H}$ so $H = \frac{D}{2\tan\theta} = \frac{0.070\,m}{2\tan 15°} = 0.131\ m$

Substituting,

$$\frac{dy}{dt} = -4\sqrt{9.81\frac{m}{s^2}\times 0.131\,m}\left(\frac{0.00312\,m}{0.070\,m}\right)^2 1000\frac{mm}{m} = -9.01\ mm/s$$

$\frac{dy}{dt}$

Alternate solution: Choose CV(2) enclosing entire funnel.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) ρ = constant, but $\forall$ changes (Note: $\forall = \frac{\pi}{3} r^2 h$ for a cone.)
(2) Neglect air
(3) Uniform flow at outlet section

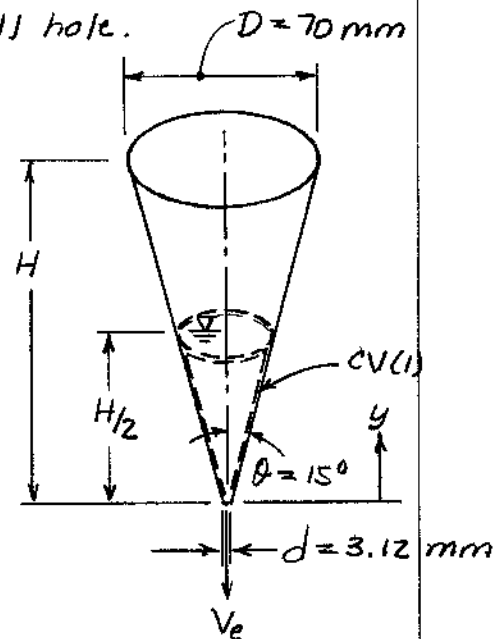
Then

$$0 = \rho\frac{\partial}{\partial t}\forall_{H_2O} + \{+|\rho V_e A_e|\} \quad \text{or} \quad \frac{d\forall}{dt} = -V_e A_e$$ (out)

The volume of water is $\forall = \frac{\pi}{3} r^2 h = \frac{\pi}{3}(y\tan\theta)^2 y = \frac{\pi y^3 \tan^2\theta}{3}$

so $\frac{d\forall}{dt} = \pi y^2 \tan^2\theta \frac{dy}{dt} = \pi\left(\frac{D}{4}\right)^2 \frac{dy}{dt}$ and $\frac{\pi D^2}{16}\frac{dy}{dt} = -V_e A_e = -\sqrt{2gy}\,\frac{\pi d^2}{4}$

Finally, since $y = H/2$, $\frac{dy}{dt} = -4\sqrt{2gH}\left(\frac{d}{D}\right)^2$ as before.

{Note: Flow is not steady in either CV. The $\partial/\partial t$ term vanishes for CV(1) because there is no change in mass inside the CV.}

Problem 4.48

Given: Steady flow of water past a porous flat plate. Suction is constant. Velocity profile at section cd is

$$\frac{u}{U_\infty} = 3\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^{1.5}$$

Find: Mass flow rate across section bc.

Solution: Apply conservation of mass using the CV shown.

Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) $\vec{V} = -v_0 \hat{j}$ along da

Then

$$0 = \int_{CS} \rho \vec{V} \cdot d\vec{A} = \int_{ab} \rho \vec{V} \cdot d\vec{A} + \dot{m}_{bc} + \int_{cd} \rho \vec{V} \cdot d\vec{A} + \int_{da} \rho \vec{V} \cdot d\vec{A}$$

or

$$0 = -\rho U_\infty w \delta + \dot{m}_{bc} + \int_0^\delta \rho U_\infty \left[3\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^{1.5}\right] w \, dy + \rho v_0 w L$$

Thus

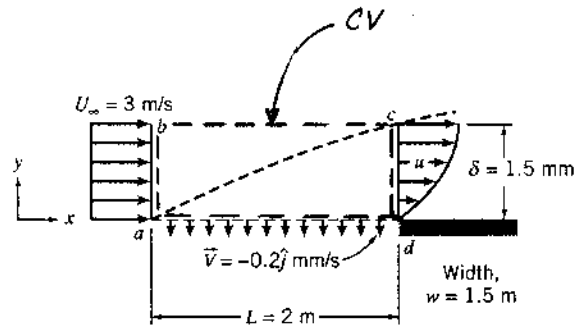
$$\dot{m}_{bc} = \rho U_\infty w \delta - \rho U_\infty w \delta \int_0^1 \left[3\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^{1.5}\right] d\left(\frac{y}{\delta}\right) - \rho v_0 w L$$

$$= \rho w \left\{ U_\infty \delta - U_\infty \delta \left[\frac{3}{2}\left(\frac{y}{\delta}\right)^2 - \frac{2}{2.5}\left(\frac{y}{\delta}\right)^{1.5}\right]_0^1 - v_0 L \right\}$$

$$= \rho w \left[U_\infty \delta - U_\infty \delta \left(\frac{3}{2} - \frac{2}{2.5}\right) - v_0 L \right] = \rho w \left(0.3 U_\infty \delta - v_0 L\right)$$

$$= 999 \frac{kg}{m^3} \times 1.5\, m \left(0.3 \times 3 \frac{m}{s} \times 0.0015\, m - 0.0002 \frac{m}{s} \times 2\, m\right)$$

$\dot{m}_{bc} = 1.42$ kg/s   ($\dot{m} > 0$, so out of CV)

$\dot{m}_{bc}$

Problem 4.49

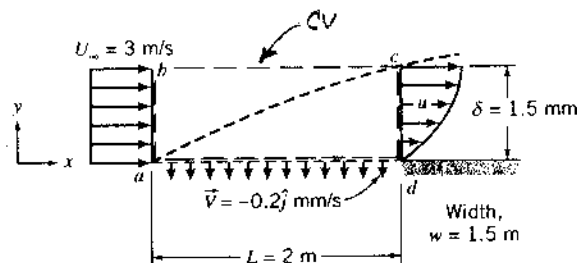
Given: Steady incompressible flow of air on porous surface shown in Fig. P4.48. Velocity profile at downstream end is parabolic. Uniform suction is applied along ad.

Find: (a) Volume flow rate across cd.
(b) Volume flow rate through porous surface (ad).
(c) Volume flow rate across bc.

Solution: Apply conservation of mass to CV shown.

Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

(the unsteady term $= 0(1)$)

Assumptions: (1) Incompressible flow
(2) Parabolic profile at section cd: $\frac{u}{U_\infty} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$

Then $$0 = \int_{CS} \vec{V} \cdot d\vec{A} = Q_{ab} + Q_{bc} + Q_{cd} + Q_{da} \quad (1)$$

$$Q_{cd} = \int_{cd} \vec{V} \cdot d\vec{A} = \int_0^\delta u w \, dy = w U_\infty \delta \int_0^1 \frac{u}{U} d\left(\frac{y}{\delta}\right) = w U_\infty \delta \int_0^1 \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] d\frac{y}{\delta}$$

$$= w U_\infty \delta \left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right]_0^1 = \frac{2}{3} w \delta U_\infty$$

$$Q_{cd} = \frac{2}{3} \times 1.5 \text{ m} \times 0.0015 \text{ m} \times 3 \frac{\text{m}}{\text{s}} = 4.50 \times 10^{-3} \text{ m}^3/\text{s} \text{ (out of CV)}$$

Q_{cd}

Flow across ad is uniform, so

$$Q_{ad} = \vec{V} \cdot \vec{A} = v\hat{j} \cdot wL(-\hat{j}) = -vwL$$

$$Q_{ad} = -0.2 \frac{\text{mm}}{\text{s}} \times 1.5 \text{ m} \times 2 \text{ m} \times \frac{\text{m}}{1000 \text{ mm}} = 6.00 \times 10^{-4} \text{ m/s (out of CV)}$$

Q_{ad}

Finally, from Eq. 1,

$$Q_{bc} = -Q_{ab} - Q_{cd} - Q_{da} \quad (2)$$

But $$Q_{ab} = \vec{U}_\infty \cdot \vec{A}_{ab} = U_\infty \hat{i} \cdot w\delta(-\hat{i}) = -w\delta U_\infty$$

$$Q_{ab} = -1.5 \text{ m} \times 0.0015 \text{ m} \times 3 \frac{\text{m}}{\text{s}} = -6.75 \times 10^{-3} \text{ m}^3/\text{s} \text{ (into CV)}$$

Substituting into Eq. 2,

$$Q_{bc} = \left[-(-6.75 \times 10^{-3}) - 4.50 \times 10^{-3} - 6.00 \times 10^{-4}\right] \text{ m}^3/\text{s}$$

$$Q_{bc} = 1.65 \times 10^{-3} \text{ m}^3/\text{s} \text{ (out of CV)}$$

Q_{bc}

Problem 4.50

Given: Tank containing brine with steady inlet stream of water. Initial density is $\rho_i > \rho_{H_2O}$.

Find: (a) Rate of change of liquid density in tank.
(b) Time required to reach density, ρ_f, where $\rho_i > \rho_f > \rho_{H_2O}$.

Solution: Apply conservation of mass using the CV shown.

Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

$\forall$ = constant; $\dot{m}_{in}$, ρ_{H_2O}; $\dot{m}_{out}$, ρ

Assumptions: (1) $\forall_{tank}$ = constant
(2) ρ uniform in tank
(3) Uniform flows at inlet and outlet sections

Then $V_1 A_1 = V_2 A_2$ since tank volume is constant, and

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \rho VA - \rho_{H_2O} VA = \frac{\partial}{\partial t}\rho\forall + (\rho - \rho_{H_2O})VA = \forall\frac{d\rho}{dt} + (\rho - \rho_{H_2O})VA$$

So that

$$\frac{d\rho}{dt} = -\frac{(\rho - \rho_{H_2O})VA}{\forall}$$

$\frac{d\rho}{dt}$

Separating variables,

$$\frac{d\rho}{\rho - \rho_{H_2O}} = -\frac{VA}{\forall} dt$$

Integrating from ρ_i at $t = 0$ to ρ_f at t,

$$\int_{\rho_i}^{\rho_f} \frac{d\rho}{\rho - \rho_{H_2O}} = \ln(\rho - \rho_{H_2O})\Big]_{\rho_i}^{\rho_f} = \ln\left(\frac{\rho_f - \rho_{H_2O}}{\rho_i - \rho_{H_2O}}\right) = \int_0^t -\frac{VA}{\forall} dt = -\frac{VA}{\forall} t$$

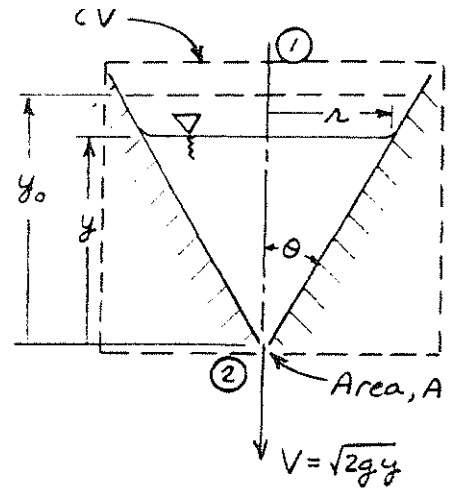
Finally,

$$t = -\frac{\forall}{VA} \ln\left(\frac{\rho_f - \rho_{H_2O}}{\rho_i - \rho_{H_2O}}\right)$$

t

{ Note that $\rho_f \to \rho_{H_2O}$ asymptotically as $t \to \infty$. }

Problem 4.51

Given: Funnel of liquid draining through a small hole of diameter $d = 5\ mm$ (area, A) as shown; y_0 is initial depth.

Find: (a) Expression for time to drain
(b) Expression for result in terms of
- initial volume $\forall_0$, and
- initial volume flow rate $Q_0 = AV_0 = A\sqrt{2gy_0}$

Plot: t as a function of y_0 ($0.1 \le y_0 \le 1\ m$) with angle θ as a parameter for $15° \le \theta \le 85°$.

Solution

Apply conservation of mass using CV shown.

Basic equation: $$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) Incompressible flow
(2) Uniform flow at each section
(3) Neglect ρ_{air} compared to ρ_{H_2O}

Then,

$$0 = \frac{\partial}{\partial t}\int_{\forall_{air}} \rho_{air}\, d\forall + \frac{\partial}{\partial t}\int_{\forall_{H_2O}} \rho_{H_2O}\, d\forall + \{-|\rho_{air} V_1 A_1|\} + \{|\rho_{H_2O} V A|\}$$

(the air terms ≈ 0 (3))

For the CV,

$$d\forall = A_s\, dy = \pi r^2\, dy = \pi (y \tan\theta)^2\, dy \quad ; \quad \forall = \pi \tan^2\theta\, \frac{y^3}{3}$$

Thus

$$0 = \rho_{H_2O} \frac{\partial}{\partial t}\left(\pi \tan^2\theta\, \frac{y^3}{3}\right) + \rho_{H_2O} A\sqrt{2gy}$$

$$0 = \pi \tan^2\theta\, y^2 \frac{dy}{dt} + A\sqrt{2g}\, y^{1/2}$$

Separating variables, $$y^{3/2}\, dy = -\frac{\sqrt{2g}\, A}{\pi \tan^2\theta}\, dt$$

Integrating from y_0 at $t=0$ to 0 at t,

$$\int_{y_0}^{0} y^{3/2}\, dy = \frac{2}{5}\left(-y_0^{5/2}\right) = -\frac{\sqrt{2g}\, A}{\pi \tan^2\theta}\, t$$

or

$$t = \frac{2}{5} \frac{\pi \tan^2\theta\, y_0^{5/2}}{\sqrt{2g}\, A} \qquad t$$

Problem 4.51

But $\forall_o = \pi \tan^2\theta \, \frac{y_o^3}{3}$ and $Q_o = A V_o = A\sqrt{2 g y_o}$, so

$$t = \frac{2}{5} \frac{\pi \tan^2\theta \, y_o^{5/2}}{\sqrt{2g}\, A} \times \frac{3}{3} \cdot \frac{y_o^{1/2}}{y_o^{1/2}} = \frac{6}{5} \frac{\forall_o}{Q_o} \qquad t$$

Since $A = \frac{\pi d^2}{4}$, we can write

$$t = \frac{2}{5} \frac{\pi \tan^2\theta \, y_o^{5/2}}{\sqrt{2g}\, \pi \frac{d^2}{4}} = \frac{8}{5} \frac{\tan^2\theta \, y_o^{5/2}}{d^2 \sqrt{2g}}$$

t is plotted as a function of y_o with θ as a parameter

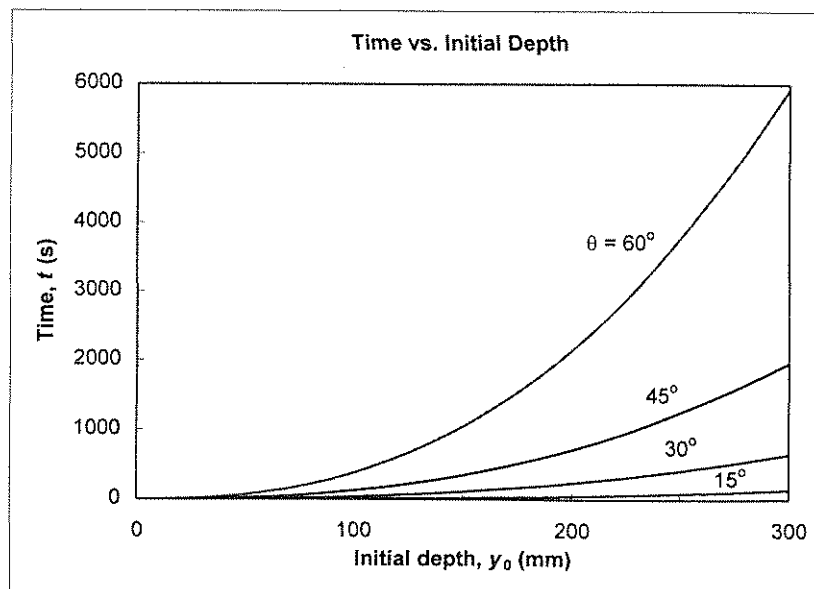
Draining of a conical liquid tank:

Input Data:

Orifice diameter: $d =$ 3 mm

Calculated Results:

Initial Height, y_0 (mm)	Cone Half Angle, θ (deg)	Drain Time, t (s)			
		60	**45**	**30**	**15**
300		5935	1978	659	142
275		4775	1592	531	114
250		3763	1254	418	90.0
225		2891	964	321	69.2
200		2154	718	239	51.5
175		1543	514	171	36.9
150		1049	350	117	25.1
125		665	222	74	15.9
100		381	127	42	9.11
75		185	62	21	4.44
50		67	22	7	1.61
25		12	4	1	0.285
0		0	0	0	0

Problem 4.52

Given: The instantaneous leakage mass flow rate $\dot{m}$ from a bicycle tire is proportional to the air density ρ in the tire and to the gage pressure p_g in the tire.
Air in the tire is nearly isothermal (because the leakage rate is slow).
The initial air pressure is $p_0 = 0.60$ MPa (gage) and the initial rate of pressure loss is 1 psi/day

Find: (a) Pressure in the tire after 30 days
(b) Accuracy of rule of thumb which says a tire loses pressure at the rate of "a pound (1 psi) a day".

Plot: the pressure as a function of time over the 30 days; show rule of thumb results for comparison.

Solution:

Apply conservation of mass to tire as the CV ($\forall$) $\to \dot{m}$

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) uniform properties in tire
(2) air inside CV behaves as ideal gas
(3) T = constant, $\forall$ = constant
(4) $\dot{m} = c(p - p_{atm})\rho$

Then we can write

$$0 = \forall \frac{\partial \rho}{\partial t} + \dot{m} = \forall \frac{\partial \rho}{\partial t} + c(p - p_{atm})\rho \quad \cdots\cdots (1)$$

But $\rho = p/RT$ and $\frac{\partial \rho}{\partial t} = \frac{1}{RT}\frac{dp}{dt}$, so

$$0 = \frac{\forall}{RT}\frac{dp}{dt} + \frac{c\,p}{RT}(p - p_{atm})$$

At $t = 0$, $p = p_0$ and $dp/dt = dp/dt)_0$. Thus

$$0 = \forall \frac{dp}{dt}\Big)_0 + c\,p_0(p_0 - p_{atm}) \quad \text{and} \quad c = -\frac{\forall}{p_0(p_0 - p_{atm})}\frac{dp}{dt}\Big)_0$$

Substituting into Eq. 1 we obtain

$$0 = \frac{dp}{dt} - \frac{p(p - p_{atm})}{p_0(p_0 - p_{atm})}\frac{dp}{dt}\Big)_0$$

Separating variables and integrating

$$\int_{p_0}^{p} \frac{dp}{p(p - p_{atm})} = \frac{dp/dt)_0}{p_0(p_0 - p_{atm})}\int_0^t dt$$

$$\frac{1}{p_{atm}}\left[\ln \frac{p_0(p - p_{atm})}{p(p_0 - p_{atm})}\right] = \frac{dp/dt)_0}{p_0(p_0 - p_{atm})}\,t$$

$$\ln\left[\frac{1 - p_{atm}/p}{1 - p_{atm}/p_0}\right] = \frac{dp/dt)_0}{p_0(p_0/p_{atm} - 1)}\,t$$

Taking antilogs,

$$1 - \frac{p_{atm}}{p} = \left(1 - \frac{p_{atm}}{p_0}\right) e^{\left\{\frac{dp/dt)_0}{p_0(p_0/p_{atm}-1)} t\right\}} = \left(1 - \frac{p_{atm}}{p_0}\right) e^{kt}$$

where

$$k = \frac{dp/dt)_0}{p_0(p_0/p_{atm}-1)} = -1\,\text{psi} \times \frac{6.895\,\text{kPa}}{\text{psi}} \times \frac{1}{701\,\text{kPa}} \frac{1}{(701/101-1)}$$

$$k = -0.00166\ \text{day}^{-1}$$

Then

$$\frac{p_{atm}}{p} = 1 - \frac{(p_0 - p_{atm})}{p_0} e^{kt}$$

and

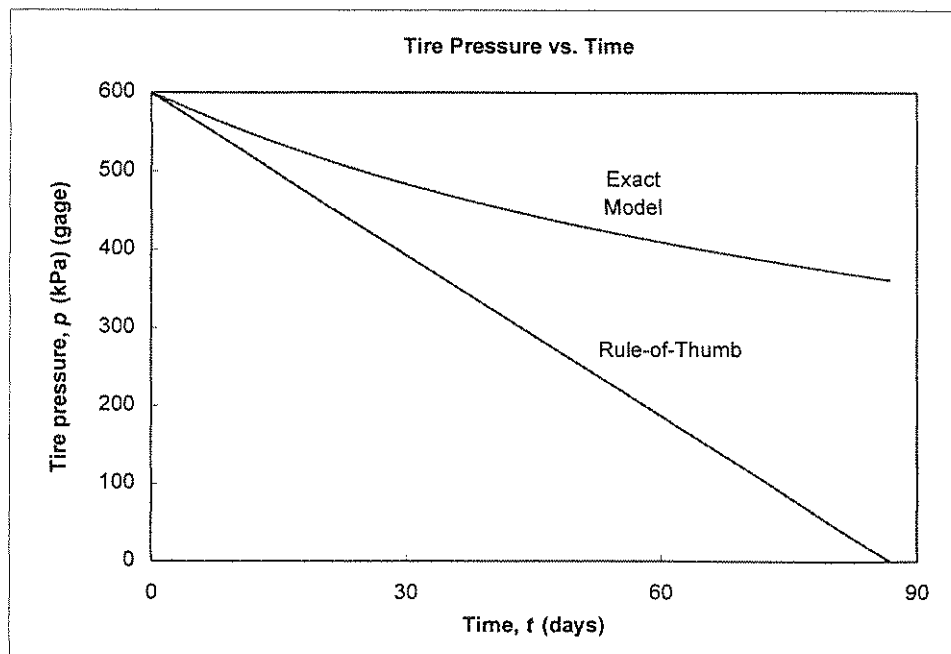
$$p = \frac{p_{atm}}{1 - \frac{(p_0 - p_{atm})}{p_0} e^{kt}} \qquad (2)$$

Evaluating at $t = 30$ days.

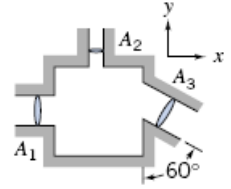
$$p = \frac{101\,\text{kPa}}{1 - \frac{600}{701} e^{-30(0.00166)}} = 544\,\text{kPa} \qquad p_{t=30\,\text{days}}$$

Rule of thumb gives $p = p_0 - 6.895\frac{\text{kPa}}{\text{day}}\,t$ $\qquad (3)$

At $t = 30$ days $\qquad p = 600\,\text{kPa} - 207\,\text{kPa} = 493\,\text{kPa}$ $\quad p_{\text{ROT.}}$

The rule of thumb predicts a larger pressure loss. Results for both models are presented below

Problem 4.53

[3]

4.53 Evaluate the net rate of flux of momentum out through the control surface of Problem 4.21.

Given: Data on flow through a control surface

Find: Net rate of momentum flux

Solution:

Basic equation: We need to evaluate $\int_{CS} \vec{V}\rho\vec{V}\cdot dA$

Assumptions: 1) Uniform flow at each section

From Problem 4.21 $V_1 = 10\cdot\frac{ft}{s}$ $A_1 = 0.5\cdot ft^2$ $V_2 = 20\cdot\frac{ft}{s}$ $A_2 = 0.1\cdot ft^2$ $A_3 = 0.6\cdot ft^2$ $V_3 = 5\cdot\frac{ft}{s}$ It is an outlet

Then for the control surface

$$\int_{CS} \vec{V}\rho\vec{V}\cdot dA = \vec{V}_1\rho\vec{V}_1\cdot\vec{A}_1 + \vec{V}_2\rho\vec{V}_2\cdot\vec{A}_2 + \vec{V}_3\rho\vec{V}_3\cdot\vec{A}_3$$

$$= V_1\hat{i}\rho\left(\vec{V}_1\cdot\vec{A}_1\right) + V_2\hat{j}\rho\left(\vec{V}_2\cdot\vec{A}_2\right) + \left[V_3\sin(60)\hat{i} - V_3\cos(60)\hat{j}\right]\rho\left(\vec{V}_3\cdot\vec{A}_3\right)$$

$$= -V_1\hat{i}\rho V_1A_1 + V_2\hat{j}\rho V_2A_2 + \left[V_3\sin(60)\hat{i} - V_3\cos(60)\hat{j}\right]\rho V_3A_3$$

$$= \rho\left[-V_1^2A_1 + V_3^2A_3\sin(60)\right]\hat{i} + \rho\left[V_2^2A_2 - V_3^2A_3\cos(60)\right]\hat{j}$$

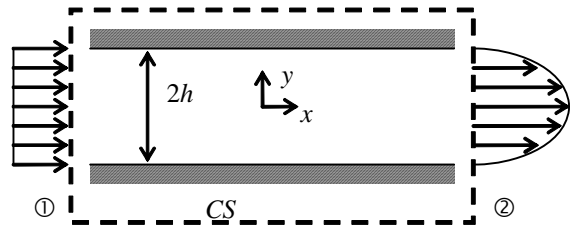
Hence the x component is

$$\rho\left[-V_1^2A_1 + V_3^2A_3\sin(60)\right] = 65\cdot\frac{lbm}{ft^3}\times\left(-10^2\times 0.5 + 5^2\times 0.6\times\sin(60\cdot deg)\right)\cdot\frac{ft^4}{s^2}\times\frac{lbf\cdot s^2}{lbm\cdot ft} = -2406\,lbf$$

and the y component is

$$\rho\left[V_2^2A_2 - V_3^2A_3\cos(60)\right] = 65\cdot\frac{lbm}{ft^3}\times\left(20^2\times 0.1 - 5^2\times 0.6\times\cos(60\cdot deg)\right)\cdot\frac{ft^4}{s^2}\times\frac{lbf\cdot s^2}{lbm\cdot ft} = 2113\,lbf$$

Problem 4.54

4.54 For the conditions of Problem 4.30, evaluate the ratio of the x-direction momentum flux at the channel outlet to that at the inlet.

Given:

Data on flow at inlet and outlet of channel

Find:

Ratio of outlet to inlet momentum flux

Solution:

Basic equation: Momentum flux in x direction at a section $mf_x = \int_A u\rho\vec{V}\cdot dA$

Assumptions: 1) Steady flow 2) Incompressible flow

Evaluating at 1 and 2 $mf_{x1} = U\cdot\rho\cdot(-U\cdot 2\cdot h)\cdot w \qquad |mf_{x1}| = 2\cdot\rho\cdot w\cdot U^2\cdot h$

Hence $mf_{x2} = \int_{-h}^{h}\rho\cdot u^2\cdot w\,dy = \rho\cdot w\cdot u_{max}^2\cdot\int_{-h}^{h}\left[1-\left(\frac{y}{h}\right)^2\right]^2 dy = \rho\cdot w\cdot u_{max}^2\cdot\int_{-h}^{h}\left[1-2\cdot\left(\frac{y}{h}\right)^2+\left(\frac{y}{h}\right)^4\right]dy$

$$|mf_{x2}| = \rho\cdot w\cdot u_{max}^2\cdot\left(2\cdot h-\frac{4}{3}\cdot h+\frac{2}{5}\cdot h\right) = \rho\cdot w\cdot u_{max}^2\cdot\frac{16}{15}\cdot h$$

Then the ratio of momentum fluxes is

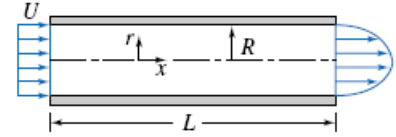
$$\frac{|mf_{x2}|}{|mf_{x1}|} = \frac{\frac{16}{15}\cdot\rho\cdot w\cdot u_{max}^2\cdot h}{2\cdot\rho\cdot w\cdot U^2\cdot h} = \frac{8}{15}\cdot\left(\frac{u_{max}}{U}\right)^2$$

But, from Problem 4.30 $u_{max} = \frac{3}{2}\cdot U \qquad \frac{|mf_{x2}|}{|mf_{x1}|} = \frac{8}{15}\cdot\left(\frac{\frac{3}{2}\cdot U}{U}\right)^2 = \frac{6}{5} = 1.2$

Hence the momentum increases as it flows in the entrance region of the channel. This appears to contradict common sense, as friction should reduce flow momentum. What happens is the pressure drops significantly along the channel so the net force on the CV is to the right.

Problem 4.55

[3]

4.55 For the conditions of Problem 4.31, evaluate the ratio of the x-direction momentum flux at the pipe outlet to that at the inlet.

Given: Data on flow at inlet and outlet of pipe

Find: Ratio of outlet to inlet momentum flux

Solution:

Basic equation: Momentum flux in x direction at a section $mf_x = \int_A u\rho\vec{V}\cdot dA$

Assumptions: 1) Steady flow 2) Incompressible flow

Evaluating at 1 and 2 $\quad mf_{x1} = U\cdot\rho\cdot\left(-U\cdot\pi\cdot R^2\right) \qquad \left|mf_{x1}\right| = \rho\cdot\pi\cdot U^2\cdot R^2$

Hence

$$mf_{x2} = \int_0^R \rho\cdot u^2\cdot 2\cdot\pi\cdot r\,dr = 2\cdot\rho\cdot\pi\cdot u_{max}^2\cdot\int_0^R r\cdot\left[1-\left(\frac{r}{R}\right)^2\right]^2 dr = 2\cdot\rho\cdot\pi\cdot u_{max}^2\cdot\int_0^R \left(r - 2\cdot\frac{r^3}{R^2} + \frac{r^5}{R^4}\right) dy$$

$$\left|mf_{x2}\right| = 2\cdot\rho\cdot\pi\cdot u_{max}^2\cdot\left(\frac{R^2}{2} - \frac{R^2}{2} + \frac{R^2}{6}\right) = \rho\cdot\pi\cdot u_{max}^2\cdot\frac{R^2}{3}$$

Then the ratio of momentum fluxes is

$$\frac{\left|mf_{x2}\right|}{\left|mf_{x1}\right|} = \frac{\frac{1}{3}\cdot\rho\cdot\pi\cdot u_{max}^2\cdot R^2}{\rho\cdot\pi\cdot U^2\cdot R^2} = \frac{1}{3}\cdot\left(\frac{u_{max}}{U}\right)^2$$

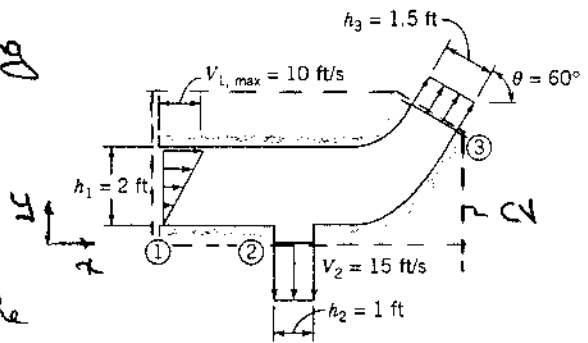
But, from Problem 4.31 $\quad u_{max} = 2\cdot U \qquad \frac{\left|mf_{x2}\right|}{\left|mf_{x1}\right|} = \frac{1}{3}\cdot\left(\frac{2\cdot U}{U}\right)^2 = \frac{4}{3} = 1.33$

Hence the momentum increases as it flows in the entrance region of the pipe This appears to contradict common sense, as friction should reduce flow momentum. What happens is the pressure drops significantly along the pipe so the net force on the CV is to the right.

Problem 4.56

Given: Two-dimensional reducing bend shown has width $w = 3$ ft.

$V_3 = 3.33$ ft/s into CV (from Problem 4.24)

$h_3 = 1.5$ ft, $V_{1,max} = 10$ ft/s, $\theta = 60°$, $h_1 = 2$ ft, $V_2 = 15$ ft/s, $h_2 = 1$ ft, CV

Find: Momentum flux through the bend.

Solution:

The momentum flux is defined as $m.f = \int \vec{V}(\rho\vec{V}\cdot d\vec{A})$

The net momentum flux through the CV is

$$m.f = \int_{A_1} \vec{V}(\rho\vec{V}\cdot d\vec{A}) + \int_{A_2} \vec{V}(\rho\vec{V}\cdot d\vec{A}) + \int_{A_3} \vec{V}(\rho\vec{V}\cdot d\vec{A})$$

where $\vec{V}_1 = V_{1max}\frac{y}{h_1}\hat{i}$, $\vec{V}_2 = -V_2\hat{j}$, $\vec{V}_3 = -V_3(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

$V_{1max} = 10$ ft/s, $V_2 = 15$ ft/s, $V_3 = 3.33$ ft/s

Assumptions: (1) incompressible flow
(2) fluid is water
(3) uniform flow at ② and ③ (given)

$$\int_{A_1} \vec{V}(\rho\vec{V}\cdot d\vec{A}) = \int_0^{h_1} V_{1max}\frac{y}{h_1}\hat{i}\,\rho\left\{-V_{1max}\frac{y}{h_1}\right\}w\,dy = -\hat{i}\rho V_{1max}^2\frac{w}{h_1^2}\int_0^{h_1} y^2 dy$$

$$\int_{A_1} \vec{V}(\rho\vec{V}\cdot d\vec{A}) = -\hat{i}\rho V_{1max}^2\frac{w}{h_1^2}\left.\frac{y^3}{3}\right]_0^{h_1} = -\hat{i}\rho V_{1max}^2\frac{wh_1}{3} \quad ----(1)$$

$$\int_{A_2} \vec{V}(\rho\vec{V}\cdot d\vec{A}) = \vec{V}_2|\rho V_2 h_2 w| = -V_2\hat{j}|\rho V_2 h_2 w| = -\hat{j}\rho V_2^2 h_2 w \quad ----(2)$$

$$\int_{A_3} \vec{V}(\rho\vec{V}\cdot d\vec{A}) = \vec{V}_3(-|\rho V_3 h_3 w|) = -V_3(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})(-|\rho V_3 h_3 w|)$$

$$\int_{A_3} \vec{V}(\rho\vec{V}\cdot d\vec{A}) = \rho V_3^2 h_3 w(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) \quad --------(3)$$

$$m.f = \hat{i}\left[\rho V_3^2 h_3 w\cos\theta - \rho V_{1max}^2\frac{wh_1}{3}\right] + \hat{j}\left[\rho V_3^2 h_3 w\sin\theta - \rho V_2^2 h_2 w\right]$$

$$m.f = \rho w\left\{\left[V_3^2 h_3\cos\theta - V_{1max}^2\frac{h_1}{3}\right]\hat{i} + \left[V_3^2 h_3\sin\theta - V_2^2 h_2\right]\hat{j}\right\}$$

Evaluating

$$m.f = 1.94\frac{\text{slug}}{\text{ft}^3}\times 3\text{ft}\times\frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}}\left\{\left[(3.33)^2\frac{\text{ft}^2}{\text{s}^2}\times 1.5\text{ft}\times\cos 60° - (10)^2\frac{\text{ft}^2}{\text{s}^2}\times\frac{2\text{ft}}{3}\right]\hat{i} + \left[(3.33)^2\frac{\text{ft}^2}{\text{s}^2}\times 1.5\text{ft}\times\sin 60° - (15)^2\frac{\text{ft}^2}{\text{s}^2}\times 1\text{ft}\right]\hat{j}\right\}$$

$$m.f = -340\hat{i} - 1230\hat{j}\ \text{lbf}$$

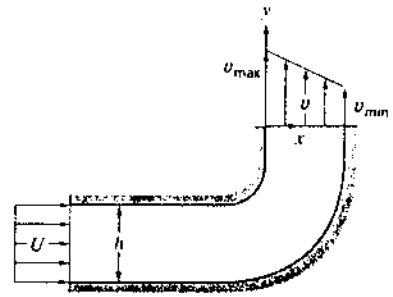
m.f.

Problem 4.57

Given: Water flow in the two-dimensional square channel shown

$\bar{U} = 7.5$ m/s, $h = w = 75.5$ mm

$v_{max} = 2\,v_{min}$

$v_{min} = 5.0$ m/s (from Problem 4.25)

Find: Momentum flux through the channel; comment on expected outlet pressure (relative to pressure at the inlet.

Solution:

The momentum flux is defined as $m.f. = \int \vec{V}(\rho\vec{V}\cdot d\vec{A})$

The net momentum flux through the CV is

$$m.f. = \int_{A_1} \vec{V}(\rho\vec{V}\cdot d\vec{A}) + \int_{A_2} \vec{V}(\rho\vec{V}\cdot d\vec{A})$$

where $\vec{V}_1 = \bar{U}\hat{i}$, $\quad \vec{V}_2 = \left\{v_{max} - (v_{max} - v_{min})\frac{x}{h}\right\}\hat{j}$

$$\vec{V}_2 = \left\{2v_{min} - v_{min}\frac{x}{h}\right\}\hat{j} = v_{min}\left(2 - \frac{x}{h}\right)\hat{j}$$

Assumptions: (1) incompressible flow
(2) uniform flow at ① (given).

$$\int_{A_1} \vec{V}(\rho\vec{V}\cdot d\vec{A}) = \vec{V}_1\{-|\rho V_1 A_1|\} = -\rho\bar{U}^2h^2\,\hat{i} \quad \text{------}(1)$$

$$\int_{A_2} \vec{V}(\rho\vec{V}\cdot d\vec{A}) = \int_0^h v_{min}\left(2 - \frac{x}{h}\right)\hat{j}\,\rho v_{min}\left(2 - \frac{x}{h}\right)h\,dx$$

$$= \hat{j}\,\rho v_{min}^2 h\int_0^h\left(4 - 4\frac{x}{h} + \frac{x^2}{h^2}\right)dx$$

$$= \hat{j}\,\rho v_{min}^2 h\left[4x - 2\frac{x^2}{h} + \frac{x^3}{3h^2}\right]_0^h = \hat{j}\,\rho v_{min}^2 h\left[4h - 2h + \frac{h}{3}\right]$$

$$= \hat{j}\,\frac{7}{3}\rho v_{min}^2 h^2$$

$$\therefore\ m.f. = -\rho\bar{U}^2h^2\hat{i} + \frac{7}{3}\rho v_{min}^2h^2\hat{j} = \rho h^2\left[-\bar{U}^2\hat{i} + \frac{7}{3}v_{min}^2\hat{j}\right]$$

Evaluating

$$m.f. = 999\,\frac{kg}{m^3}\times(0.0755)^2 m^2\left[-(7.5)^2\frac{m^2}{s^2}\hat{i} + \frac{7}{3}(5)^2\frac{m^2}{s^2}\hat{j}\right]\times\frac{N\cdot s^2}{kg\cdot m}$$

$$m.f. = -320\,\hat{i} + 332\,\hat{j}\ N \qquad\qquad m.f.$$

For viscous (real) flow friction causes a pressure drop in the direction of flow (Chapter 8)
For flow in a bend streamline curvature results in a pressure gradient normal to the flow (Chapter 6)

Problem 4.58

[2]

4.58 What force (lbf) will a horizontal 2-in.-diameter stream of water moving at 20 ft/s generate upon hitting a vertical flat plate?

Given: Water jet hitting wall

Find: Force generated on wall

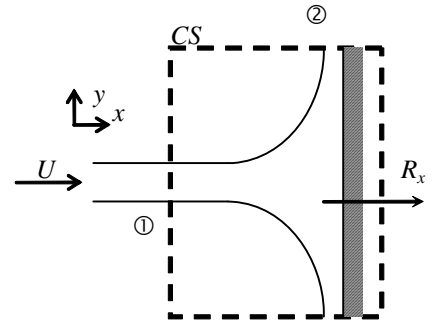
Solution:

Basic equation: Momentum flux in x direction

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure throughout 4) Uniform flow 5) Water leaves vertically

Hence $\quad R_x = u_1\cdot\rho\cdot\left(-u_1\cdot A_1\right) = -\rho\cdot U^2\cdot A = -\rho\cdot U^2\cdot\frac{\pi\cdot D^2}{4}$

$$R_x = -1.94\cdot\frac{\text{slug}}{\text{ft}^3}\times\left(20\cdot\frac{\text{ft}}{\text{s}}\right)^2\times\frac{\pi\cdot\left(\frac{1}{6}\cdot\text{ft}\right)^2}{4}\times\frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \qquad R_x = -16.9\cdot\text{lbf}$$

Problem 4.59

[1]

4.59 Considering that in the fully developed region of a pipe, the integral of the axial momentum is the same at all cross sections, explain the reason for the pressure drop along the pipe.

Given: Fully developed flow in pipe

Find: Why pressure drops if momentum is constant

Solution:

Basic equation: Momentum flux in x direction

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV\!\!\!\!/ + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Fully developed flow

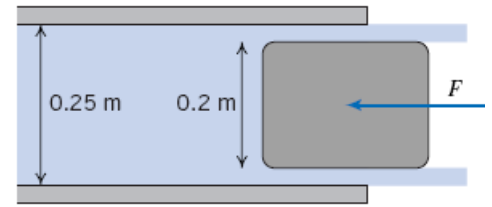
Hence $\quad F_x = \frac{\Delta p}{L} - \tau_w \cdot A_s = 0 \qquad \Delta p = L \cdot \tau_w \cdot A_s$

where Δp is the pressure drop over length L, τ_w is the wall friction and As is the pipe surface area

The sum of forces in the x direction is zero. The friction force on the fluid is in the negative x direction, so the net pressure force must be in the positive direction. Hence pressure drops in the x direction so that pressure and friction forces balance

Problem 4.60

[2]

4.60 Find the force required to hold the plug in place at the exit of the water pipe. The flow rate is 1.5 m^3/s, and the upstream pressure is 3.5 MPa.

Given:
Data on flow and system geometry

Find:
Force required to hold plug

Solution:

The given data is $D_1 = 0.25\cdot m$ $\quad D_2 = 0.2\cdot m$ $\quad Q = 1.5\cdot\frac{m^3}{s}$ $\quad p_1 = 3500\cdot kPa$ $\quad \rho = 999\cdot\frac{kg}{m^3}$

Then $A_1 = \frac{\pi\cdot D_1^{\,2}}{4}$ $\qquad A_1 = 0.0491\,m^2$

$A_2 = \frac{\pi}{4}\cdot\left(D_1^{\,2} - D_2^{\,2}\right)$ $\qquad A_2 = 0.0177\,m^2$

$V_1 = \frac{Q}{A_1}$ $\qquad V_1 = 30.6\frac{m}{s}$

$V_2 = \frac{Q}{A_2}$ $\qquad V_2 = 84.9\frac{m}{s}$

Governing equation:

Momentum
$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,d\mathcal{V} + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A} \qquad (4.18a)$$

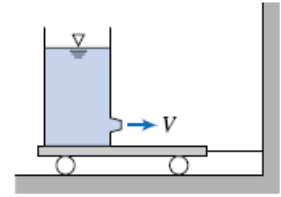
Applying this to the current system

$$-F + p_1\cdot A_2 - p_2\cdot A_2 = 0 + V_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) + V_2\cdot\left(\rho\cdot V_2\cdot A_2\right)$$ and $p_2 = 0$ (gage)

Hence
$$F = p_1\cdot A_1 + \rho\cdot\left(V_1^{\,2}\cdot A_1 - V_2^{\,2}\cdot A_2\right)$$

$$F = 3500\times\frac{kN}{m^2}\cdot 0.0491\cdot m^2 + 999\cdot\frac{kg}{m^3}\times\left[\left(30.6\cdot\frac{m}{s}\right)^2\cdot 0.0491\cdot m^2 - \left(84.9\cdot\frac{m}{s}\right)^2\cdot 0.0177\cdot m^2\right] \qquad F = 90.4\,kN$$

Problem 4.61

4.61 A large tank of height $h = 1$ m and diameter $D = 0.75$ m is affixed to a cart as shown. Water issues from the tank through a nozzle of diameter $d = 15$ mm. The speed of the liquid leaving the tank is approximately $V = \sqrt{2gy}$ where y is the height from the nozzle to the free surface. Determine the tension in the wire when $y = 0.9$ m. Plot the tension in the wire as a function of water depth for $0 \le y \le 0.9$ m.

Given: Large tank with nozzle and wire

Find: Tension in wire; plot for range of water depths

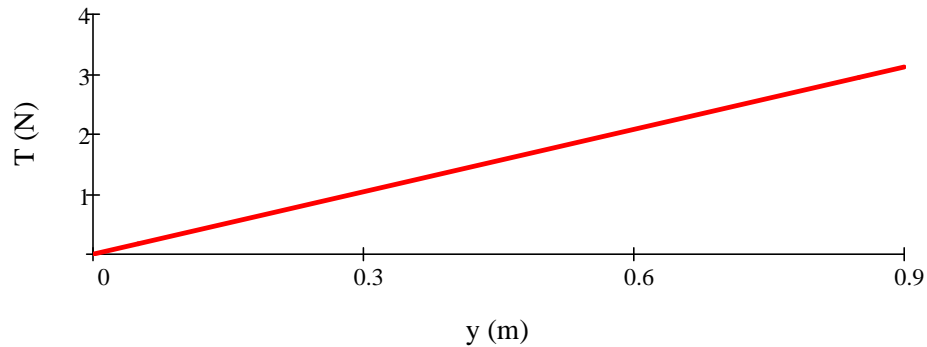
Solution:

Basic equation: Momentum flux in x direction for the tank $F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure throughout 4) Uniform flow

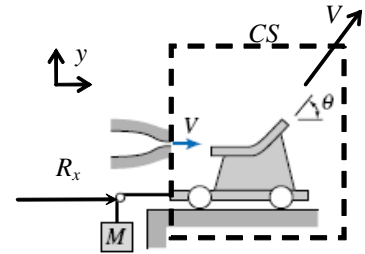
Hence $R_x = T = V\cdot\rho\cdot(V\cdot A) = \rho\cdot V^2\cdot A = \rho\cdot(2\cdot g\cdot y)\cdot\frac{\pi\cdot d^2}{4}$ $\qquad T = \frac{1}{2}\cdot\rho\cdot g\cdot y\cdot\pi\cdot d^2 \qquad (1)$

When y = 0.9 m $\quad T = \frac{\pi}{2}\times 1000\cdot\frac{kg}{m^3}\times 9.81\cdot\frac{m}{s^2}\times 0.9\cdot m\times(0.015\cdot m)^2\times\frac{N\cdot s^2}{kg\cdot m} \qquad T = 3.12\,N$

From Eq 1

This graph can be plotted in *Excel*

Problem 4.62

[2]

4.62 A jet of water issuing from a stationary nozzle at 10 m/s (A_j = 0.1 m^2) strikes a turning vane mounted on a cart as shown. The vane turns the jet through angle $\theta = 40°$. Determine the value of M required to hold the cart stationary. If the vane angle θ is adjustable, plot the mass, M, needed to hold the cart stationary versus θ for $0 \le \theta \le 180°$.

Given: Nozzle hitting stationary cart

Find: Value of M to hold stationary; plot M versu θ

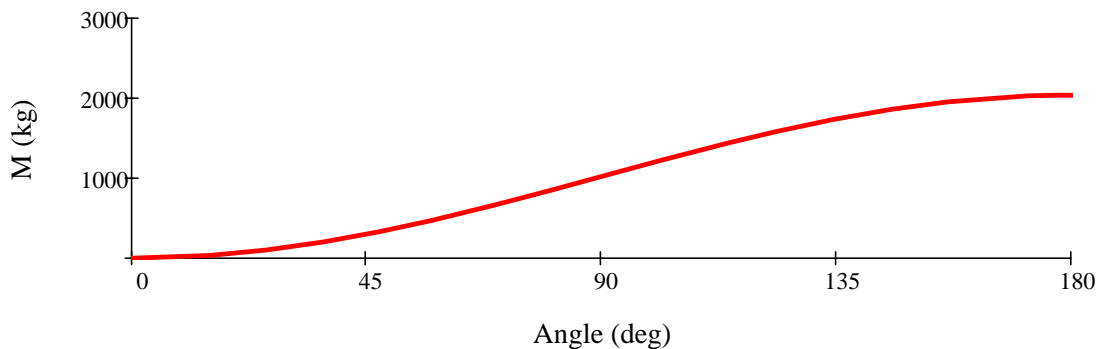
Solution:

Basic equation: Momentum flux in x direction for the tank $F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$

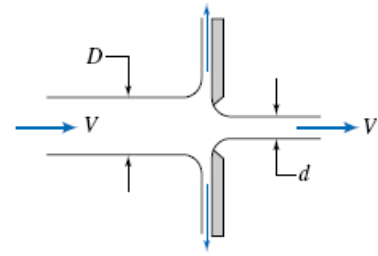
Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure throughout 4) Uniform flow 5) Exit velocity is V

Hence $R_x = -M\cdot g = V\cdot\rho\cdot(-V\cdot A) + V\cdot\cos(\theta)\cdot(V\cdot A) = \rho\cdot V^2\cdot A\cdot(\cos(\theta) - 1)$ $\qquad M = \frac{\rho\cdot V^2\cdot A}{g}\cdot(1 - \cos(\theta))$ (1)

When θ = 40º $M = \frac{s^2}{9.81\cdot m} \times 1000\cdot\frac{kg}{m^3} \times \left(10\cdot\frac{m}{s}\right)^2 \times 0.1\cdot m^2 \times (1 - \cos(40\cdot deg))$ $\qquad M = 238\,kg$

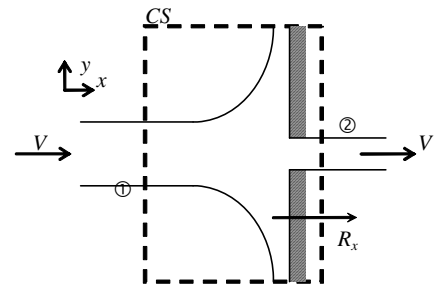
From Eq 1

This graph can be plotted in *Excel*

Problem 4.63

4.63 A vertical plate has a sharp-edged orifice at its center. A water jet of speed V strikes the plate concentrically. Obtain an expression for the external force needed to hold the plate in place, if the jet leaving the orifice also has speed V. Evaluate the force for $V = 15$ ft/s, $D = 4$ in., and $d = 1$ in. Plot the required force as a function of diameter ratio for a suitable range of diameter d.

Given: Water jet hitting plate with opening

Find: Force generated on plate; plot force versus diameter d

Solution:

Basic equation: Momentum flux in x direction

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV\!\!\!/ + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

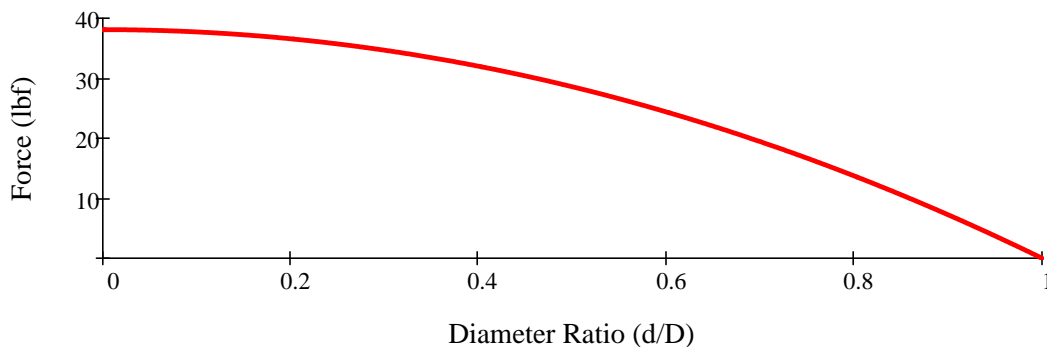
Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure throughout 4) Uniform flow

Hence $\quad R_x = u_1\cdot\rho\cdot\left(-u_1\cdot A_1\right) + u_2\cdot\rho\cdot\left(u_2\cdot A_2\right) = -\rho\cdot V^2\cdot\frac{\pi\cdot D^2}{4} + \rho\cdot V^2\cdot\frac{\pi\cdot d^2}{4}$

$$R_x = -\frac{\pi\cdot\rho\cdot V^2\cdot D^2}{4}\cdot\left[1-\left(\frac{d}{D}\right)^2\right] \qquad (1)$$

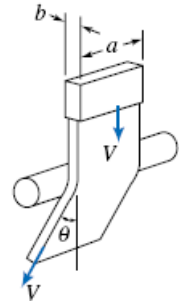
For given data $\quad R_x = -\frac{\pi}{4}\cdot 1.94\cdot\frac{slug}{ft^3}\times\left(15\cdot\frac{ft}{s}\right)^2\times\left(\frac{1}{3}\cdot ft\right)^2\times\left[1-\left(\frac{1}{4}\right)^2\right]\times\frac{lbf\cdot s^2}{slug\cdot ft}$

$$R_x = -35.7\cdot lbf$$

From Eq 1 (using the absolute value of R_x)

This graph can be plotted in *Excel*

Problem 4.64

[3]

4.64 A circular cylinder inserted across a stream of flowing water deflects the stream through angle θ, as shown. (This is termed the "Coanda effect.") For $a = 12.5$ mm, $b = 2.5$ mm, $V = 3$ m/s, and $\theta = 20°$, determine the horizontal component of the force on the cylinder caused by the flowing water.

Given: Water flowing past cylinder

Find: Horizontal force on cylinder

Solution:

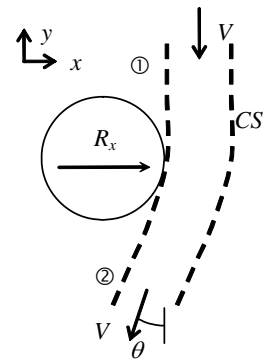
Basic equation: Momentum flux in x direction

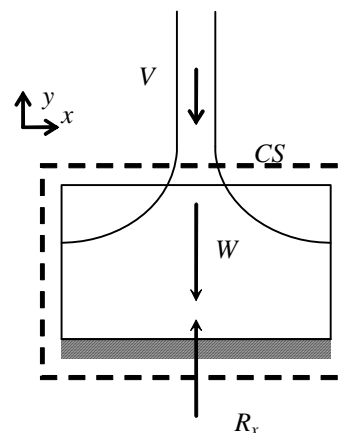
$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV\!\!\!\!- + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure throughout 4) Uniform flow

Hence $\quad R_x = u_1\cdot\rho\cdot\left(-u_1\cdot A_1\right) + u_2\cdot\rho\cdot\left(u_2\cdot A_2\right) = 0 + \rho\cdot(-V\cdot\sin(\theta))\cdot(V\cdot a\cdot b) \qquad R_x = -\rho\cdot V^2\cdot a\cdot b\cdot\sin(\theta)$

For given data $\quad R_x = -1000\cdot\frac{kg}{m^3}\times\left(3\cdot\frac{m}{s}\right)^2\times 0.0125\cdot m\times 0.0025\cdot m\times\sin(20\cdot deg)\times\frac{N\cdot s^2}{kg\cdot m} \qquad R_x = -0.0962\,N$

This is the force on the fluid (it is to the left). Hence the force on the cylinder is $\quad R_x = -R_x \qquad R_x = 0.0962\,N$

Problem 4.65

4.65 In a laboratory experiment, the water flow rate is to be measured catching the water as it vertically exits a pipe into an empty open cylindrical (3-ft diameter) tank that is on a zeroed balance. The tank bottom is 5 ft directly below the pipe exit, and the pipe diameter is 2 in. One student obtains a flow rate by noting that after 30 seconds the volume of water (at 50°F) in the tank was 15 ft^3. Another student obtains a flow rate by reading the instantaneous weight of 960 lb indicated at the 30-second point. Find the mass flow rate each student computes. Why do they disagree? Which one is more accurate? Show that the magnitude of the discrepancy can be explained by any concept you may have.

Given: Water flowing into tank

Find: Mass flow rates estimated by students. Explain discrepancy

Solution:

Basic equation: Momentum flux in y direction

$$F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\,\rho\,dV + \int_{CS} v\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure throughout 4) Uniform flow

For the first student $m_1 = \frac{\rho \cdot V}{t}$ where m_1 represents mass flow rate (software cannot render a dot above it!)

$$m_1 = 1.94 \cdot \frac{slug}{ft^3} \times 15 \cdot ft^3 \times \frac{1}{30 \cdot s} \qquad m_1 = 0.97 \cdot \frac{slug}{s} \qquad m_1 = 31.2 \cdot \frac{lbm}{s}$$

For the second student $m_2 = \frac{M}{t}$ where m_2 represents mass flow rate

$$m_2 = 960 \cdot lb \times \frac{1}{30 \cdot s} \qquad m_2 = 0.995 \cdot \frac{slug}{s} \qquad m_2 = 32 \cdot \frac{lbm}{s}$$

There is a discrepancy because the second student is measuring instantaneous weight PLUS the force generated as the pipe flow momentum is "killed".

To analyse this we first need to find the speed at which the water stream enters the tank, 5 ft below the pipe exit. This would be a good place to use the Bernoulli equation, but this problem is in the set before Bernoulli is covered. Instead we use the simple concept that the fluid is falling under gravity (a conclusion supported by the Bernoulli equation). From the equations for falling under gravity:

$$V_{tank}^2 = V_{pipe}^2 + 2 \cdot g \cdot h$$

where V_{tank} is the speed entering the tank, V_{pipe} is the speed at the pipe, and h = 5 ft is the distance traveled. V_{pipe} is obtained from

$$V_{pipe} = \frac{m_1}{\rho \cdot \frac{\pi \cdot d_{pipe}^2}{4}} = \frac{4 \cdot m_1}{\pi \cdot \rho \cdot d_{pipe}^2}$$

$$V_{pipe} = \frac{4}{\pi} \times 31.2 \cdot \frac{lbm}{s} \times \frac{ft^3}{1.94 \cdot slug} \times \frac{1 \cdot slug}{32.2 \cdot lbm} \times \left(\frac{1}{\frac{1}{6} \cdot ft}\right)^2 \qquad V_{pipe} = 22.9 \frac{ft}{s}$$

Then

$$V_{tank} = \sqrt{V_{pipe}^2 + 2 \cdot g \cdot h} \qquad V_{tank} = \sqrt{\left(22.9 \cdot \frac{ft}{s}\right)^2 + 2 \times 32.2 \cdot \frac{ft}{s^2} \times 5ft} \qquad V_{tank} = 29.1 \frac{ft}{s}$$

We can now use the y momentum equation for the CS shown above

$$R_y - W = -V_{tank}\cdot\rho\cdot\left(-V_{tank}\cdot A_{tank}\right)$$

where A_{tank} is the area of the water flow as it enters the tank. But for the water flow $V_{tank}\cdot A_{tank} = V_{pipe}\cdot A_{pipe}$

Hence $\Delta W = R_y - W = \rho\cdot V_{tank}\cdot V_{pipe}\cdot\frac{\pi\cdot d_{pipe}^{\,2}}{4}$

This equation indicate the instantaneous difference ΔW between the scale reading (R_y) and the actual weight of water (W) in the tank

$$\Delta W = 1.94\cdot\frac{slug}{ft^3}\times 29.1\cdot\frac{ft}{s}\times 22.9\cdot\frac{ft}{s}\times\frac{\pi}{4}\times\left(\frac{1}{6}\cdot ft\right)^2\times\frac{lbf\cdot s^2}{slug\cdot ft} \qquad \Delta W = 28.2\,lbf$$

Hence the scale overestimates the weight of water by 28.2 lbf, or a mass of 28.2 lbm

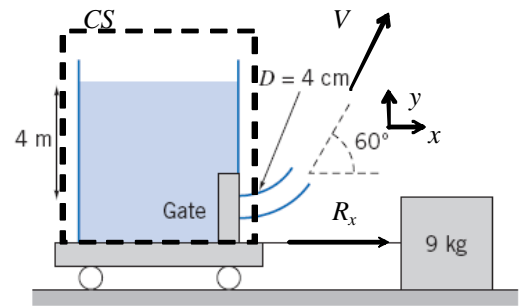
For the second student $M = 960\cdot lbm - 28.2\cdot lbm = 932\cdot lbm$

Hence $m_2 = \frac{M}{t}$ where m_2 represents mass flow rate

$$m_2 = 932\cdot lb\times\frac{1}{30\cdot s} \qquad m_2 = 0.966\cdot\frac{slug}{s} \qquad m_2 = 31.1\cdot\frac{lbm}{s}$$

Comparing with the answer obtained from student 1, we see the students now agree! The discrepancy was entirely caused by the fact that t second student was measuring the weight of tank water PLUS the momentum lost by the water as it entered the tank!

Problem 4.66

4.66 A tank of water sits on a cart with frictionless wheels as shown. The cart is attached using a cable to a 9 kg mass, and the coefficient of static friction of the mass with the ground is 0.5. At time $t = 0$, a second cable is used to remove a gate blocking the tank exit. Will the resulting exit flow be sufficient to start the tank moving? (Assume the water flow is frictionless.)

Given:
Water tank attached to mass

Find:
Whether tank starts moving

Solution:

Basic equation: Momentum flux in x direction for the tank $F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure at exit 4) Uniform flow

Hence $R_x = V\cdot\cos(\theta)\cdot\rho\cdot(V\cdot A) = \rho\cdot V^2\cdot\frac{\pi\cdot D^2}{4}\cdot\cos(\theta)$

We need to find V. We could use the Bernoulli equation, but here it is known that $V = \sqrt{2\cdot g\cdot h}$ where h = 4 m is the height of fluid in the tank

$$V = \sqrt{2\times 9.81\cdot\frac{m}{s^2}\times 4\cdot m} \qquad V = 8.86\,\frac{m}{s}$$

Hence $R_x = 1000\cdot\frac{kg}{m^3}\times\left(8.86\cdot\frac{m}{s}\right)^2\times\frac{\pi}{4}\times(0.04\cdot m)^2\times\cos(60\cdot deg)$ $\qquad R_x = 49.3\,N$

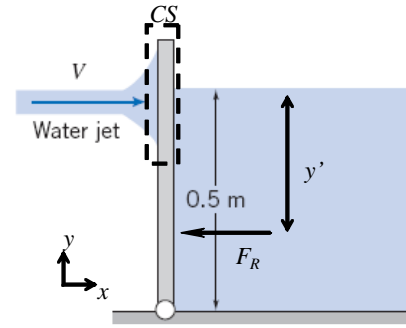
This force is equal to the tension T in the wire $\qquad T = R_x \qquad T = 49.3\,N$

For the block, the maximum friction force a mass of M = 9 kg can generate is $\qquad F_{max} = M\cdot g\cdot\mu$ where μ is static friction

$$F_{max} = 9\cdot kg\times 9.81\cdot\frac{m}{s^2}\times 0.5\times\frac{N\cdot s^2}{kg\cdot m} \qquad F_{max} = 44.1\,N$$

Hence the tension T created by the water jet is larger than the maximum friction F_{max}; the tank starts to move

Problem 4.67

4.67 A gate is 0.5 m wide and 0.6 m tall, and is hinged at the bottom. On one side the gate holds back a 0.5-m deep body of water. On the other side, a 10-cm diameter water jet hits the gate at a height of 0.5 m. What jet speed V is required to hold the gate vertical? What will the speed be if the body of water is lowered to 0.25 m? What will the speed be if the water level is at the top of the gate?

Given: Gate held in place by water jet

Find: Required jet speed for various water depths

Solution:

Basic equation: Momentum flux in x direction for the wall $F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$

Note: We use this equation ONLY for the jet impacting the wall. For the hydrostatic force and location we use computing equations

$$F_R = p_c \cdot A \qquad y' = y_c + \frac{I_{xx}}{A\cdot y_c}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Uniform flow

Hence $\quad R_x = V\cdot\rho\cdot\left(-V\cdot A_{jet}\right) = -\rho\cdot V^2\cdot\frac{\pi\cdot D^2}{4}$

This force is the force generated by the wall on the jet; the force of the jet hitting the wall is then

$$F_{jet} = -R_x = \rho\cdot V^2\cdot\frac{\pi\cdot D^2}{4} \qquad \text{where D is the jet diameter}$$

For the hydrostatic force $\quad F_R = p_c\cdot A = \rho\cdot g\cdot\frac{h}{2}\cdot h\cdot w = \frac{1}{2}\cdot\rho\cdot g\cdot w\cdot h^2 \qquad y' = y_c + \frac{I_{xx}}{A\cdot y_c} = \frac{h}{2} + \frac{\frac{w\cdot h^3}{12}}{w\cdot h\cdot\frac{h}{2}} = \frac{2}{3}\cdot h$

where h is the water depth and w is the gate width

For the gate, we can take moments about the hinge to obtain $\quad -F_{jet}\cdot h_{jet} + F_R\cdot(h - y') = -F_{jet}\cdot h_{jet} + F_R\cdot\frac{h}{3} = 0$

where h_{jet} is the height of the jet from the ground

Hence $\quad F_{jet} = \rho\cdot V^2\cdot\frac{\pi\cdot D^2}{4}\cdot h_{jet} = F_R\cdot\frac{h}{3} = \frac{1}{2}\cdot\rho\cdot g\cdot w\cdot h^2\cdot\frac{h}{3} \qquad V = \sqrt{\frac{2\cdot g\cdot w\cdot h^3}{3\cdot\pi\cdot D^2\cdot h_j}}$

For the first case (h = 0.5 m) $\quad V = \sqrt{\frac{2}{3\cdot\pi}\times 9.81\cdot\frac{m}{s^2}\times 0.5\cdot m\times(0.5\cdot m)^3\times\left(\frac{1}{0.01\cdot m}\right)^2\times\frac{1}{0.5\cdot m}} \qquad V = 51\,\frac{m}{s}$

For the second case (h = 0.25 m) $\quad V = \sqrt{\frac{2}{3\cdot\pi}\times 9.81\cdot\frac{m}{s^2}\times 0.5\cdot m\times(0.25\cdot m)^3\times\left(\frac{1}{0.01\cdot m}\right)^2\times\frac{1}{0.5\cdot m}} \qquad V = 18\,\frac{m}{s}$

For the first case (h = 0.6 m) $\quad V = \sqrt{\frac{2}{3\cdot\pi}\times 9.81\cdot\frac{m}{s^2}\times 0.5\cdot m\times(0.6\cdot m)^3\times\left(\frac{1}{0.01\cdot m}\right)^2\times\frac{1}{0.5\cdot m}} \qquad V = 67.1\,\frac{m}{s}$

Problem 4.68

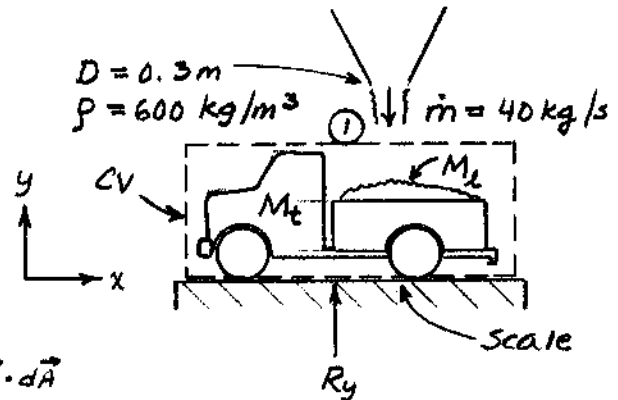
Given: Farmer purchases 675 kg of bulk grain. The grain is loaded into a pickup truck from a hopper as shown. Grain flow is terminated when the scale reading reaches the desired gross value.

Find: The true payload.

$D = 0.3\,m$
$\rho = 600\ kg/m^3$
$\dot{m} = 40\ kg/s$

Solution: Apply the y component of momentum equation using CV shown.

Basic equation:

$$F_{S_y} + F_{B_y} = \cancel{\frac{\partial}{\partial t}\int_{CV} v\,\rho\, d\forall}^{\simeq 0(2)} + \int_{CS} v\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: (1) No net pressure force; $F_{S_y} = R_y$
(2) Neglect v inside CV
(3) Uniform flow of grain at inlet section ①

Then

$$R_y - (M_t + M_\ell)g = v_1\{-|\dot{m}|\}$$

$$v_1 = -V_1 = -\frac{\dot{m}}{\rho A}$$

or

$$R_y = (M_t + M_\ell)g + \frac{\dot{m}^2}{\rho A} \quad \text{(indicated during grain flow)}$$

Loading is terminated when

$$\frac{R_y}{g} - M_t = M_\ell + \frac{\dot{m}^2}{\rho g A} = 675\ kg$$

Thus

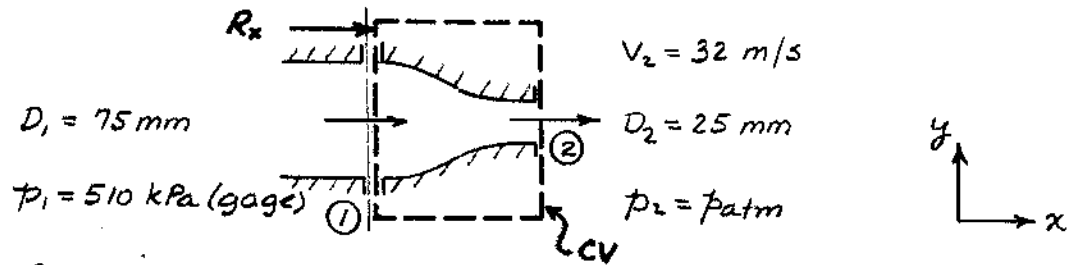
$$M_\ell = 675\,kg - \frac{\dot{m}^2}{\rho g A}$$

$$= 675\ kg - (40)^2\frac{kg^2}{s^2} \times \frac{m^3}{600\ kg} \times \frac{s^2}{9.81\ m} \times \frac{4}{\pi}\frac{1}{(0.3)^2\,m^2}$$

$$M_\ell = 671\ kg$$

M_ℓ

Problem 4.69

Given: Water flow through a fire hose and nozzle.

Find: (a) Coupling force, R_x
(b) Indicate if in tension or compression.

Solution: Apply continuity and x component of momentum equation to inertial CV shown; use gage pressures to cancel p_{atm}.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} \qquad (=0(1))$$

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A} \qquad (=0(4)),\ (=0(1))$$

Assumptions: (1) Steady flow
(2) Uniform flow at each section
(3) Incompressible flow
(4) $F_{B_x} = 0$

Then

$$0 = \{-|\rho V_1 A_1|\} + \{|\rho V_2 A_2|\} = -\rho V_1 A_1 + \rho V_2 A_2$$

$$V_1 = V_2 \frac{A_2}{A_1} = V_2 \left(\frac{D_2}{D_1}\right)^2 = 32 \frac{m}{s} \times \left(\frac{25 \text{ mm}}{75 \text{ mm}}\right)^2 = 3.56 \text{ m/s}$$

and

$$R_x + p_{1g} A_1 = u_1 \{-|\rho V_1 A_1|\} + u_2 \{|\rho V_2 A_2|\}$$

$$u_1 = V_1 \qquad u_2 = V_2$$

$$R_x = -p_{1g} A_1 - V_1 \rho V_1 A_1 + V_2 \rho V_2 A_2 = -p_{1g} A_1 + \rho V_2 A_2 (V_2 - V_1)$$

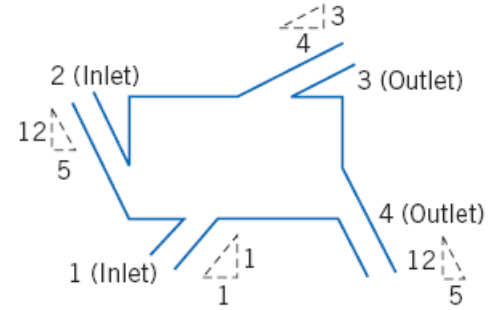
$$= -510 \times 10^3 \frac{N}{m^2} \times \frac{\pi}{4}(0.075)^2 m^2 + 999 \frac{kg}{m^3} \times 32 \frac{m}{s} \times \frac{\pi}{4}(0.025)^2 m^2 (32.0 - 3.56) \frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m}$$

$R_x = -1.81$ kN (i.e. force on CV is to the left) ← $\qquad R_x$

Thus the coupling must be in tension.

Problem 4.70

[3]

4.70 Obtain expressions for the rate of change in mass of the control volume shown, as well as the horizontal and vertical forces required to hold it in place, in terms of $p_1, A_1, V_1, p_2, A_2, V_2, p_3, A_3, V_3, p_4, A_4, V_4$, and the constant density ρ.

Given: Flow into and out of CV

Find: Expressions for rate of change of mass, and force

Solution:

Basic equations: Mass and momentum flux
$$\frac{\partial}{\partial t}\int_{CV}\rho\, dV\!\!\!/ + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0$$

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV\!\!\!/ + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

$$F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\,\rho\, dV\!\!\!/ + \int_{CS} v\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Incompressible flow 2) Uniform flow

For the mass equation
$$\frac{dM_{CV}}{dt} + \sum_{CS}\left(\rho\cdot\vec{V}\cdot\vec{A}\right) = \frac{dM_{CV}}{dt} + \rho\cdot\left(-V_1\cdot A_1 - V_2\cdot A_2 + V_3\cdot A_3 + V_4\cdot A_4\right) = 0$$

$$\frac{dM_{CV}}{dt} = \rho\cdot\left(V_1\cdot A_1 + V_2\cdot A_2 - V_3\cdot A_3 - V_4\cdot A_4\right)$$

For the x momentum
$$F_x + \frac{p_1\cdot A_1}{\sqrt{2}} + \frac{5}{13}\cdot p_2\cdot A_2 - \frac{4}{5}\cdot p_3\cdot A_3 - \frac{5}{13}\cdot p_4\cdot A_4 = 0 + \frac{V_1}{\sqrt{2}}\cdot\left(-\rho\cdot V_1\cdot A_1\right) + \frac{5}{13}\cdot V_2\cdot\left(-\rho\cdot V_2\cdot A_2\right) \ldots$$
$$+\frac{4}{5}\cdot V_3\cdot\left(\rho\cdot V_3\cdot A_3\right) + \frac{5}{13}\cdot V_3\cdot\left(\rho\cdot V_3\cdot A_3\right)$$

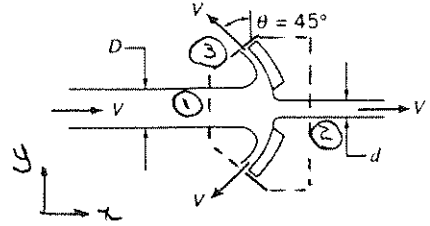
$$F_x = -\frac{p_1\cdot A_1}{\sqrt{2}} - \frac{5}{13}\cdot p_2\cdot A_2 + \frac{4}{5}\cdot p_3\cdot A_3 + \frac{5}{13}\cdot p_4\cdot A_4 + \rho\cdot\left(-\frac{1}{\sqrt{2}}\cdot V_1^2\cdot A_1 - \frac{5}{13}\cdot V_2^2\cdot A_2 + \frac{4}{5}\cdot V_3^2\cdot A_3 + \frac{5}{13}\cdot V_3^2\cdot A_3\right)$$

For the y momentum
$$F_y + \frac{p_1\cdot A_1}{\sqrt{2}} - \frac{12}{13}\cdot p_2\cdot A_2 - \frac{3}{5}\cdot p_3\cdot A_3 + \frac{12}{13}\cdot p_4\cdot A_4 = 0 + \frac{V_1}{\sqrt{2}}\cdot\left(-\rho\cdot V_1\cdot A_1\right) - \frac{12}{13}\cdot V_2\cdot\left(-\rho\cdot V_2\cdot A_2\right) \ldots$$
$$+\frac{3}{5}\cdot V_3\cdot\left(\rho\cdot V_3\cdot A_3\right) - \frac{12}{13}\cdot V_3\cdot\left(\rho\cdot V_3\cdot A_3\right)$$

$$F_y = -\frac{p_1\cdot A_1}{\sqrt{2}} + \frac{12}{13}\cdot p_2\cdot A_2 + \frac{3}{5}\cdot p_3\cdot A_3 - \frac{12}{13}\cdot p_4\cdot A_4 + \rho\cdot\left(-\frac{1}{\sqrt{2}}\cdot V_1^2\cdot A_1 - \frac{12}{13}\cdot V_2^2\cdot A_2 + \frac{3}{5}\cdot V_3^2\cdot A_3 - \frac{12}{13}\cdot V_3^2\cdot A_3\right)$$

Problem 4.71

Given: Circular dish with central orifice struck concentrically by water jet as shown

Find: (a) Expression for force needed to hold the dish in place
(b) Value of force for $V = 5$ m/s, $D = 100$ mm, and $d = 20$ mm

Plot: required force as a function of θ ($0 \le \theta \le 90°$) with d/D as a parameter.

Solution:

Apply the x component of the momentum equation to the inertial CV shown.

Basic equation: $F_{S_x} + \cancel{F_{B_x}} = \cancel{\frac{\partial}{\partial t}\int_{CV} u \rho \, dV} + \int_{CS} u(\rho \vec{V} \cdot d\vec{A})$ ($F_{B_x} = 0$ (2), $\partial/\partial t = 0$ (3))

Assumptions: (1) atmospheric pressure acts on all CV surfaces
(2) $F_{B_x} = 0$
(3) steady flow
(4) uniform flow at each section
(5) incompressible flow
(6) no change in jet speed on dish: $V_1 = V_2 = V_3 = V$

Then,

$$R_x = u_1\{-|\rho V_1 A_1|\} + u_2\{|\rho V_2 A_2|\} + u_3\{|\rho V_3 A_3|\}$$

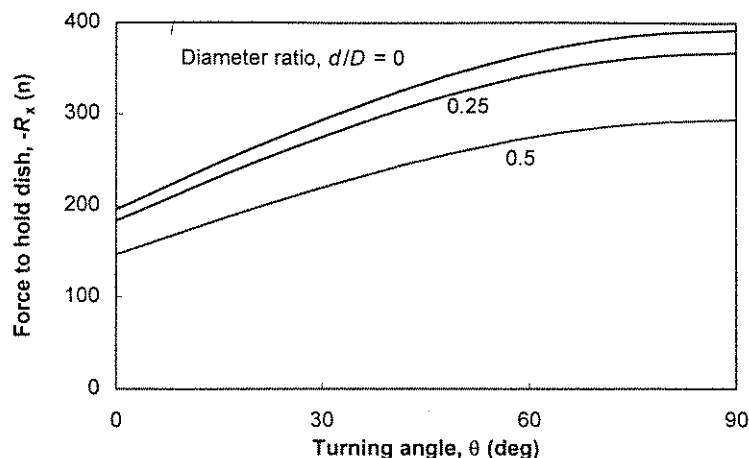
$u_1 = V \quad A_1 = \frac{\pi D^2}{4} \quad u_2 = V \quad A_2 = \frac{\pi d^2}{4} \quad u_3 = -V\sin\theta \quad A_3 = A_1 - A_2$

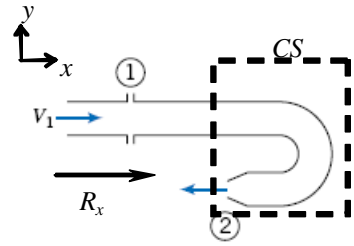
$$R_x = -\rho V^2 \frac{\pi D^2}{4} + \rho V^2 \frac{\pi d^2}{4} - \rho V^2 \sin\theta \frac{\pi}{4}(D^2 - d^2) = \rho V^2 \frac{\pi}{4}(1 + \sin\theta)(d^2 - D^2)$$

$$R_x = -\rho V^2 \frac{\pi D^2}{4}(1 + \sin\theta)\left[1 - \left(\frac{d}{D}\right)^2\right] \qquad R_x$$

Evaluating for $d = 25$ mm

$$R_x = -\frac{\pi}{4} \times 999\frac{\text{kg}}{\text{m}^3} \times (5)^2\frac{\text{m}^2}{\text{s}^2} \times (0.10)^2\text{m}^2\,(1 + \sin 45°)\left[1 - \left(\frac{25}{100}\right)^2\right]\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} = -314\text{ N} \qquad R_x$$

Since $R_x < 0$, it must be applied to the left. R_x is plotted as a function of θ for different values of d/D

Problem 4.72

4.72 Water is flowing steadily through the 180° elbow shown. At the inlet to the elbow the gage pressure is 15 psi. The water discharges to atmospheric pressure. Assume properties are uniform over the inlet and outlet areas: $A_1 = 4$ in.^{2}, $A_2 = 1$ in.^{2}, and $V_1 = 10$ ft/s. Find the horizontal component of force required to hold the elbow in place.

Given: Water flow through elbow

Find: Force to hold elbow

Solution:

Basic equation: Momentum flux in x direction for the elbow $F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure at exit 4) Uniform flow

Hence $R_x + p_{1g}\cdot A_1 = V_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) - V_2\cdot\left(\rho\cdot V_2\cdot A_2\right)$ $\qquad R_x = -p_{1g}\cdot A_1 - \rho\cdot\left(V_1^{\,2}\cdot A_1 + V_2^{\,2}\cdot A_2\right)$

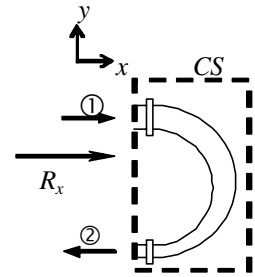
From continuity $V_2\cdot A_2 = V_1\cdot A_1$ so $V_2 = V_1\cdot\frac{A_1}{A_2}$ $\qquad V_2 = 10\cdot\frac{ft}{s}\cdot\frac{4}{1}$ $\qquad V_2 = 40\,\frac{ft}{s}$

Hence $R_x = -15\cdot\frac{lbf}{in^2}\times 4\cdot in^2 - 1.94\cdot\frac{slug}{ft^3}\times\left[\left(10\cdot\frac{ft}{s}\right)^2\cdot 4\cdot in^2 + \left(40\cdot\frac{ft}{s}\right)^2\cdot 1\cdot in^2\right]\times\left(\frac{1\cdot ft}{12\cdot in}\right)^2\times\frac{lbf\cdot s^2}{slug\,ft}$ $\qquad R_x = -86.9\cdot lbf$

The force is to the left: It is needed to hold the elbow on against the high pressure, plus it generates the large change in x momentum

Problem 4.73

[2]

4.73 A 180° elbow takes in water at an average velocity of 0.8 m/s and a pressure of 350 kPa (gage) at the inlet, where the diameter is 0.2 m. The exit pressure is 75 kPa, and the diameter is 0.04 m. What is the force required to hold the elbow in place?

Given: Water flow through elbow

Find: Force to hold elbow

Solution:

Basic equation: Momentum flux in x direction for the elbow $F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Uniform flow

Hence $R_x + p_{1g}\cdot A_1 + p_{2g}\cdot A_2 = V_1\cdot(-\rho\cdot V_1\cdot A_1) - V_2\cdot(\rho\cdot V_2\cdot A_2)$ $\qquad R_x = -p_{1g}\cdot A_1 - p_{2g}\cdot A_2 - \rho\cdot\left(V_1^2\cdot A_1 + V_2^2\cdot A_2\right)$

From continuity $V_2\cdot A_2 = V_1\cdot A_1$ so $V_2 = V_1\cdot\frac{A_1}{A_2} = V_1\cdot\left(\frac{D_1}{D_2}\right)^2$ $\qquad V_2 = 0.8\cdot\frac{m}{s}\cdot\left(\frac{0.2}{0.04}\right)^2$ $\qquad V_2 = 20\frac{m}{s}$

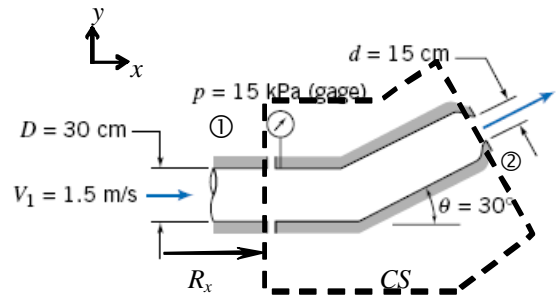
Hence
$$R_x = -350\times10^3\cdot\frac{N}{m^2}\times\frac{\pi\cdot(0.2\cdot m)^2}{4} - 75\times10^3\cdot\frac{N}{m^2}\times\frac{\pi\cdot(0.04\cdot m)^2}{4}\ ...$$
$$+ -1000\cdot\frac{kg}{m^3}\times\left[\left(0.8\cdot\frac{m}{s}\right)^2\times\frac{\pi\cdot(0.2\cdot m)^2}{4} + \left(20\cdot\frac{m}{s}\right)^2\times\frac{\pi\cdot(.04\cdot m)^2}{4}\right]\times\frac{N\cdot s^2}{kg\cdot m}$$

$R_x = -11.6\cdot kN$

The force is to the left: It is needed to hold the elbow on against the high pressures, plus it generates the large change in x momentum

Problem 4.74

[2]

4.74 Water flows steadily through the nozzle shown, discharging to atmosphere. Calculate the horizontal component of force in the flanged joint. Indicate whether the joint is in tension or compression.

Given: Water flow through nozzle

Find: Force to hold nozzle

Solution:

Basic equation: Momentum flux in x direction for the elbow $F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Uniform flow

Hence $R_x + p_{1g}\cdot A_1 + p_{2g}\cdot A_2 = V_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) + V_2\cdot\cos(\theta)\cdot\left(\rho\cdot V_2\cdot A_2\right)$ $\qquad R_x = -p_{1g}\cdot A_1 + \rho\cdot\left(V_2^2\cdot A_2\cdot\cos(\theta) - V_1^2\cdot A_1\right)$

From continuity $V_2\cdot A_2 = V_1\cdot A_1$ so $V_2 = V_1\cdot\frac{A_1}{A_2} = V_1\cdot\left(\frac{D_1}{D_2}\right)^2$ $\qquad V_2 = 1.5\cdot\frac{m}{s}\cdot\left(\frac{30}{15}\right)^2$ $\qquad V_2 = 6\cdot\frac{m}{s}$

Hence $R_x = -15\times 10^3\cdot\frac{N}{m^2}\times\frac{\pi\cdot(0.3\cdot m)^2}{4} + 1000\cdot\frac{kg}{m^3}\times\left[\left(6\cdot\frac{m}{s}\right)^2\times\frac{\pi\cdot(0.15\cdot m)^2}{4}\cdot\cos(30\cdot deg) - \left(1.5\cdot\frac{m}{s}\right)^2\times\frac{\pi\cdot(.3\cdot m)^2}{4}\right]\times\frac{N\cdot s^2}{kg\cdot m}$

$R_x = -668\cdot N$ The joint is in tension: It is needed to hold the elbow on against the high pressure, plus it generates the large change in x momentum

Problem 4.75

Given: Two-dimensional square bend shown is a segment of a larger channel, lies in horizontal plane.

$U = 7.5\ m/s$, $h = w = 75.5\ mm$

$p_1 = 170\ kPa(abs)$, $p_2 = 130\ kPa(abs)$

$v_{max} = 2v_{min}$; $v_{min} = 5.0\ m/s$ (from Problem 4.25)

Find: Force required to hold the bend in place.

Solution:

Basic equation: $\vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V}\rho\, d\forall + \int_{CS} \vec{V}(\rho\vec{V}\cdot d\vec{A})$ $\quad = 0(1)$

Assumptions: (1) steady flow
(2) $F_{B_x} = F_{B_y} = 0$
(3) incompressible flow
(4) atmospheric pressure acts on outside surfaces.

The x-momentum equation becomes

$R_x + P_1A_1 + F_{B_x} = \int_{CS} u(\rho\vec{V}\cdot d\vec{A}) = U\{-|\rho U A_1|\}$ $\quad (F_{B_x} = 0(2))$

$R_x = -P_1A_1 - \rho U^2 A_1 = -h^2(P_1 + \rho U^2)$

$R_x = -(0.0755)^2 m^2\left[(170-101)10^3 \frac{N}{m^2} + 999\frac{kg}{m^3}\times(7.5)^2\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}\right] = -714\ N$

The y-momentum equation becomes

$R_y - P_2A_2 + F_{B_y} = \int_{CS} v(\rho\vec{V}\cdot d\vec{A})$ $\quad (F_{B_y} = 0(2))$

$$v_2 = V_2 = v_{max} - (v_{max} - v_{min})\frac{x}{h} = 2v_{min} - v_{min}\frac{x}{h} = v_{min}\left(2 - \frac{x}{h}\right)$$

$R_y - P_2A_2 = \int_0^h v_{min}\left(2-\frac{x}{h}\right)\rho v_{min}\left(2-\frac{x}{h}\right)h\,dx$

$R_y = P_2A_2 + \rho v_{min}^2 h\int_0^h\left(4 - 4\frac{x}{h} + \frac{x^2}{h^2}\right)dx$

$\quad = P_2A_2 + \rho v_{min}^2 h\left[4x - 2\frac{x^2}{h} + \frac{x^3}{3h^2}\right]_0^h$

$R_y = P_2A_2 + \rho v_{min}^2 h\left[4h - 2h + \frac{h}{3}\right] = P_2A_2 + \frac{7}{3}\rho v_{min}^2 h^2$

$R_y = h^2\left(P_2 + \frac{7}{3}\rho v_{min}^2\right)$

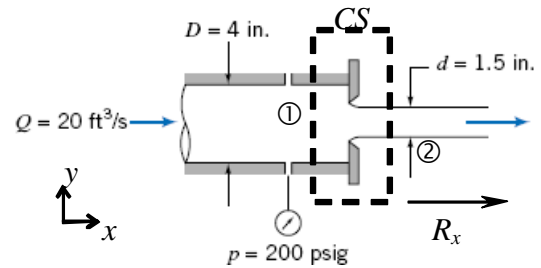
$\quad = (0.0755)^2 m^2\left[(130-101)10^3\frac{N}{m^2} + \frac{7}{3}\times 999\frac{kg}{m^3}\times(5.0)^2\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}\right]$

$R_y = 498\ N$

$\therefore \vec{R} = -714\hat{i} + 498\hat{j}\ N$ $\quad \vec{R}$

Problem 4.76

[2]

4.76 A flat plate orifice of 2 in. diameter is located at the end of a 4-in. diameter pipe. Water flows through the pipe and orifice at 20 ft^3/s. The diameter of the water jet downstream from the orifice is 1.5 in. Calculate the external force required to hold the orifice in place. Neglect friction on the pipe wall.

Given: Water flow through orifice plate

Find: Force to hold plate

Solution:

Basic equation: Momentum flux in x direction for the elbow
$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV\!\!\!\!/ + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Uniform flow

Hence $\quad R_x + p_{1g}\cdot A_1 - p_{2g}\cdot A_2 = V_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) + V_2\cdot\left(\rho\cdot V_2\cdot A_2\right) \qquad R_x = -p_{1g}\cdot A_1 + \rho\cdot\left(V_2^{\,2}\cdot A_2 - V_1^{\,2}\cdot A_1\right)$

From continuity $\quad Q = V_1\cdot A_1 = V_2\cdot A_2$

so $\quad V_1 = \frac{Q}{A_1} = 20\cdot\frac{ft^3}{s}\times\frac{4}{\pi\cdot\left(\frac{1}{3}\cdot ft\right)^2} = 229\cdot\frac{ft}{s}$ $\quad$ and $\quad V_2 = V_1\cdot\frac{A_1}{A_2} = V_1\cdot\left(\frac{D}{d}\right)^2 = 229\cdot\frac{ft}{s}\times\left(\frac{4}{1.5}\right)^2 = 1628\cdot\frac{ft}{s}$

NOTE: problem has an error: Flow rate should be 2 ft^3/s not 20 ft^3/s! We will provide answers to both

Hence $\quad R_x = -200\cdot\frac{lbf}{in^2}\times\frac{\pi\cdot(4\cdot in)^2}{4} + 1.94\cdot\frac{slug}{ft^3}\times\left[\left(1628\cdot\frac{ft}{s}\right)^2\times\frac{\pi\cdot(1.5\cdot in)^2}{4} - \left(229\cdot\frac{ft}{s}\right)^2\times\frac{\pi\cdot(4\cdot in)^2}{4}\right]\times\left(\frac{1\cdot ft}{12\cdot in}\right)^2\times\frac{lbf\cdot s^2}{slug\,ft}$

$R_x = 51707\cdot lbf$

With more realistic velocities

Hence $\quad R_x = -200\cdot\frac{lbf}{in^2}\times\frac{\pi\cdot(4\cdot in)^2}{4} + 1.94\cdot\frac{slug}{ft^3}\times\left[\left(163\cdot\frac{ft}{s}\right)^2\times\frac{\pi\cdot(1.5\cdot in)^2}{4} - \left(22.9\cdot\frac{ft}{s}\right)^2\times\frac{\pi\cdot(4\cdot in)^2}{4}\right]\times\left(\frac{1\cdot ft}{12\cdot in}\right)^2\times\frac{lbf\cdot s^2}{slug\cdot ft}$

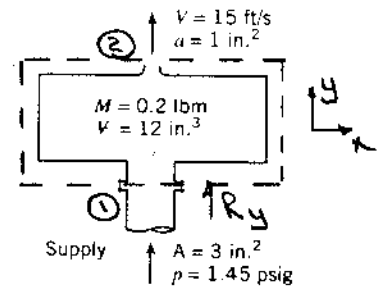
$R_x = -1970\cdot lbf$

Problem 4.77

Given: Spray system, of mass $M = 0.200$ lbm and internal volume $\forall = 12\ in^3$ operates under steady state conditions shown.

Find: the vertical force exerted on the supply pipe by the spray system

Solution:

Apply the y component of the momentum equation to the fixed control volume shown.

Basic Equation:

$$F_{Sy} + F_{By} = \frac{\partial}{\partial t}\int_{CV} v \rho \, d\forall + \int_{CS} v \rho \vec{V} \cdot d\vec{A} \quad (1)$$

(the unsteady term is marked $= 0(1)$)

Assumptions:
(1) steady flow
(2) incompressible flow
(3) uniform flow at each section
(4) calculation of surface forces is simplified through use of gage pressures

From continuity, $0 = \frac{\partial}{\partial t}\int \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$, for given conditions

$$0 = -|\rho V_1 A_1| + |\rho V_2 A_2| \quad \text{and} \quad V_1 = V_2 \frac{A_2}{A_1} = V \frac{a}{A}$$

The momentum flux is

$$\int_{CS} v \rho \vec{V} \cdot d\vec{A} = v_1\{-|\rho V_1 A_1|\} + v_2\{|\rho V_2 A_2|\} = V_1(-\rho V_1 A_1) + V(\rho V a)$$

$$= V\frac{a}{A}(-\rho V a) + V(\rho V a) = \rho V^2 a\left(1 - \frac{a}{A}\right)$$

Then from eq (1) we can write

$$R_y + p_{1g}A - \rho \forall g - Mg = \rho V^2 a\left(1 - \frac{a}{A}\right). \quad \text{Solving for } R_y,$$

$$R_y = -p_{1g}A + \rho \forall g + Mg + \rho V^2 a\left(1 - \frac{a}{A}\right)$$

$$= -1.45 \frac{lbf}{in^2} \times 3\,in^2 + 1.94 \frac{slug}{ft^3} \times 12\,in^3 \times 32.2 \frac{ft}{s^2} \times \frac{ft^3}{1728\,in^3} \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

$$+ 0.2\,lbm \times 32.2 \frac{ft}{s^2} \times \frac{slug}{32.2\,lbm} \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

$$+ 1.94 \frac{slug}{ft^3} \times (15)^2 \frac{ft^2}{s^2} \times 1\,in^2 \times \frac{ft^2}{144\,in^2} \times \frac{lbf \cdot s^2}{slug \cdot ft}\left(1 - \frac{1\,in^2}{3\,in^2}\right)$$

$$R_y = -1.70 \text{ lbf}$$

The force of the spray system on the supply pipe is

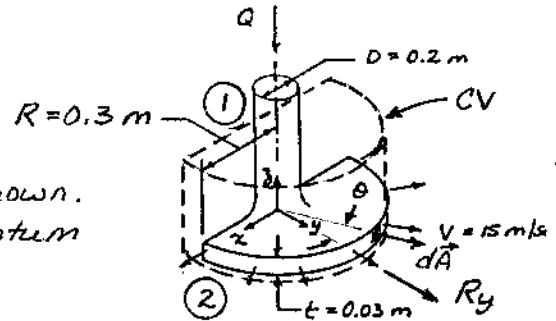
$$K_y = -R_y = 1.70 \text{ lbf (upward)}$$

K_y

Problem 4.78

Given: Flow through semi-circular nozzle, as shown.

Find: (a) Volume flow rate
(b) y-component of force required to hold in place

Solution: Choose CV and coordinates shown. Apply continuity and momentum equation in y-direction.

Basic equations: $Q = \int_A \vec{V} \cdot d\vec{A}$

$$F_{Sy} + F_{By} = \frac{\partial}{\partial t}\int_{CV} v \rho \, d\forall + \int_{CS} v \rho \vec{V} \cdot d\vec{A}$$

($F_{By} = 0$ (2); $\frac{\partial}{\partial t} = 0$ (3))

Assumptions: (1) Flow uniform across exit section
(2) $F_{By} = 0$
(3) Steady flow

At section ②, $\vec{V} \cdot d\vec{A} = VRt\,d\theta$, since flow out of CV. Then

$$Q = \int_{-\pi/2}^{\pi/2} VRt\,d\theta = VRt[\theta]_{-\pi/2}^{\pi/2} = VRt\pi$$

$$Q = 15\frac{m}{s} \times 0.3\,m \times 0.03\,m \times \pi = 0.424\ m^3/s$$

Q

From momentum

$$R_y = \int_{CS} v \rho \vec{V} \cdot d\vec{A} = \int_{A_1} v_1 \{-|\rho V_1 dA_1|\} + \int_{A_2} v_2 \{+|\rho V_2 dA_2|\}$$

with $\quad v_1 = 0 \quad\quad v_2 = V\cos\theta$

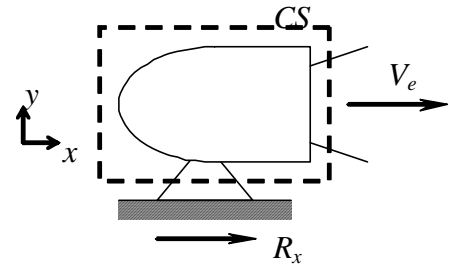
$$R_y = \int_{-\pi/2}^{\pi/2} V\cos\theta \,\rho V R t\,d\theta = \rho V^2 R t[\sin\theta]_{-\pi/2}^{\pi/2} = 2\rho V^2 R t$$

$$R_y = 2 \times 999\frac{kg}{m^3} \times (15)^2\frac{m^2}{s^2} \times 0.3\,m \times 0.03\,m \times \frac{N \cdot s^2}{kg \cdot m} = 4.05\ kN$$

R_y

Problem 4.79

[2]

4.79 At rated thrust, a liquid-fueled rocket motor consumes 80 kg/s of nitric acid as oxidizer and 32 kg/s of aniline as fuel. Flow leaves axially at 180 m/s relative to the nozzle and at 110 kPa. The nozzle exit diameter is $D = 0.6$ m. Calculate the thrust produced by the motor on a test stand at standard sea-level pressure.

Given: Data on rocket motor

Find: Thrust produced

Solution:

Basic equation: Momentum flux in x direction for the elbow

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Neglect change of momentum within CV 3) Uniform flow

Hence $\quad R_x - p_{eg}\cdot A_e = V_e\cdot\left(\rho_e\cdot V_e\cdot A_e\right) = m_e\cdot V_e \qquad R_x = p_{eg}\cdot A_e + m_e\cdot V_e$

where p_{eg} is the exit pressure (gage), m_e is the mass flow rate at the exit (software cannot render dot over m!) and V_e is the xit velocity

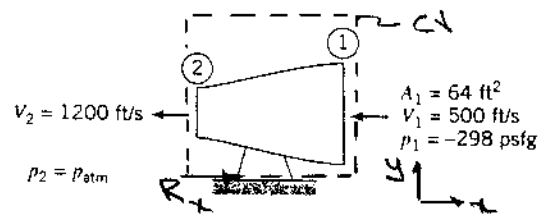
For the mass flow rate $\quad m_e = m_{nitricacid} + m_{aniline} = 80\cdot\frac{kg}{s} + 32\cdot\frac{kg}{s} \qquad m_e = 112\cdot\frac{kg}{s}$

Hence $\quad R_x = (110 - 101)\times 10^3\cdot\frac{N}{m^2}\times\frac{\pi\cdot(0.6\cdot m)^2}{4} + 112\cdot\frac{kg}{s}\times 180\cdot\frac{m}{s}\times\frac{N\cdot s^2}{kg\cdot m} \qquad R_x = 22.7\,kN$

Problem 4.80

[2]

Given: Jet engine on test stand. Fuel enters vertically at rate

$\dot{m}_{fuel} = 0.02\,\dot{m}_{air}$

Find: (a) Air flow rate
(b) Estimate of engine thrust.

Solution:

Apply x-component of the momentum equation to CV shown

Basic equations: $F_{S_x} + \cancel{F_{B_x}} = \cancel{\frac{\partial}{\partial t}\int_{CV} u \rho\, d\forall} + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$ ($=0(1)$, $=0(2)$)

$\dot{m}_{air} = \rho_1 V_1 A_1 \quad , \quad \rho = p/RT$

Assumptions: (1) $F_{B_x} = 0$
(2) steady flow
(3) uniform flow at inlet and outlet sections.
(4) air behaves as ideal gas; $T = 70°F$
(5) fuel enters vertically (given).

$$\rho_1 = \frac{p_1}{RT_1} = \left(14.7\frac{lbf}{in^2} \times 144\frac{in^2}{ft^2} - 298\frac{lbf}{ft^2}\right) \times \frac{lbm\,°R}{53.3\,ft\cdot lbf} \times \frac{1}{530°R} = 0.0644\,\frac{lbm}{ft^3}$$

$$\dot{m}_{air} = \rho_1 V_1 A_1 = 0.0644\,\frac{lbm}{ft^3} \times 500\frac{ft}{s} \times 64\,ft^2 = 2060\ lbm/s \qquad \dot{m}$$

From the x-momentum equation

$$R_x - p_{1g}A_1 + \cancel{p_{2g}A_2} = u_1\{-\dot{m}_1\} + u_2\{\dot{m}_2\} + \cancel{u_f}\{-\dot{m}_f\}$$ ($=0$, $=0(5)$)

$u_1 = -V_1 \quad , u_2 = -V_2 \quad , \quad \dot{m}_2 = \dot{m}_1 + \dot{m}_f$

Also thrust $T = K_x$ (force of engine on surroundings) $= -R_x$

So

$$-T - p_{1g}A_1 = \dot{m}_1 V_1 - \dot{m}_2 V_2 = \dot{m}_1 V_1 - (1.02\dot{m}_1)V_2$$

$$T = \dot{m}_1(1.02V_2 - V_1) - p_{1g}A_1$$

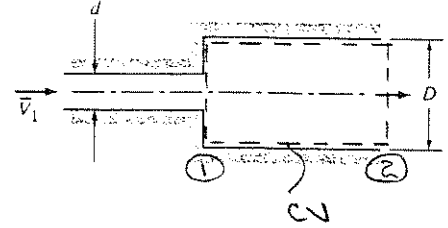
$$T = 2060\frac{lbm}{s}\left[1.02 \times 1200\frac{ft}{s} - 500\frac{ft}{s}\right] \times \frac{slug}{32.2\,lbm} \times \frac{lbf\cdot s^2}{ft\cdot slug} - \left(-298\frac{lbf}{ft^2}\right)64\,ft^2$$

$$T = 65{,}400\ lbf \qquad T$$

Problem 4.81

Given: Incompressible, frictionless flow through a sudden expansion as shown.

Show: Pressure rise, $\Delta p = p_2 - p_1$, is given by

$$\frac{\Delta p}{\frac{1}{2}\rho \bar{V}_1^2} = 2\left(\frac{d}{D}\right)^2\left[1-\left(\frac{d}{D}\right)^2\right]$$

Plot: the nondimensional pressure rise vs d/D to determine the optimum d/D and corresponding nondimensional pressure rise

Solution:

Apply x component of momentum equation, using fixed CV shown

Basic equation: $F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u(\rho \vec{V} \cdot d\vec{A})$ (with $F_{B_x} = 0$ (2), $\frac{\partial}{\partial t} = 0$ (3))

Assumptions: (1) no friction, so surface force due to pressure only
(2) $F_{B_x} = 0$
(3) steady flow (4) incompressible flow (given)
(5) uniform flow at sections ① and ②
(6) uniform pressure p_1 on vertical surface of expansion

Then,

$$p_1 A_2 - p_2 A_2 = u_1\{-|\rho \bar{V}_1 A_1|\} + u_2\{|\rho \bar{V}_2 A_2|\} \qquad u_1 = \bar{V}_1, \; u_2 = \bar{V}_2$$

From continuity for uniform flow, $\dot{m} = \rho A_1 \bar{V}_1 = \rho A_2 \bar{V}_2$; $\bar{V}_2 = \bar{V}_1 \frac{A_1}{A_2}$

Thus,

$$p_2 - p_1 = \rho \bar{V}_1 \frac{A_1}{A_2} V_1 - \rho \bar{V}_1 \frac{A_1}{A_2} V_2 = \rho \bar{V}_1 \frac{A_1}{A_2}(\bar{V}_1 - \bar{V}_2)$$

$$p_2 - p_1 = \rho \bar{V}_1^2 \frac{A_1}{A_2}\left(1 - \frac{\bar{V}_2}{\bar{V}_1}\right) = \rho \bar{V}_1^2 \frac{A_1}{A_2}\left(1 - \frac{A_1}{A_2}\right)$$

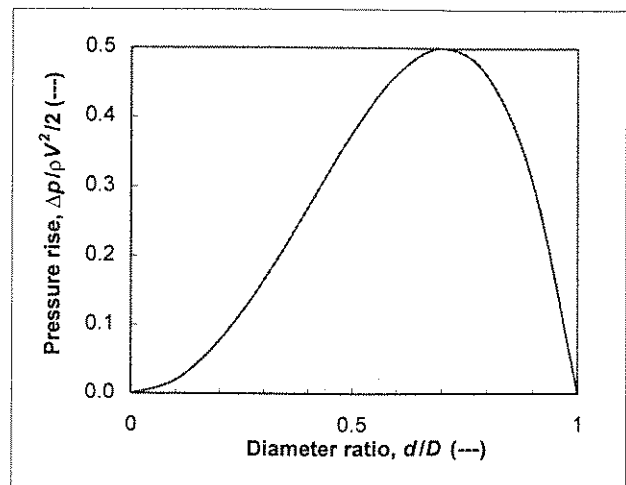
and

$$\frac{p_2 - p_1}{\frac{1}{2}\rho \bar{V}_1^2} = 2\frac{A_1}{A_2}\left(1 - \frac{A_1}{A_2}\right) = 2\left(\frac{d}{D}\right)^2\left[1 - \left(\frac{d}{D}\right)^2\right]$$

Q.E.D.

From the plot below we see that $\frac{\Delta p}{\frac{1}{2}\rho \bar{V}_1^2}$ has an optimum value of ≈ 0.5 at $d/D = 0.70$

Note: As expected
- for $d = D$, $\Delta p = 0$ for straight pipe
- for $\frac{d}{D} \to 0$, $\Delta p = 0$ for free jet

Also note that the location of section ② would have to be chosen with care to make assumption (5) reasonable.

Problem 4.82

[2]

4.82 A free jet of water with constant cross-section area 0.005 m^2 is deflected by a hinged plate of length 2 m supported by a spring with spring constant $k = 1$ N/m and uncompressed length $x_0 = 1$ m. Find and plot the deflection angle θ as a function of jet speed V. What jet speed has a deflection of 10°?

Given: Data on flow and system geometry

Find: Deflection angle as a function of speed; jet speed for 10º deflection

Solution:

The given data is $\rho = 999\cdot\frac{kg}{m^3}$ $\quad A = 0.005\cdot m^2$ $\quad L = 2\cdot m$ $\quad k = 1\cdot\frac{N}{m}$ $\quad x_0 = 1\cdot m$

Governing equation:

y -momentum

$$F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\,\rho\,d\forall + \int_{CS} v\,\rho\,\vec{V}\cdot d\vec{A} \qquad (4.18b)$$

Applying this to the current system in the vertical direction

$$F_{spring} = V\cdot\sin(\theta)\cdot(\rho\cdot V\cdot A)$$

But $\quad F_{spring} = k\cdot x = k\cdot\left(x_0 - L\cdot\sin(\theta)\right)$

Hence $\quad k\cdot\left(x_0 - L\cdot\sin(\theta)\right) = \rho\cdot V^2\cdot A\cdot\sin(\theta)$

Solving for θ $\quad \theta = \mathrm{asin}\left(\frac{k\cdot x_0}{k\cdot L + \rho\cdot A\cdot V^2}\right)$

For the speed at which θ = 10º, solve $\quad V = \sqrt{\frac{k\cdot\left(x_0 - L\cdot\sin(\theta)\right)}{\rho\cdot A\cdot\sin(\theta)}}$ $\quad V = \sqrt{\frac{1\cdot\frac{N}{m}\cdot(1 - 2\cdot\sin(10\cdot deg))\cdot m}{999\cdot\frac{kg}{m^3}\cdot 0.005\cdot m^2\cdot\sin(10\cdot deg)}\cdot\frac{kg\cdot m}{N\cdot s^2}}$ $\quad V = 0.867\frac{m}{s}$

The deflection is plotted in the corresponding *Excel* workbook, where the above velocity is obtained using *Goal Seek*

Problem 4.82

4.82 A free jet of water with constant cross-section area 0.005 m^2 is deflected by a hinged plate of length 2 m supported by a spring with spring constant $k = 1$ N/m and uncompressed length $x_0 = 1$ m. Find and plot the deflection angle θ as a function of jet speed V. What jet speed has a deflection of 10°?

Given: Data on flow and system geometry

Find: Deflection angle as a function of speed; jet speed for 10° deflection

Solution:

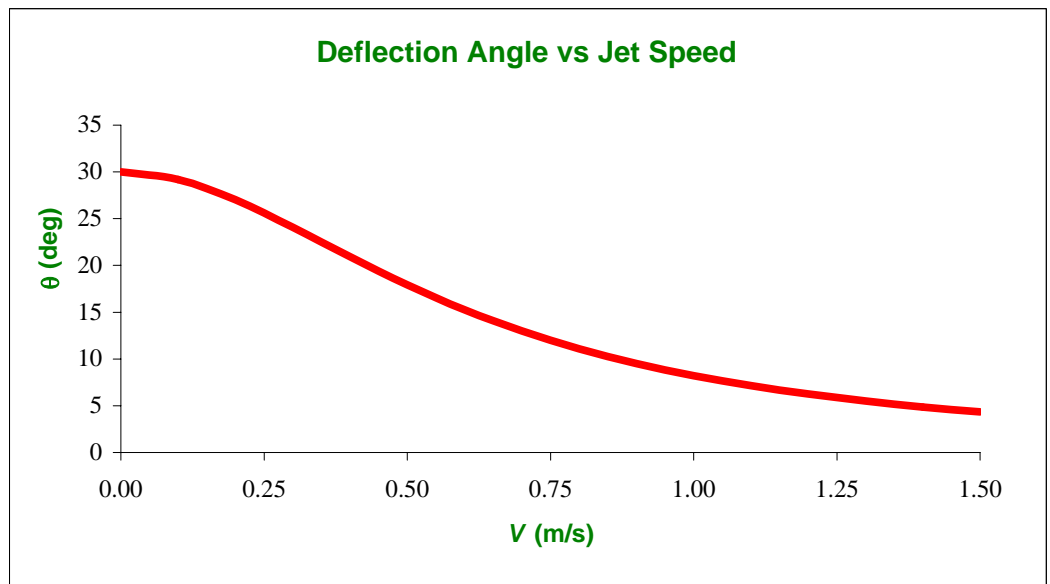
Solving for θ

$$\theta = \operatorname{asin}\left(\frac{k \cdot x_0}{k \cdot L + \rho \cdot A \cdot V^2}\right)$$

ρ =	999	kg/m³
x_o =	1	m
L =	2	m
k =	1	N/m
A =	0.005	m²

To find when θ = 10°, use *Goal Seek*

V (m/s)	θ (°)
0.867	10

V (m/s)	θ (°)
0.0	30.0
0.1	29.2
0.2	27.0
0.3	24.1
0.4	20.9
0.5	17.9
0.6	15.3
0.7	13.0
0.8	11.1
0.9	9.52
1.0	8.22
1.1	7.14
1.2	6.25
1.3	5.50
1.4	4.87
1.5	4.33

Problem 4.83

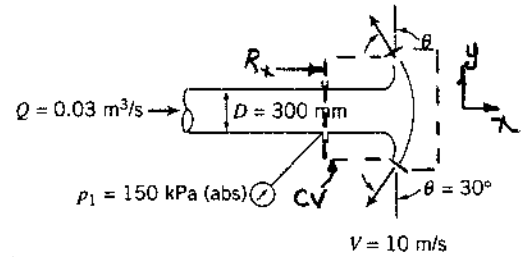
Given: Conical spray head discharging water, as shown.

Find: (a) Thickness of spray sheet at $R = 400$ mm radius.
(b) Axial force exerted on supply pipe.

Solution: Apply continuity and the x component of the momentum equation, using the CV, CS shown.

Basic equation:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

($F_{B_x} = 0$ (1); $\frac{\partial}{\partial t} = 0$ (2))

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow
(3) Incompressible flow
(4) Uniform flow at each section
(5) Use gage pressure to cancel p_{atm}

From continuity,

$$V_1 = \frac{Q}{A_1} = \frac{4Q}{\pi D_1^2} = \frac{4}{\pi}\times 0.03\frac{m^3}{sec}\times\frac{1}{(0.3)^2 m^2} = 0.424\ m/s$$

Assume velocity in jet sheet is constant at $V = 10$ m/s. Then

$$Q = 2\pi R t V\ ;\ t = \frac{Q}{2\pi R V} = \frac{1}{2\pi}\times 0.03\frac{m^3}{s}\times\frac{1}{0.4\ m}\times\frac{s}{10\ m}\times 1000\frac{mm}{m} = 1.19\ mm$$

t

From momentum,

$$R_x + p_{1g}A_1 = u_1\{-\rho Q\} + u_2\{+\rho Q\}$$

$$u_1 = V_1 \qquad u_2 = -V\sin\theta$$

$$R_x + p_{1g}A_1 = -(V_1 + V\sin\theta)\rho Q$$

or

$$R_x = -p_{1g}A_1 - (V_1 + V\sin\theta)\rho Q$$

$$= -(150-101)10^3\frac{N}{m^2}\times\frac{\pi}{4}(0.3)^2 m^2 - (0.424 + 10\sin 30°)\frac{m}{s}\times 999\frac{kg}{m^3}\times 0.03\frac{m^3}{s}\times\frac{N\cdot s^2}{kg\cdot m}$$

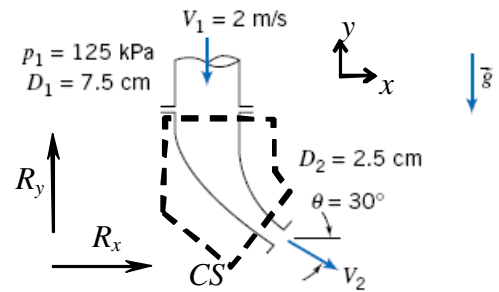
$$R_x = -3.63\ kN$$

But R_x is force on CV; force on supply pipe is K_x,

$$K_x = -R_x = 3.63\ kN \text{ (to the right)}$$

K_x

Problem 4.84

4.84 A curved nozzle assembly that discharges to the atmosphere is shown. The nozzle mass is 4.5 kg and its internal volume is 0.002 m³. The fluid is water. Determine the reaction force exerted by the nozzle on the coupling to the inlet pipe.

Given: Data on nozzle assembly

Find: Reaction force

Solution:

Basic equation: Momentum flux in x and y directions

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

$$F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\,\rho\,dV + \int_{CS} v\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow CV 3) Uniform flow

For x momentum $\quad R_x = V_2\cdot\cos(\theta)\cdot(\rho\cdot V_2\cdot A_2) = \rho\cdot V_2^2\cdot\frac{\pi\cdot D_2^2}{4}\cdot\cos(\theta)$

From continuity $\quad A_1\cdot V_1 = A_2\cdot V_2 \qquad V_2 = V_1\cdot\frac{A_1}{A_2} = V_1\cdot\left(\frac{D_1}{D_2}\right)^2 \qquad V_2 = 2\cdot\frac{m}{s}\times\left(\frac{7.5}{2.5}\right)^2 \qquad V_2 = 18\frac{m}{s}$

Hence $\quad R_x = 1000\cdot\frac{kg}{m^3}\times\left(18\cdot\frac{m}{s}\right)^2\times\frac{\pi}{4}\times(0.025\cdot m)^2\times\cos(30\cdot deg)\times\frac{N\cdot s^2}{kg\cdot m} \qquad R_x = 138\,N$

For y momentum $\quad R_y - p_1\cdot A_1 - W - \rho\cdot Vol\cdot g = -V_1\cdot(-\rho\cdot V_1\cdot A_1) - V_2\cdot\sin(\theta)\cdot(\rho\cdot V_2\cdot A_2)$

$$R_y = p_1\cdot\frac{\pi\cdot D_1^2}{4} + W + \rho\cdot Vol\cdot g + \frac{\rho\cdot\pi}{4}\cdot\left(V_1^2\cdot D_1^2 - V_2^2\cdot D_2^2\cdot\sin(\theta)\right)$$

where $\quad W = 4.5\cdot kg\times 9.81\cdot\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} \qquad W = 44.1\,N \qquad Vol = 0.002\cdot m^3$

Hence $\quad R_y = 125\times 10^3\cdot\frac{N}{m^2}\times\frac{\pi\cdot(0.075\cdot m)^2}{4} + 44.1\cdot N + 1000\cdot\frac{kg}{m^3}\times 0.002\cdot m^3\times 9.81\cdot\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}\ ...$

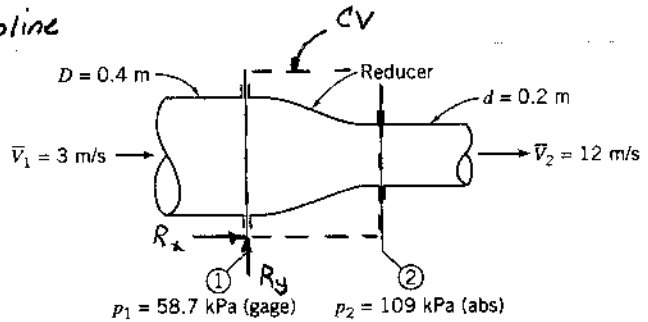
$$+1000\cdot\frac{kg}{m^3}\times\frac{\pi}{4}\times\left[\left(2\cdot\frac{m}{s}\right)^2\times(0.075\cdot m)^2 - \left(18\cdot\frac{m}{s}\right)^2\times(0.025\cdot m)^2\times\sin(30\cdot deg)\right]\times\frac{N\cdot s^2}{kg\cdot m}$$

$$R_y = 554\,N$$

Problem 4.85

Given: Flow through reducer in gasoline piping system, as shown.

$M = 25\ kg \quad V\!\!\!/ = 0.2\ m^3$

Find: Force needed to hold reducer in place.

Solution: Apply the x and y components of the momentum equation, using the CV and coordinates shown. Use gage pressures to cancel p_{atm}.

Basic equations:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, dV\!\!\!/ + \int_{CS} u\rho \vec{V}\cdot d\vec{A} \qquad (F_{B_x}=0\ (1),\ \tfrac{\partial}{\partial t}=0\ (2))$$

$$F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\rho\, dV\!\!\!/ + \int_{CS} v\rho \vec{V}\cdot d\vec{A} \qquad (=0\ (2))$$

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow
(3) Uniform flow at each section
(4) Incompressible flow, $SG = 0.72$ {Table A.2, Appendix A}

From the x component of momentum,

$$R_x + p_{1g}A_1 - p_{2g}A_2 = u_1\{-|\rho V_1 A_1|\} + u_2\{+|\rho V_2 A_2|\} = (V_2 - V_1)\rho V_1 A_1$$

$$u_1 = V_1 \qquad u_2 = V_2$$

$$R_x = p_{2g}A_2 - p_{1g}A_1 + (\bar{V}_2 - \bar{V}_1)\rho \bar{V}_1 A_1 \qquad \text{Note } \rho = SG\,\rho_{H_2O}$$

$$= (109-101)10^3\frac{N}{m^2}\times\frac{\pi}{4}(0.2)^2 m^2 - 58.7\times 10^3\frac{N}{m^2}\times\frac{\pi}{4}(0.4)^2 m^2$$
$$+ (12-3)\frac{m}{s}\times(0.72)1000\frac{kg}{m^3}\times 3\frac{m}{s}\times\frac{\pi}{4}(0.4)^2 m^2\times\frac{N\cdot s^2}{kg\cdot m}$$

$R_x = -4.68\ kN$ (force must be applied to left) $\qquad R_x$

From the y component of momentum,

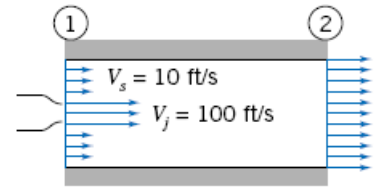
$$R_y - Mg - \rho g V\!\!\!/ = v_1\{-|\rho V_1 A_1|\} + v_2\{+|\rho V_2 A_2|\} \qquad (v_1 = 0,\ v_2 = 0)$$

$$R_y = Mg + \rho g V\!\!\!/$$

$$= 25\ kg\times 9.81\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} + (0.72)1000\frac{kg}{m^3}\times 9.81\frac{m}{s^2}\times 0.2\ m^3\times\frac{N\cdot s^2}{kg\cdot m}$$

$R_y = 1.66\ kN$ (force must be applied up) $\qquad R_y$

Problem 4.86

4.86 A water jet pump has jet area 0.1 ft^2 and jet speed 100 ft/s. The jet is within a secondary stream of water having speed V_s = 10 ft/s. The total area of the duct (the sum of the jet and secondary stream areas) is 0.75 ft^2. The water is thoroughly mixed and leaves the jet pump in a uniform stream. The pressures of the jet and secondary stream are the same at the pump inlet. Determine the speed at the pump exit and the pressure rise, $p_2 - p_1$.

Given: Data on water jet pump

Find: Speed at pump exit; pressure rise

Solution:

Basic equation: Continuity, and momentum flux in x direction

$$\frac{\partial}{\partial t}\int_{CV}\rho\, dV + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0 \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow CV 3) Uniform flow

From continuity
$$-\rho\cdot V_s\cdot A_s - \rho\cdot V_j\cdot A_j + \rho\cdot V_2\cdot A_2 = 0 \qquad V_2 = V_s\cdot\frac{A_s}{A_2} + V_j\cdot\frac{A_j}{A_2} = V_s\cdot\left(\frac{A_2 - A_j}{A_2}\right) + V_j\cdot\frac{A_j}{A_2}$$

$$V_2 = 10\cdot\frac{ft}{s}\times\left(\frac{0.75 - 0.1}{0.75}\right) + 100\cdot\frac{ft}{s}\times\frac{0.1}{0.75} \qquad V_2 = 22\,\frac{ft}{s}$$

For x momentum
$$p_1\cdot A_2 - p_2\cdot A_2 = V_j\cdot\left(-\rho\cdot V_j\cdot A_j\right) + V_s\cdot\left(-\rho\cdot V_s\cdot A_s\right) + V_2\cdot\left(\rho\cdot V_2\cdot A_2\right)$$

$$\Delta p = p_2 - p_1 = \rho\cdot\left(V_j^2\cdot\frac{A_j}{A_2} + V_s^2\cdot\frac{A_s}{A_2} - V_2^2\right)$$

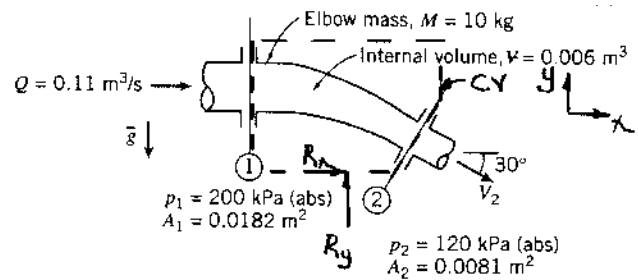
$$\Delta p = 1.94\cdot\frac{slug}{ft^3}\times\left[\left(100\cdot\frac{ft}{s}\right)^2\times\frac{0.1}{0.75} + \left(10\cdot\frac{ft}{s}\right)^2\times\frac{(0.75-0.1)}{0.75} - \left(22\cdot\frac{ft}{s}\right)^2\right]\times\frac{lbf\cdot s^2}{slug\cdot ft}$$

Hence
$$\Delta p = 1816\,\frac{lbf}{ft^2} \qquad \Delta p = 12.6\,psi$$

Problem 4.87

Given: Reducing elbow shown.

Fluid is water.

Find: Force components needed to keep elbow from moving.

Solution: Apply the x and y components of the momentum equation using the CS and CV shown.

Basic equations:

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A} \quad (F_{Bx}=0(4),\ \tfrac{\partial}{\partial t}=0(1))$$

$$F_{Sy} + F_{By} = \frac{\partial}{\partial t}\int_{CV} v\rho\, d\forall + \int_{CS} v\rho \vec{V}\cdot d\vec{A} \quad (\tfrac{\partial}{\partial t}=0(1))$$

Assumptions: (1) Steady flow (2) Uniform flow at each section (3) Use gage pressures (4) x horizontal

x comp: $R_x + p_{1g}A_1 - p_{2g}A_2\cos\theta = u_1\{-|\rho Q|\} + u_2\{+|\rho Q|\}$

$u_1 = V_1 \qquad u_2 = V_2\cos\theta$

$$R_x = (-V_1 + V_2\cos\theta)\rho Q - p_{1g}A_1 + p_{2g}A_2\cos\theta$$

$V_1 = \frac{Q}{A_1} = 0.11\frac{m^3}{s}\times\frac{1}{0.0182\,m^2} = 6.04\frac{m}{s}$

$V_2 = \frac{Q}{A_2} = 0.11\frac{m^3}{s}\times\frac{1}{0.0081\,m^2} = 13.6\frac{m}{s}$

$$= \left(-6.04\frac{m}{s} + 13.6\frac{m}{s}\times\cos 30°\right)999\frac{kg}{m^3}\times 0.11\frac{m^3}{s}\times\frac{N\cdot s^2}{kg\cdot m} - (200-101)10^3\frac{N}{m^2}\times 0.0182\,m^2 + (120-101)10^3\frac{N}{m^2}\times 0.0081\,m^2\times\cos 30°$$

$$R_x = +631 - 1800 + 133\ N = -1040\ N$$

R_x

y comp: $R_y + p_{2g}A_2\sin\theta - Mg - \rho\forall g = v_1\{-|\rho Q|\} + v_2\{+|\rho Q|\}$

$v_1 = 0 \qquad v_2 = -V_2\sin\theta$

$$R_y = -V_2\sin\theta\,\rho Q + Mg + \rho\forall g - p_{2g}A_2\sin\theta$$

$$= -13.6\frac{m}{s}\times\sin 30°\times 999\frac{kg}{m^3}\times 0.11\frac{m^3}{s}\times\frac{N\cdot s^2}{kg\cdot m} + 10\,kg\times 9.81\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}$$

$$+ 999\frac{kg}{m^3}\times 0.006\,m^3\times 9.81\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} - (120-101)10^3\frac{N}{m^2}\times 0.0081\,m^2\times\sin 30°$$

$$R_y = -747 + 98.1 + 58.8 - 77 = -667\ N$$

R_y

{ R_x and R_y are the horizontal and vertical components of force that must be supplied by the adjacent pipes to keep the elbow (the control volume) from moving. }

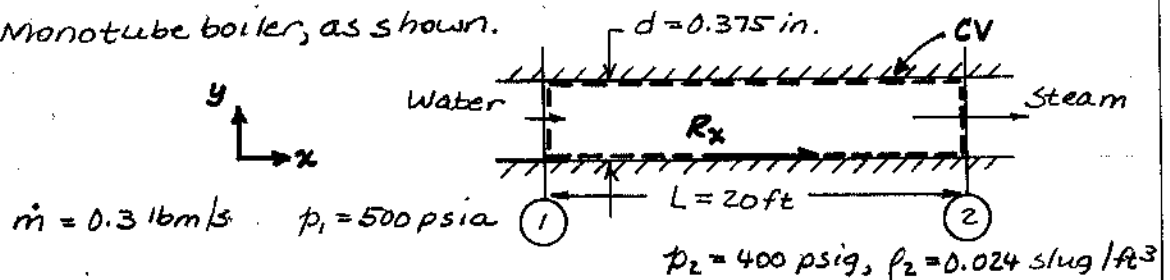
Problem 4.88

Given: Monotube boiler, as shown.

$d = 0.375$ in.

Water, Steam, CV, R_x, $L = 20$ ft, (1), (2)

$\dot{m} = 0.3$ lbm/s $\quad p_1 = 500$ psia

$p_2 = 400$ psig, $\rho_2 = 0.024$ slug/ft^3

Find: Magnitude and direction of force exerted by fluid on tube.

Solution: Apply the x component of the momentum equation, using the CV and coordinates shown.

Basic equation:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

($F_{B_x} = 0$ (1), $\frac{\partial}{\partial t}\int_{CV} = 0$ (2))

Assumptions: (1) $F_{B_x} = 0$
(2) Steady flow
(3) Uniform flow at each section
(4) Use gage pressures to cancel p_{atm}

From continuity,

$$\dot{m} = \rho_1 V_1 A_1 = \rho_2 V_2 A_2 \;;\; A = \text{constant, so } \rho_1 V_1 = \rho_2 V_2. \text{ Thus}$$

$$V_1 = \frac{\dot{m}}{\rho_1 A} = 0.3\,\frac{\text{lbm}}{\text{s}} \times \frac{\text{ft}^3}{1.94\,\text{slug}} \times \frac{4}{\pi}\frac{1}{(0.375)^2\,\text{in.}^2} \times \frac{\text{slug}}{32.2\,\text{lbm}} \times 144\,\frac{\text{in.}^2}{\text{ft}^2} = 6.26\ \text{ft/s}$$

and

$$V_2 = V_1\frac{\rho_1}{\rho_2} = 6.26\,\frac{\text{ft}}{\text{s}} \times 1.94\,\frac{\text{slug}}{\text{ft}^3} \times \frac{\text{ft}^3}{0.024\,\text{slug}} = 506\ \text{ft/s}$$

From momentum,

$$R_x + p_{1g}A_1 - p_{2g}A_2 = u_1\{-\dot{m}\} + u_2\{+\dot{m}\} = (V_2 - V_1)\dot{m}$$

$$u_1 = V_1 \qquad u_2 = V_2$$

$$R_x = (p_{2g} - p_{1g})A + (V_2 - V_1)\dot{m}$$

$$= [400 - (500 - 14.7)]\,\frac{\text{lbf}}{\text{in.}^2} \times \frac{\pi}{4}(0.375)^2\,\text{in.}^2 + (506 - 6.26)\frac{\text{ft}}{\text{s}} \times 0.3\,\frac{\text{lbm}}{\text{s}} \times \frac{\text{slug}}{32.2\,\text{lbm}} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

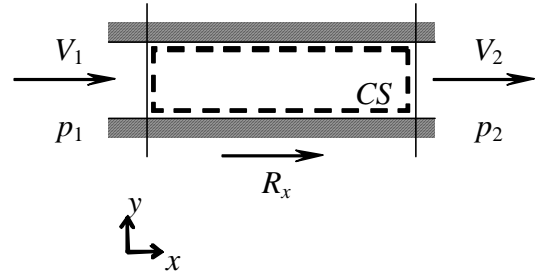
$$R_x = -4.77\ \text{lbf}$$

But R_x is force on CV; force on pipe is K_x,

$$K_x = -R_x = 4.77\ \text{lbf (to right)}$$

K_x

Problem 4.89

4.89 Consider the steady adiabatic flow of air through a long straight pipe with 0.05 m² cross-sectional area. At the inlet, the air is at 200 kPa (gage), 60°C and has a velocity of 150 m/s. At the exit, the air is at 80 kPa and has a velocity of 300 m/s. Calculate the axial force of the air on the pipe. (Be sure to make the direction clear.)

Given: Data on adiabatic flow of air

Find: Force of air on pipe

Solution:

Basic equation: Continuity, and momentum flux in x direction, plus ideal gas equation

$$\frac{\partial}{\partial t}\int_{CV}\rho\, dV + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0 \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A} \qquad p = \rho\cdot R\cdot T$$

Assumptions: 1) Steady flow 2) Ideal gas CV 3) Uniform flow

From continuity $\quad -\rho_1\cdot V_1\cdot A_1 + \rho_2\cdot V_2\cdot A_2 = 0 \qquad \rho_1\cdot V_1\cdot A = \rho_2\cdot V_2\cdot A \qquad \rho_1\cdot V_1 = \rho_2\cdot V_2$

For x momentum $\quad R_x + p_1\cdot A - p_2\cdot A = V_1\cdot\left(-\rho_1\cdot V_1\cdot A\right) + V_2\cdot\left(\rho_2\cdot V_2\cdot A\right) = \rho_1\cdot V_1\cdot A\cdot\left(V_2 - V_1\right)$

$$R_x = \left(p_2 - p_1\right)\cdot A + \rho_1\cdot V_1\cdot A\cdot\left(V_2 - V_1\right)$$

For the air $\quad \rho_1 = \frac{P_1}{R_{air}\cdot T_1} \qquad \rho_1 = (200 + 101)\times 10^3\cdot\frac{N}{m^2}\times\frac{kg\cdot K}{286.9\cdot N\cdot m}\times\frac{1}{(60 + 273)\cdot K} \qquad \rho_1 = 3.15\frac{kg}{m^3}$

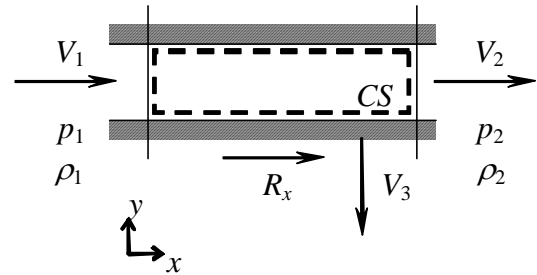
$$R_x = (80 - 200)\times 10^3\cdot\frac{N}{m^2}\times 0.05\cdot m^2 + 3.15\cdot\frac{kg}{m^3}\times 150\cdot\frac{m}{s}\times 0.05\cdot m^2\times(300 - 150)\cdot\frac{m}{s}\times\frac{N\cdot s^2}{kg\cdot m}$$

Hence $\quad R_x = -2456\,N$

This is the force of the pipe on the air; the pipe is opposing flow. Hence the force of the air on the pipe is $\quad F_{pipe} = -R_x$

$F_{pipe} = 2456\,N \qquad$ The air is dragging the pipe to the right

Problem 4.90

4.90 A gas flows steadily through a heated porous pipe of constant 0.15 m^2 cross-sectional area. At the pipe inlet, the absolute pressure is 400 kPa, the density is 6 kg/m^3, and the mean velocity is 170 m/s. The fluid passing through the porous wall leaves in a direction normal to the pipe axis, and the total flow rate through the porous wall is 20 kg/s. At the pipe outlet, the absolute pressure is 300 kPa and the density is 2.75 kg/m^3. Determine the axial force of the fluid on the pipe.

Given: Data on heated flow of gas

Find: Force of gas on pipe

Solution:

Basic equation: Continuity, and momentum flux in x direction

$$\frac{\partial}{\partial t}\int_{CV}\rho\,dV + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0 \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV}u\,\rho\,dV + \int_{CS}u\,\rho\vec{V}\cdot d\vec{A} \qquad p = \rho\cdot R\cdot T$$

Assumptions: 1) Steady flow 2) Uniform flow

From continuity $\quad -\rho_1\cdot V_1\cdot A_1 + \rho_2\cdot V_2\cdot A_2 + m_3 = 0 \qquad V_2 = V_1\cdot\frac{\rho_1}{\rho_2} - \frac{m_3}{\rho_2\cdot A}$ where m_3 = 20 kg/s is the mass leaving througl the walls (the software does not allow a dot)

$$V_2 = 170\cdot\frac{m}{s}\times\frac{6}{2.75} - 20\cdot\frac{kg}{s}\times\frac{m^3}{2.75\cdot kg}\times\frac{1}{0.15\cdot m^2} \qquad V_2 = 322\frac{m}{s}$$

For x momentum $\quad R_x + p_1\cdot A - p_2\cdot A = V_1\cdot\left(-\rho_1\cdot V_1\cdot A\right) + V_2\cdot\left(\rho_2\cdot V_2\cdot A\right)$

$$R_x = \left[\left(p_2 - p_1\right) + \rho_2\cdot V_2^2 - \rho_1\cdot V_1^2\right]\cdot A$$

$$R_x = \left[(300 - 400)\times 10^3\cdot\frac{N}{m^2} + \left[2.75\cdot\frac{kg}{m^3}\times\left(322\cdot\frac{m}{s}\right)^2 - 6\cdot\frac{kg}{m^3}\times\left(170\cdot\frac{m}{s}\right)^2\right]\times\frac{N\cdot s^2}{kg\cdot m}\right]\times 0.15\cdot m^2$$

Hence $\quad R_x = 1760\,N$

Problem 4.91

Given: Water flow discharging nonuniformly from slot, as shown.

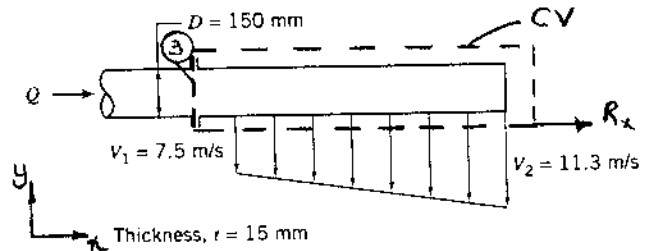
$p_{1g} = 30$ kPa

Find: (a) Volume flow rate.
(b) Forces to hold pipe.

Solution: Apply x, y components of momentum, using the CV, CS shown.

Basic equations:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A} \; ; \; F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\rho\, d\forall + \int_{CS} v\rho \vec{V}\cdot d\vec{A}$$

($F_{B_x} = 0$ (1), $\frac{\partial}{\partial t} = 0$ (2); $F_{B_y} = 0$ (1), $\frac{\partial}{\partial t} = 0$ (2))

Assumptions: (1) $F_{Bx} = F_{By} = 0$
(2) Steady flow
(3) Uniform flow at inlet section
(4) Use gage pressures to cancel p_{atm}

From continuity,

$$Q = \bar{V}A = \frac{1}{2}(V_1 + V_2)Lt = \frac{1}{2}(7.5 + 11.3)\frac{m}{s} \times 1\,m \times 0.015\,m = 0.141\ m^3/s$$

Q

$$V_3 = \frac{Q}{A_3} = 0.141\frac{m^3}{s} \times \frac{4}{\pi}\frac{1}{(0.15)^2 m^2} = 7.98\ m/s$$

From x momentum, since flow leaves slot vertically ($u = 0$),

$$R_x + p_{3g}A_3 = u_3\{-\rho Q\} = -V_3\rho Q \; ; \; R_x = -p_{3g}A_3 - V_3\rho Q$$

$$R_x = -30\times10^3\frac{N}{m^2}\times\frac{\pi}{4}(0.15)^2 m^2 - 7.98\frac{m}{s}\times 999\frac{kg}{m^3}\times 0.141\frac{m^3}{s}\times\frac{N\cdot s^2}{kg\cdot m}$$

$$R_x = -1.65\ kN \text{ (to left)}$$

R_x

From y momentum, since $v_3 = 0$,

$$R_y = v_3\{-\rho Q\} + \int_0^L v\rho V t\, dx = -\rho t\int_0^L \left(V_1 + \frac{V_2 - V_1}{L}x\right)^2 dx$$

($v_3 = 0$)

$$= -\rho t\left[V_1^2 x + 2V_1\left(\frac{V_2 - V_1}{L}\right)\frac{x^2}{2} + \left(\frac{V_2 - V_1}{L}\right)^2\frac{x^3}{3}\right]_0^L$$

$$= -999\frac{kg}{m^3}\times 0.015\,m\left[(7.5)^2\frac{m^2}{s^2} + 7.5\frac{m}{s}\times(11.3 - 7.5)\frac{m}{s}\times\frac{1}{1\,m}\times(1)^2 m^2 + (11.3 - 7.5)^2\frac{m^2}{s^2}\times\frac{1}{(1)^2 m^2}\times\frac{(1)^3 m^3}{3}\right]$$

$$R_y = -1.34\ kN \text{ (down)}$$

R_y

{A moment also would be required at the coupling.}

Problem 4.92

Given: Steady flow of water through square channel shown
$V_{max} = 2V_{min}$, $U = 7.5$ m/s, $p_1 = 185$ kPa (gage), $p_2 = p_{atm}$
$M_c = 2.05$ kg, $\forall_c = 0.00355$ m^3, $h = 75.5$ mm $= w$

Find: Force exerted by channel assembly on the supply duct.

Solution: Apply conservation of mass + momentum equations to the CV shown.

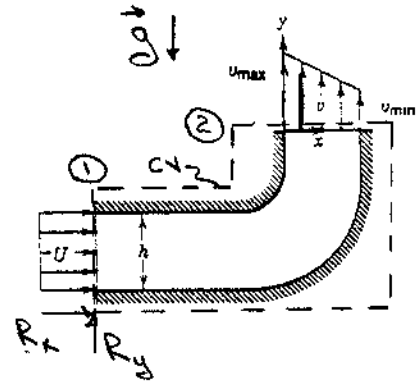
Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} \quad (1)$$ [$\frac{\partial}{\partial t}$ term $= 0$ (1)]

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A} \quad (2)$$ [$\frac{\partial}{\partial t}$ term $= 0$ (1)]

$$F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\rho\, d\forall + \int_{CS} v\rho\vec{V}\cdot d\vec{A} \quad (3)$$ [$\frac{\partial}{\partial t}$ term $= 0$ (1)]

$\vec{g}\downarrow$

Assumptions:
(1) steady flow (2) incompressible flow
(3) uniform flow at inlet
(4) use gage pressures

From continuity, $0 = \vec{V}_1\cdot\vec{A}_1 + \int \vec{V}_2\cdot d\vec{A}_2 = -Uwh + \int_0^h v\,w\,dx$

$\therefore Uh = \int_0^h v\,dx = \int_0^h v_{min}\left(2 - \frac{x}{h}\right)dx = v_{min}\left[2x - \frac{x^2}{2h}\right]_0^h = \frac{3}{2} v_{min} h$

and $v_{min} = \frac{2}{3}U = \frac{2}{3}\times 7.5\frac{m}{s} = 5.0$ m/s

From Eq. 2,

$R_x + p_{1g}A_1 = u_1\{-\rho U A_1\} + \int_0^h u_2\, \rho\, v_{min}\left(2 - \frac{x}{h}\right) w\,dx = -\rho U^2 A_1$ [$u_2 = 0$]

$R_x = -p_{1g}A_1 - \rho U^2 A_1 = -(185-101)10^3\frac{N}{m^2}\times(0.0755)^2 m^2 - 999\frac{kg}{m^3}\times(7.5)^2\frac{m^2}{s^2}\times(0.0755)^2 m^2$

$R_x = -479\text{ N} - 320\frac{kg\cdot m}{s^2}\cdot\frac{N\cdot s^2}{kg\cdot m} = -479\text{ N} - 320\text{ N} = -799\text{ N}$

$K_x = -R_x = 799$ N (on supply duct to the right) $\longleftarrow K_x$

From Eq. 3,

$R_y - M_c g - \rho\forall g = v_1\{-\rho U A_1\} + \int_0^h v_2\{\rho v_2 w\,dx\}$ [$v_1 = 0$]

$R_y - M_c g - \rho\forall g = \int_0^h v_{min}\left(2 - \frac{x}{h}\right)\rho\, v_{min}\left(2 - \frac{x}{h}\right) w\,dx$

$= \rho v_{min}^2 w \int_0^h \left(4 - 4\frac{x}{h} + \frac{x^2}{h^2}\right)dx$

$= \rho v_{min}^2 w\left[4x - 2\frac{x^2}{h} + \frac{x^3}{3h^2}\right]_0^h = \rho v_{min}^2 w h \frac{7}{3}$

$\therefore R_y = \left[2.05\,kg\times 9.81\frac{m}{s^2} + 999\frac{kg}{m^3}\times(0.00355\,m^3)\,9.81\frac{m}{s^2} + \frac{7}{3}\times 999\frac{kg}{m^3}\times(5.0)^2\frac{m^2}{s^2}\times(0.0755)^2 m^2\right]\frac{N\cdot s^2}{kg\cdot m}$

$R_y = (20.1 + 34.8 + 332)\text{ N} = 387\text{ N}$ (on CV)

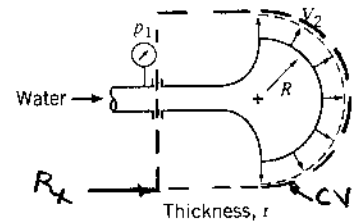
$K_y = -R_y = -387$ N (on supply duct, down) $\longleftarrow K_y$

Problem 4.93

Given: Nozzle discharging flat, radial sheet of water, as shown.

Find: Axial force of nozzle on coupling.

$D_1 = 35\ mm$

Solution: Apply the x component of momentum, using CV and coordinates shown.

Basic equation:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

($F_{B_x} = 0(1)$; $\frac{\partial}{\partial t} = 0(2)$)

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow
(3) Uniform flow at each section
(4) Use gage pressure to cancel p_{atm}

From continuity

$$Q = V_1 A_1 = V_2 A_2 = V_2 \pi R t = \pi \times 10\,\frac{m}{sec} \times 0.05\,m \times 0.0015\,m = 0.00236\ m^3/s$$

$$V_1 = \frac{Q}{A_1} = \frac{4Q}{\pi D_1^2} = \frac{4}{\pi} \times 0.00236\,\frac{m^3}{sec} \times \frac{1}{(0.035)^2 m^2} = 2.45\ m/s$$

From momentum $\quad \left\{ \text{Note } A_1 = \frac{\pi D_1^2}{4} = 0.000962\ m^2 \right\}$

$$R_x + p_{1g} A_1 = u_1\{-\rho Q\} + \int_{A_2} u_2 \rho V_2\, dA_2$$

$$u_1 = V_1 \qquad u_2 = V_2 \cos\theta;\ dA_2 = Rt\,d\theta$$

$$\int_{A_2} = \int_{-\pi/2}^{\pi/2} V_2 \cos\theta\, \rho V_2 R t\, d\theta = 2\rho V_2^2 R t \int_0^{\pi/2} \cos\theta\, d\theta = 2\rho V_2^2 R t$$

Thus

$$R_x = -p_{1g} A_1 - V_1 \rho Q + 2\rho V_2^2 R t$$

$$= -(150-101)10^3\,\frac{N}{m^2} \times 0.000962\,m^2 - 2.45\,\frac{m}{sec} \times 999\,\frac{kg}{m^3} \times 0.00236\,\frac{m^3}{sec} \times \frac{N\cdot sec^2}{kg\cdot m}$$
$$+ 2 \times 999\,\frac{kg}{m^3} \times (10)^2\,\frac{m^2}{sec^2} \times 0.05\,m \times 0.0015\,m \times \frac{N\cdot sec^2}{kg\cdot m}$$

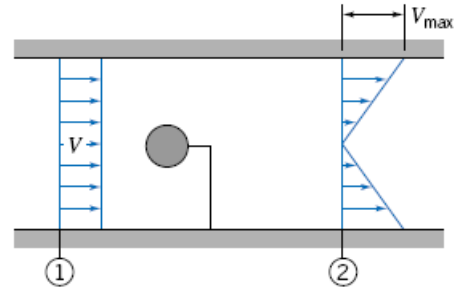
$$R_x = -37.9\ N$$

But R_x is force on CV; force on coupling is K_x,

$$K_x = -R_x = 37.9\ N \text{ (to right)}$$

K_x

Problem 4.94

4.94 A small round object is tested in a 0.75-m diameter wind tunnel. The pressure is uniform across sections ① and ②. The upstream pressure is 30 mm H_2O (gage), the downstream pressure is 15 mm H_2O (gage), and the mean air speed is 12.5 m/s. The velocity profile at section ② is linear; it varies from zero at the tunnel centerline to a maximum at the tunnel wall. Calculate (a) the mass flow rate in the wind tunnel, (b) the maximum velocity at section ②, and (c) the drag of the object and its supporting vane. Neglect viscous resistance at the tunnel wall.

Given:
Data on flow in wind tunnel

Find:
Mass flow rate in tunnel; Maximum velocity at section 2; Drag on object

Solution:

Basic equations: Continuity, and momentum flux in x direction; ideal gas equation

$$\frac{\partial}{\partial t}\int_{CV}\rho\, dV + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0 \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A} \qquad p = \rho\cdot R\cdot T$$

Assumptions: 1) Steady flow 2) Uniform density at each section

From continuity $\quad m_{flow} = \rho_1\cdot V_1\cdot A_1 = \rho_1\cdot V_1\cdot\frac{\pi\cdot D_1^2}{4}\quad$ where m_{flow} is the mass flow rate

We take ambient conditions for the air density $\quad \rho_{air} = \frac{p_{atm}}{R_{air}\cdot T_{atm}} \qquad \rho_{air} = 101000\cdot\frac{N}{m^2}\times\frac{kg\cdot K}{286.9\cdot N\cdot m}\times\frac{1}{293\cdot K} \qquad \rho_{air} = 1.2\frac{kg}{m^3}$

$$m_{flow} = 1.2\cdot\frac{kg}{m^3}\times 12.5\cdot\frac{m}{s}\times\frac{\pi\cdot(0.75\cdot m)^2}{4} \qquad m_{flow} = 6.63\frac{kg}{s}$$

Also

$$m_{flow} = \int \rho_2\cdot u_2\, dA_2 = \rho_{air}\cdot\int_0^R V_{max}\cdot\frac{r}{R}\cdot 2\cdot\pi\cdot r\, dr = \frac{2\cdot\pi\cdot\rho_{air}\cdot V_{max}}{R}\cdot\int_0^R r^2\, dr = \frac{2\cdot\pi\cdot\rho_{air}\cdot V_{max}\cdot R^2}{3}$$

$$V_{max} = \frac{3\cdot m_{flow}}{2\cdot\pi\cdot\rho_{air}\cdot R^2} \qquad V_{max} = \frac{3}{2\cdot\pi}\times 6.63\cdot\frac{kg}{s}\times\frac{m^3}{1.2\cdot kg}\times\left(\frac{1}{0.375\cdot m}\right)^2 \qquad V_{max} = 18.8\frac{m}{s}$$

For x momentum $\quad R_x + p_1\cdot A - p_2\cdot A = V_1\cdot(-\rho_1\cdot V_1\cdot A) + \int \rho_2\cdot u_2\cdot u_2\, dA_2$

$$R_x = (p_2 - p_1)\cdot A - V_1\cdot m_{flow} + \int_0^R \rho_{air}\cdot\left(V_{max}\cdot\frac{r}{R}\right)^2\cdot 2\cdot\pi\cdot r\, dr = (p_2 - p_1)\cdot A - V_1\cdot m_{flow} + \frac{2\cdot\pi\cdot\rho_{air}\cdot V_{max}^2}{R^2}\cdot\int_0^R r^3\, dr$$

$$R_x = (p_2 - p_1)\cdot A - V_1\cdot m_{flow} + \frac{\pi}{2}\cdot\rho_{air}\cdot V_{max}^2\cdot R^2$$

We also have $\quad p_1 = \rho\cdot g\cdot h_1 \qquad p_1 = 1000\cdot\frac{kg}{m^3}\times 9.81\cdot\frac{m}{s^2}\times 0.03\cdot m \qquad p_1 = 294\,Pa \qquad p_2 = \rho\cdot g\cdot h_2 \qquad p_2 = 147\cdot Pa$

Hence

$$R_x = (147 - 294)\cdot\frac{N}{m^2}\times\frac{\pi\cdot(0.75\cdot m)^2}{4} + \left[-6.63\cdot\frac{kg}{s}\times 12.5\cdot\frac{m}{s} + \frac{\pi}{2}\times 1.2\cdot\frac{kg}{m^3}\times\left(18.8\cdot\frac{m}{s}\right)^2\times(0.375\cdot m)^2\right]\times\frac{N\cdot s^2}{kg\cdot m}$$

$R_x = -54\,N \quad$ The drag on the object is equal and opposite $\quad F_{drag} = -R_x \qquad F_{drag} = 54.1\,N$

Problem 4.95

4.95 The horizontal velocity in the wake behind an object in an air stream of velocity U is given by

$$u(r) = U\left[1 - \cos^2\left(\frac{\pi r}{2}\right)\right] \qquad |r| \le 1$$

$$u(r) = U \qquad |r| > 1$$

where r is the non-dimensional radial coordinate, measured perpendicular to the flow. Find an expression for the drag on the object.

Given:

Data on wake behind object

Find:

An expression for the drag

Solution:

Governing equation:

Momentum

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, d\maltese + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A} \qquad (4.18a)$$

Applying this to the horizontal motion

$$-F = U\cdot\left(-\rho\cdot\pi\cdot 1^2\cdot U\right) + \int_0^1 u(r)\cdot\rho\cdot 2\cdot\pi\cdot r\cdot u(r)\,dr \qquad F = \pi\,\rho\cdot\left(U^2 - 2\cdot\int_0^1 r\cdot u(r)^2\,dr\right)$$

$$F = \pi\,\rho\cdot U^2\cdot\left[1 - 2\cdot\int_0^1 r\cdot\left(1 - \cos\left(\frac{\pi\cdot r}{2}\right)^2\right)^2 dr\right]$$

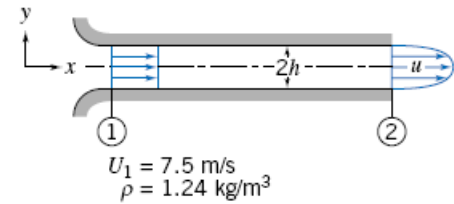
$$F = \pi\,\rho\cdot U^2\cdot\left(1 - 2\cdot\int_0^1 r - 2\cdot r\cdot\cos\left(\frac{\pi\cdot r}{2}\right)^2 + r\cdot\cos\left(\frac{\pi\cdot r}{2}\right)^4 dr\right)$$

Integrating and using the limits $F = \pi\,\rho\cdot U^2\cdot\left[1 - \left(\frac{3}{8} + \frac{2}{\pi^2}\right)\right]$ $\qquad F = \left(\frac{5\cdot\pi}{8} - \frac{2}{\pi}\right)\cdot\rho\cdot U^2$

Problem 4.96

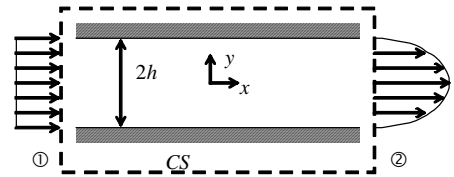
4.96 An incompressible fluid flows steadily in the entrance region of a two-dimensional channel of height $2h$. The uniform velocity at the channel entrance is $U_1 = 7.5$ m/s. The velocity distribution at a section downstream is

$$\frac{u}{u_{max}} = 1 - \left[\frac{y}{h}\right]^2$$

Evaluate the maximum velocity at the downstream section. Calculate the pressure drop that would exist in the channel if viscous friction at the walls could be neglected.

Given: Data on flow in 2D channel

Find: Maximum velocity; Pressure drop

Solution:

Basic equations: Continuity, and momentum flux in x direction; ideal gas equation

$$\frac{\partial}{\partial t}\int_{CV} \rho\, dV + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0 \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV + \int_{CS} u\,\rho \vec{V} \cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Neglect frition

From continuity
$$-\rho \cdot U_1 \cdot A_1 + \int \rho \cdot u_2 \, dA = 0$$

$$U_1 \cdot 2 \cdot h \cdot w = w \cdot \int_{-h}^{h} u_{max} \cdot \left(1 - \frac{y^2}{h^2}\right) dy = w \cdot u_{max} \cdot \left[[h - (-h)] - \left[\frac{h}{3} - \left(-\frac{h}{3}\right)\right]\right] = w \cdot u_{max} \cdot \frac{4}{3} \cdot h$$

Hence
$$u_{max} = \frac{3}{2} \cdot U_1 \qquad u_{max} = \frac{3}{2} \times 7.5 \cdot \frac{m}{s} \qquad u_{max} = 11.3 \frac{m}{s}$$

For x momentum
$$p_1 \cdot A - p_2 \cdot A = V_1 \cdot \left(-\rho_1 \cdot V_1 \cdot A\right) + \int \rho_2 \cdot u_2 \cdot u_2 \, dA_2$$
Note that there is no R_x (no friction)

$$p_1 - p_2 = -\rho \cdot U_1^2 + \frac{w}{A} \cdot \int_{-h}^{h} \rho \cdot u_{max}^2 \cdot \left(1 - \frac{y^2}{h^2}\right)^2 dy = -\rho \cdot U_1^2 + \frac{\rho \cdot u_{max}^2}{h} \cdot \left[2 \cdot h - 2 \cdot \left(\frac{2}{3} \cdot h\right) + 2 \cdot \left(\frac{1}{5} \cdot h\right)\right]$$

$$\Delta p = p_1 - p_2 = -\rho \cdot U_1^2 + \frac{8}{15} \cdot \rho \cdot u_{max}^2 = \rho \cdot U_1 \cdot \left[\frac{8}{15} \cdot \left(\frac{3}{2}\right)^2 - 1\right] = \frac{1}{5} \cdot \rho \cdot U_1$$

Hence
$$\Delta p = \frac{1}{5} \times 1.24 \cdot \frac{kg}{m^3} \times \left(7.5 \cdot \frac{m}{s}\right)^2 \times \frac{N \cdot s^2}{kg \cdot m} \qquad \Delta p = 14\, Pa$$

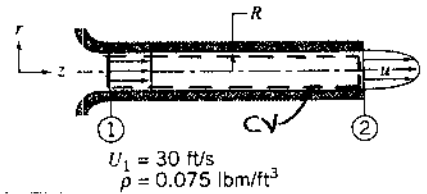
Problem 4.97

[3]

Given: Incompressible flow in entrance region of circular tube of radius, R.

$$u_2 = u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

Find: (a) Maximum velocity at section ②.
(b) Pressure drop if viscous friction could be neglected.

Solution: Apply continuity and the x direction momentum equations. Use the CV and CS shown.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} \qquad (=0(1))$$

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A} \qquad (F_{B_x}=0(3),\ \tfrac{\partial}{\partial t}=0(1))$$

Assumptions: (1) Steady flow
(2) Uniform flow at section ①
(3) $F_{Bx} = 0$
(4) Neglect friction at duct wall
(5) Incompressible flow

Then

$$0 = \{-|\rho U_1 \pi R^2|\} + \int_0^R \rho u_{max}\left[1-\left(\frac{r}{R}\right)^2\right] 2\pi r\, dr$$

or

$$\pi \rho U_1 R^2 = 2\pi\rho u_{max} R^2 \int_0^1 \left[1-\left(\frac{r}{R}\right)^2\right]\left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) = 2\pi\rho u_{max} R^2\left[\frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4\right]_0^1$$

Thus $u_{max} = 2U_1 = 2 \times 30 \frac{ft}{s} = 60\ ft/s$ — u_{ma}

From the momentum equation,

$$p_1 \pi R^2 - p_2 \pi R^2 = u_1\{-\rho U_1 \pi R^2\} + \int_0^R u_2 \rho u_2\, dA_2 = -\rho U_1 \pi R^2 + \rho u_{max}^2 2\pi R^2 \int_0^1 \left[1-\left(\frac{r}{R}\right)^2\right]^2 \left(\frac{r}{R}\right) d\left(\frac{r}{R}\right)$$

$$u_1 = U_1 \qquad u_2 = u_{max}\left[1-\left(\frac{r}{R}\right)^2\right]$$

or

$$p_1 - p_2 = -\rho U_1^2 + 2\rho u_{max}^2 \int_0^1 (1-\eta^2)^2 \eta\, d\eta \ ; \ \eta = \frac{r}{R}$$

But $\int_0^1 (1-\eta^2)\eta\, d\eta = \int_0^1 (1 - 2\eta^2 + \eta^4)\eta\, d\eta = \frac{1}{2}\eta^2 - \frac{1}{2}\eta^4 + \frac{1}{6}\eta^6\Big]_0^1 = \frac{1}{6}$

and $u_{max}^2 = (2U_1)^2 = 4U_1^2$, so

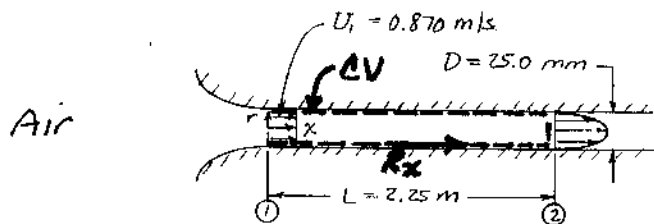
$$p_1 - p_2 = -\rho U_1^2 + \frac{8}{6}\rho U_1^2 = \rho U_1^2\left(\frac{4}{3} - 1\right) = \frac{1}{3}\rho U_1^2$$

$$= \frac{1}{3} \times 0.075 \frac{lbm}{ft^3} \times (30)^2 \frac{ft^2}{s^2} \times \frac{slug}{32.2\ lbm} \times \frac{lbf\cdot s^2}{slug\cdot ft}$$

$$p_1 - p_2 = 0.699\ lbf/ft^2$$ — $p_1 - p_2$

Problem 4.98

[3]

Given: Uniform flow into, fully developed flow from duct shown.

$$\frac{u(r)}{U_c} = 1 - \left(\frac{r}{R}\right)^2 \quad \text{at } (2)$$

$$p_1 - p_2 = 1.92 \text{ N/m}^2$$

Find: Total force exerted by tube on the flowing air.

Solution: Apply continuity and momentum to CV, CS shown.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} \quad (=0(1))$$

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A} \quad (=0(4), =0(1))$$

Assumptions: (1) Steady flow (2) Incompressible flow (3) Uniform flow at inlet (4) $F_{B_x} = 0$

Then

$$0 = \{-|\rho U_1 A_1|\} + \int_{(2)} \rho u \, dA = -\rho U_1 \pi R^2 + \int_0^R \rho U_c \left[1 - \left(\frac{r}{R}\right)^2\right] 2\pi r \, dr$$

$$0 = -\rho U_1 \pi R^2 + 2\rho \pi R^2 U_c \int_0^1 (1-\lambda^2)\lambda \, d\lambda \quad \text{or} \quad 0 = -U_1 + 2U_c \left[\frac{\lambda^2}{2} - \frac{\lambda^4}{4}\right]_0^1$$

Thus $0 = -U_1 + \frac{1}{2}U_c$ or $U_c = 2U_1$ $\qquad (\lambda = r/R)$

From momentum $R_x + p_1 A_1 - p_2 A_2 = u_1\{-|\rho U_1 A_1|\} + \int_{(2)} u_2 \{+\rho u_2 \, dA_2\}$

$$u_1 = U_1 \qquad u_2 = U_c\left[1 - \left(\frac{r}{R}\right)^2\right]$$

so

$$\int_{(2)} = \int_0^R U_c\left[1 - \left(\frac{r}{R}\right)^2\right] \rho U_c \left[1 - \left(\frac{r}{R}\right)^2\right] 2\pi r \, dr = 2\pi \rho U_c^2 R^2 \int_0^1 (1-\lambda^2)(1-\lambda^2)\lambda \, d\lambda$$

$$= 2\pi \rho U_c^2 R^2 \int_0^1 (1 - 2\lambda^2 + \lambda^4)\lambda \, d\lambda = 2\pi \rho U_c^2 R^2 \left[\frac{\lambda^2}{2} - \frac{\lambda^4}{2} + \frac{\lambda^6}{6}\right]_0^1 = \frac{1}{3}\pi \rho U_c^2 R^2$$

Substituting,

$$R_x + (p_1 - p_2)\pi R^2 = -\pi \rho U_1^2 R^2 + \frac{1}{3}\pi \rho U_c^2 R^2 = -\pi \rho U_1^2 R^2 + \frac{1}{3}\pi \rho (2U_1)^2 R^2$$

$$R_x = -(p_1 - p_2)\frac{\pi D^2}{4} + \frac{1}{3}\rho U_1^2 \frac{\pi D^2}{4}$$

$$= -1.92 \frac{\text{N}}{\text{m}^2} \times \frac{\pi}{4}(0.025)^2 \text{m}^2 + \frac{1}{3} \times 1.23 \frac{\text{kg}}{\text{m}^3} \times (0.870)^2 \frac{\text{m}^2}{\text{s}^2} \times \frac{\pi}{4}(0.025)^2 \text{m}^2 \times \frac{\text{N} \cdot \text{s}^2}{\text{kg} \cdot \text{m}}$$

$$R_x = -7.90 \times 10^{-4} \text{ N} \quad (\text{to left } \underline{\text{on}} \text{ CV, since } < 0)$$

R_x

Problem 4.99

Given: Incompressible flow in boundary layer.

$w = 0.6\ m$

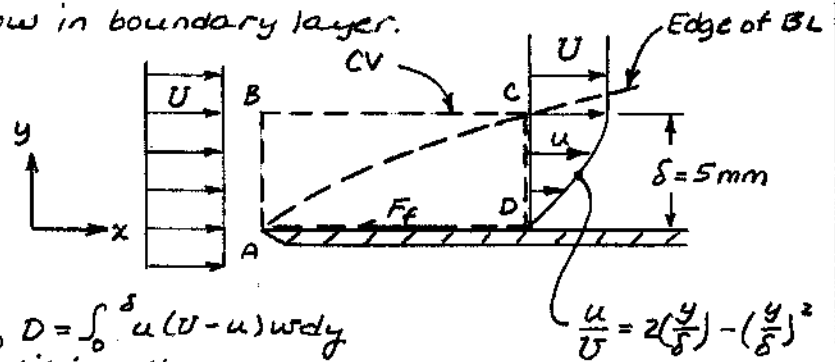
$U = 30\ m/s$

$\rho = 1.24\ kg/m^3$

Edge of BL

CV

$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$

$\delta = 5\ mm$

Find: (a) Show that drag, $D = \int_0^\delta u(U-u)w\,dy$

(b) Evaluate for conditions shown.

Solution: Apply continuity and x component of momentum using CV shown.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} \quad (=0(1))$$

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A} \quad (F_{B_x}=0(3),\ \partial/\partial t = 0(1))$$

Assumptions: (1) Steady flow

(2) No net pressure force; $F_{S_x} = -F_f$

(3) $F_{B_x} = 0$

(4) Uniform flow at section (AB)

(5) Incompressible flow

Then from continuity

$$0 = \{-|\rho U w \delta|\} + \{|\int_0^\delta \rho u w\,dy|\} + \dot{m}_{BC}\ ;\ \delta = \int_0^\delta dy\ ;\ \dot{m}_{BC} = \rho\int_0^\delta (U-u)w\,dy$$

From momentum

$$-F_f = U\{-|\rho U w \delta|\} + \{\int_0^\delta \rho u^2 w\,dy|\} + U\dot{m}_{BC} = \rho\int_0^\delta [-U^2 + u^2 + U(U-u)]w\,dy$$

$$Drag = F_f = \int_0^\delta \rho u(U-u)w\,dy$$

Drag

At CD, $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 = 2\eta - \eta^2;\ dy = \delta\, d\left(\frac{y}{\delta}\right) = \delta\, d\eta$

$$Drag = \int_0^\delta \rho U\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]\left(U - U\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]\right)w\,dy = \rho U^2 w\delta \int_0^1 (2\eta-\eta^2)(1-2\eta+\eta^2)\,d\eta$$

$$= \rho U^2 w\delta\int_0^1 (2\eta - 5\eta^2 + 4\eta^3 - \eta^4)\,d\eta = \rho U^2 w\delta\left[\eta^2 - \frac{5}{3}\eta^3 + \eta^4 - \frac{1}{5}\eta^5\right]_0^1$$

$$= \frac{2}{15}\rho U^2 w\delta$$

$$Drag = \frac{2}{15} \times 1.24\frac{kg}{m^3} \times (30)^2\frac{m^2}{s^2} \times 0.6\,m \times 0.005\,m \times \frac{N\cdot s^2}{kg\cdot m} = 0.446\ N$$

Drag

Problem 4.100

4.100 Air at standard conditions flows along a flat plate. The undisturbed freestream speed is $U_0 = 30$ ft/s. At $L = 6$ in. downstream from the leading edge of the plate, the boundary-layer thickness is $\delta = 0.1$ in. The velocity profile at this location is

$$\frac{u}{U_0} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left[\frac{y}{\delta}\right]^3$$

Calculate the horizontal component of force per unit width required to hold the plate stationary.

Given: Data on flow of boundary layer

Find: Force on plate per unit width

Solution:

Basic equations: Continuity, and momentum flux in x direction

$$\frac{\partial}{\partial t}\int_{CV} \rho\, dV + \int_{CS} \rho \vec{V}\cdot d\vec{A} = 0 \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV + \int_{CS} u\,\rho \vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible 3) No net pressure force

From continuity $\quad -\rho\cdot U_0\cdot w\cdot\delta + m_{bc} + \int_0^\delta \rho\cdot u\cdot w\, dy = 0 \quad$ where m_{bc} is the mass flow rate across bc (Note: sotware cannot render a dot!)

Hence $\quad m_{bc} = \int_0^\delta \rho\cdot(U_0 - u)\cdot w\, dy$

For x momentum $\quad -F_f = U_0\cdot(-\rho\cdot U_0\cdot w\cdot\delta) + U_0\cdot m_{bc} + \int_0^\delta u\cdot\rho\cdot u\cdot w\, dy = \int_0^\delta \left[-U_0^2 + u^2 + U_0\cdot(U_0 - u)\right]\cdot w\, dy$

Then the drag force is $\quad F_f = \int_0^\delta \rho\cdot u\cdot(U_0 - u)\cdot w\, dy = \int_0^\delta \rho\cdot U_0^2\cdot\frac{u}{U_0}\cdot\left(1 - \frac{u}{U_0}\right) dy$

But we have $\quad \frac{u}{U_0} = \frac{3}{2}\cdot\eta - \frac{1}{2}\cdot\eta^3 \quad$ where we have used substitution $\quad y = \delta\cdot\eta$

$$\frac{F_f}{w} = \int_0^{\eta=1} \rho\cdot U_0^2\cdot\delta\cdot\frac{u}{U_0}\cdot\left(1 - \frac{u}{U_0}\right) d\eta = \rho\cdot U_0^2\cdot\delta\cdot\int_0^1 \left(\frac{3}{2}\cdot\eta - \frac{9}{4}\cdot\eta^2 - \frac{1}{2}\cdot\eta^3 + \frac{3}{2}\cdot\eta^4 - \frac{1}{4}\cdot\eta^6\right) d\eta$$

$$\frac{F_f}{w} = \rho\cdot U_0^2\cdot\delta\cdot\left(\frac{3}{4} - \frac{3}{4} - \frac{1}{8} + \frac{3}{10} - \frac{1}{28}\right) = 0.139\cdot\rho\cdot U_0^2\cdot\delta$$

Hence $\quad \frac{F_f}{w} = 0.139 \times 0.002377\cdot\frac{slug}{ft^3} \times \left(30\cdot\frac{ft}{s}\right)^2 \times \frac{0.1}{12}\cdot ft \times \frac{lbf\cdot s^2}{slug\cdot ft} \quad$ (using standard atmosphere density)

$$\frac{F_f}{w} = 2.48 \times 10^{-3}\cdot\frac{lbf}{ft}$$

Problem 4.101

4.101 Air at standard conditions flows along a flat plate. The undisturbed freestream speed is $U_0 = 20$ m/s. At $L = 0.4$ m downstream from the leading edge of the plate, the boundary-layer thickness is $\delta = 2$ mm. The velocity profile at this location is approximated as $u/U_0 = y/\delta$. Calculate the horizontal component of force per unit width required to hold the plate stationary.

Given: Data on flow of boundary layer

Find: Force on plate per unit width

Solution:

Basic equations: Continuity, and momentum flux in x direction

$$\frac{\partial}{\partial t}\int_{CV}\rho\, dV + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0 \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible 3) No net pressure force

From continuity $\quad -\rho\cdot U_0\cdot w\cdot\delta + m_{bc} + \int_0^{\delta}\rho\cdot u\cdot w\,dy = 0 \quad$ where m_{bc} is the mass flow rate across bc (Note: sotware cannot render a dot!)

Hence $\quad m_{bc} = \int_0^{\delta}\rho\cdot\left(U_0 - u\right)\cdot w\,dy$

For x momentum $\quad -F_f = U_0\cdot\left(-\rho\cdot U_0\cdot w\cdot\delta\right) + U_0\cdot m_{bc} + \int_0^{\delta} u\cdot\rho\cdot u\cdot w\,dy = \int_0^{\delta}\left[-U_0^{\,2} + u^2 + U_0\cdot\left(U_0 - u\right)\right]\cdot w\,dy$

Then the drag force is $\quad F_f = \int_0^{\delta}\rho\cdot u\cdot\left(U_0 - u\right)\cdot w\,dy = \int_0^{\delta}\rho\cdot U_0^{\,2}\cdot\frac{u}{U_0}\cdot\left(1 - \frac{u}{U_0}\right)dy$

But we have $\quad \frac{u}{U_0} = \frac{y}{\delta} \quad$ where we have used substitution $\quad y = \delta\cdot\eta$

$$\frac{F_f}{w} = \int_0^{\eta=1}\rho\cdot U_0^{\,2}\cdot\delta\cdot\frac{u}{U_0}\cdot\left(1 - \frac{u}{U_0}\right)d\eta = \rho\cdot U_0^{\,2}\cdot\delta\cdot\int_0^1 \eta\cdot(1-\eta)\,d\eta$$

$$\frac{F_f}{w} = \rho\cdot U_0^{\,2}\cdot\delta\cdot\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{6}\cdot\rho\cdot U_0^{\,2}\cdot\delta$$

Hence $\quad \frac{F_f}{w} = \frac{1}{6}\times 1.225\cdot\frac{kg}{m^3}\times\left(20\cdot\frac{m}{s}\right)^2\times\frac{2}{1000}\cdot m\times\frac{N\cdot s^2}{kg\cdot m} \quad$ (using standard atmosphere density)

$$\frac{F_f}{w} = 0.163\cdot\frac{N}{m}$$

Given: Flow of flat jet over sharp-edged splitter plate, as shown. Neglect friction force between water and plate; $0 \le \alpha \le 0.5$.

Find: (a) Expression for angle θ as a function of α
(b) Expression for force R_x needed to hold splitter plate in place.

Plot: both θ and R_x as functions of θ.

Solution

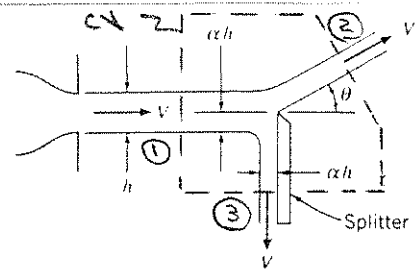
Apply the x and y components of the momentum equation to the CV shown.

Basic equations:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u(\rho \vec{V} \cdot d\vec{A})$$

with $F_{B_x} = 0(3)$, $\frac{\partial}{\partial t} = 0(4)$

$$F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v \rho \, d\forall + \int_{CS} v(\rho \vec{V} \cdot d\vec{A})$$

with $F_{S_y} = 0(2)$, $F_{B_y} = 0(3)$, $\frac{\partial}{\partial t} = 0(4)$

Assumptions: (1) no net pressure forces on CV
(2) no friction in y direction, so $F_{S_y} = 0$
(3) neglect body forces
(4) steady flow
(5) no change in jet speed: $V_1 = V_2 = V_3 = V$
(6) uniform flow at each section

Then from the y equation

$$0 = v_1\{-|\rho V_1 A_1|\} + v_2\{|\rho V_2 A_2|\} + v_3\{|\rho V_3 A_3|\}$$

$v_1 = 0$ $\quad v_2 = V\sin\theta$ $\quad v_3 = -V$

{w is depth} $\quad A_1 = wh$ $\quad A_2 = w(1-\alpha)h$ $\quad A_3 = w\alpha h$

$$0 = 0 + \rho V^2 \sin\theta \, w(1-\alpha)h - \rho V^2 w \alpha h$$

Thus

$$\sin\theta = \frac{\rho V^2 w \alpha h}{\rho V^2 w(1-\alpha)h} = \frac{\alpha}{(1-\alpha)} \; ; \quad \theta = \sin^{-1}\left(\frac{\alpha}{1-\alpha}\right) \qquad \theta(\alpha)$$

From the x equation

$$R_x = u_1\{-|\rho_1 V_1 A_1|\} + u_2\{|\rho_2 V_2 A_2|\} + u_3\{|\rho_3 V_3 A_3|\}$$

$u_1 = V$ $\quad u_2 = V\cos\theta$ $\quad u_3 = 0$

$$R_x = -\rho V^2 wh + \rho V^2 \cos\theta \, w(1-\alpha)h = \rho V^2 wh\left[\cos\theta(1-\alpha) - 1\right]$$

But

$$\cos\theta = (1-\sin^2\theta)^{1/2} = \left(1 - \frac{\alpha^2}{(1-\alpha)^2}\right)^{1/2} = \frac{(1-2\alpha)^{1/2}}{(1-\alpha)}$$

$$\therefore R_x = -\rho V^2 wh\left[1 - (1-2\alpha)^{1/2}\right] \qquad (R_x < 0; \text{ so to left}) \qquad R_x$$

{ Check: $\alpha = 0$, $R_x = 0$ ✓ ; $\alpha = \frac{1}{2}$, $R_x = -\rho V^2 wh$ ✓ }

Problem 4.102

Plots of: $\theta = \sin^{-1}\left(\frac{\alpha}{1-\alpha}\right)$ and

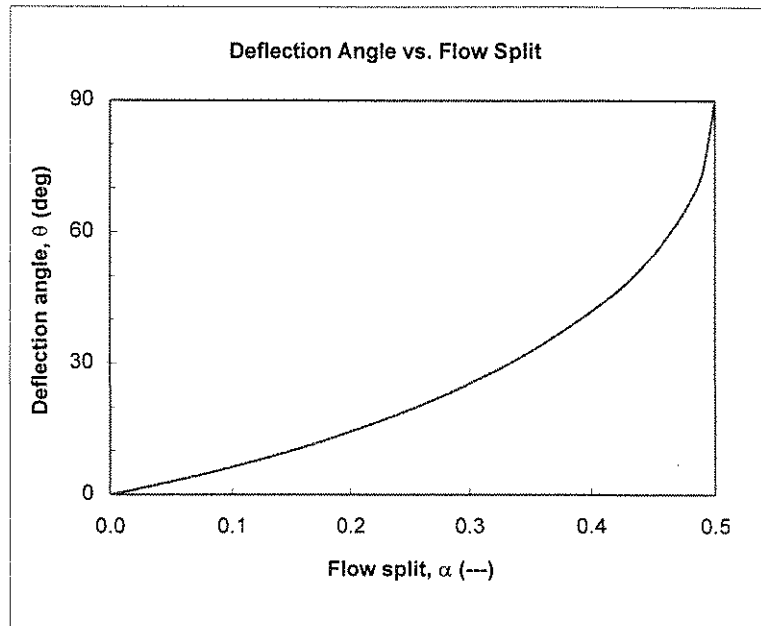
$$\frac{R_x}{R_{x\,\alpha=0.5}} = 1 - \sqrt{1-2\alpha}$$

are presented below

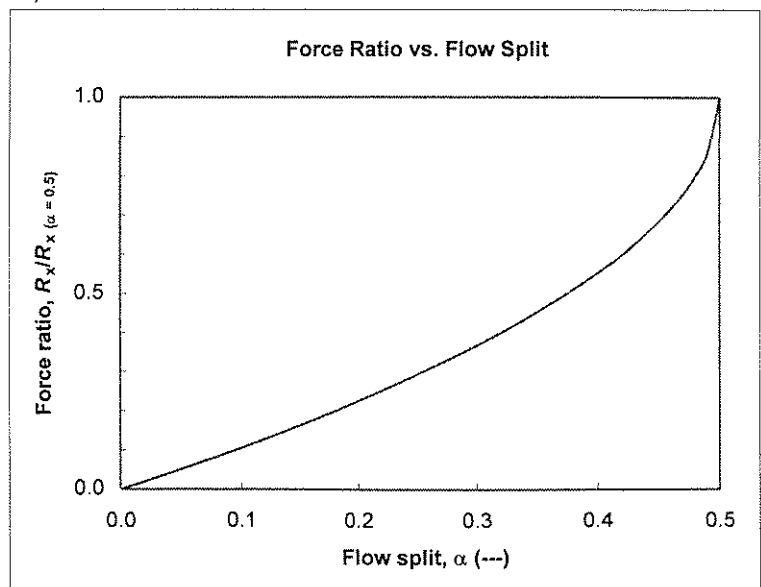
Flow deflection by sharp-edged splitter:

α = fraction of jet intercepted by splitter

Calculated Results: Deflection angle

α (---)	θ (deg)
0	0
0.05	3.02
0.10	6.38
0.15	10.2
0.20	14.5
0.25	19.5
0.30	25.4
0.35	32.6
0.40	41.8
0.425	47.7
0.45	54.9
0.470	62.5
0.480	67.4
0.490	73.9
0.50	90.0

Calculated Results: Force over maximum force

α (---)	$R_x/R_{x\ (\alpha=0.5)}$
0	0
0.05	0.0513
0.10	0.106
0.15	0.163
0.20	0.225
0.25	0.293
0.30	0.368
0.35	0.452
0.40	0.553
0.425	0.613
0.45	0.684
0.470	0.755
0.480	0.800
0.490	0.859
0.50	1.00

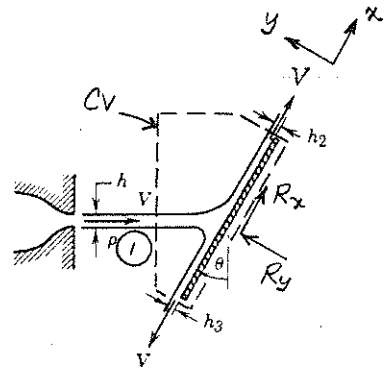
Problem 4.103

Given: Plane jet striking inclined plate, as shown. No frictional force along plate surface.

Find: (a) Expression for h_2/h as a function of θ.
(b) Plot of results.
(c) Comment on limiting cases, $\theta = 0$ and $\theta = 90°$

Solution: Apply the x component of the momentum equation using the CV and coordinates shown.

Basic equation:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

($F_{S_x} = 0(1)$, $F_{B_x} = 0(2)$, $\frac{\partial}{\partial t} = 0(3)$)

Assumptions: (1) No surface force on CV
(2) Neglect body forces
(3) Steady flow
(4) No change in jet speed: $V_1 = V_2 = V_3 = V$
(5) Uniform flow at each section

From continuity for uniform incompressible flow $0 = -\rho V w h + \rho V w h_2 + \rho V w h_3$

or

$$h = h_2 + h_3 = h_1 \quad \text{or} \quad h_3 = h_1 - h_2$$

From momentum

$$0 = u_1\{-|\rho V w h_1|\} + u_2\{+|\rho V w h_2|\} + u_3\{+|\rho V w h_3|\}$$

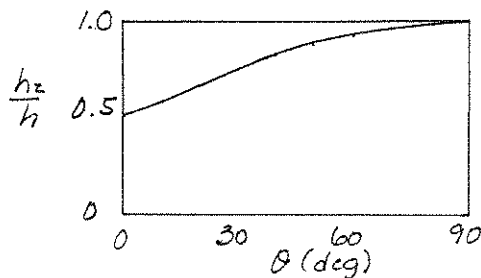
$$u_1 = V\sin\theta \qquad u_2 = V \qquad u_3 = -V$$

$$0 = -\rho V^2 \sin\theta\, w h_1 + \rho V^2 w h_2 - \rho V^2 w h_3$$

Substituting from continuity and simplifying

$$0 = -\sin\theta\, h_1 + h_2 - (h_1 - h_2) \quad \text{so} \quad \frac{h_2}{h} = \frac{h_2}{h_1} = \frac{1+\sin\theta}{2}$$

$\frac{h_2}{h}$

Plot:

At $\theta = 0$, $\frac{h_2}{h} = 0.5$; flow is equally split when plate is $\perp$ to jet.

At $\theta = 90°$, $\frac{h_2}{h} = 1.0$; plate has no effect on flow.

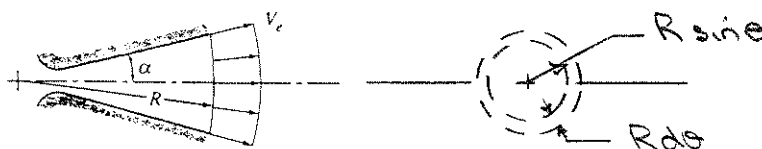
Problem 4.104

Given: Model gas flow in a propulsion nozzle as a spherical source; V_e = constant

Find: (a) Expression for axial thrust, T_a, and compare to the 1-D approximation, $T = \dot{m}V_e$
(b) Percent error for $\alpha = 15°$.

Plot: the percent error vs α for $0 \leq \alpha \leq 22.5°$.

Solution:

Apply definitions $\dot{m} = \int_A \rho V dA$, $T_a = \int_A u \rho V dA$. Use spherically symmetric flow.

The mass flow rate is [assuming $\rho_e \neq \rho_e(\theta)$]

$$\dot{m} = \int_A \rho V dA = \int_0^\alpha \rho_e V_e (2\pi R\sin\theta) R d\theta = 2\pi \rho_e V_e R^2 \left[-\cos\theta\right]_0^\alpha = 2\pi\rho_e V_e R^2 (1-\cos\alpha)$$

The one-dimensional approximation for thrust is then

$$T = \dot{m}V_e = 2\pi \rho_e V_e^2 R^2 (1-\cos\alpha) \qquad T_{1\text{-}D}$$

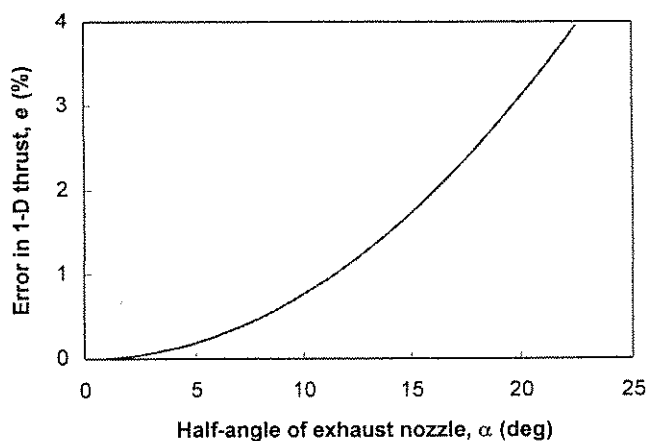
The axial thrust is given by

$$T_a = \int u \rho V dA = \int_0^\alpha V_e \cos\theta \, \rho_e V_e (2\pi R \sin\theta) R d\theta = 2\pi \rho_e V_e^2 R^2 \int_0^\alpha \sin\theta\cos\theta d\theta$$

$$T_a = 2\pi \rho_e V_e^2 R^2 \left[\frac{\sin^2\theta}{2}\right]_0^\alpha = \pi \rho_e V_e^2 R^2 \sin^2\alpha \qquad T_a$$

The error in the one-dimensional approximation is

$$e = \frac{T_{1\text{-}D} - T_a}{T_a} = \frac{T_{1\text{-}D}}{T_a} - 1 = \frac{2\pi\rho_e V_e^2 R^2(1-\cos\alpha)}{\pi \rho_e V_e^2 R^2 \sin^2\alpha} - 1 = \frac{2(1-\cos\alpha)}{\sin^2\alpha} - 1 \quad \text{---(1)}$$

The percent error is plotted as a function of α

For $\alpha = 15°$

$$e_{15} = \frac{2(1-\cos 15)}{\sin^2 15°} - 1$$

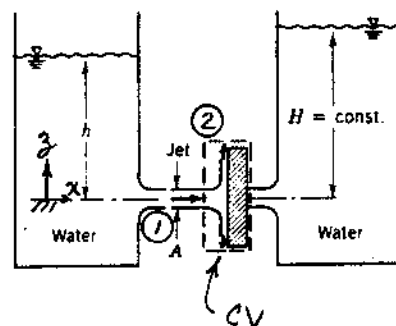
$$e_{15} = 0.0173 \text{ or } 1.73\% \qquad e_{15}$$

Problem *4.105

[4]

Given: Tanks and flat plate shown.

Find: Minimum height h needed to keep plate in place.

Solution: Apply Bernoulli and momentum equations. Use CV enclosing plate, as shown.

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

($F_{B_x} = 0(5)$, $\frac{\partial}{\partial t} = 0(1)$)

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Flow along a streamline
(4) No friction
(5) $F_{Bx} = 0$

Apply Bernoulli from water surface to jet

$$\frac{p}{\rho} + \frac{V_s^2}{2} + gh = \frac{p}{\rho} + \frac{V^2}{2} + g(0) \quad \text{so that} \quad V^2 = 2gh \text{ or } V = \sqrt{2gh}$$

($V_s^2 \ll V^2$)

From fluid statics, $p_{3g} = \rho g H$

From momentum

$$-p_{3g}A = -\rho g H A = u_1\{-\rho V A\} + u_2\{+\rho V A\} = -\rho V^2 A$$

$$u_1 = V \qquad u_2 = 0$$

Thus, using Bernoulli,

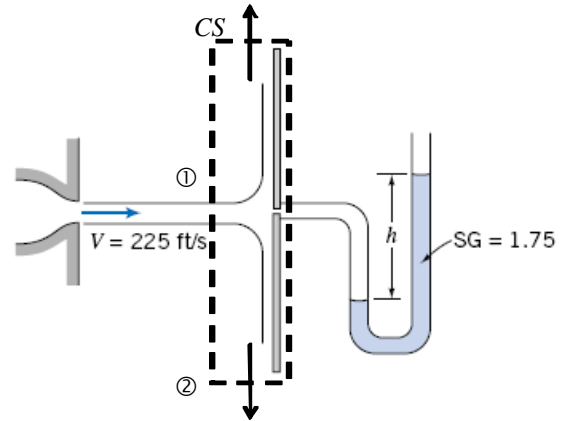
$$\rho g H A = \rho V^2 A = \rho(2gh)A = 2\rho g h A$$

and

$$h = \frac{H}{2}$$

h

Problem *4.106

*4.106 A horizontal axisymmetric jet of air with 0.5 in. diameter strikes a stationary vertical disk of 8 in. diameter. The jet speed is 225 ft/s at the nozzle exit. A manometer is connected to the center of the disk. Calculate (a) the deflection, h, if the manometer liquid has SG = 1.75 and (b) the force exerted by the jet on the disk.

Given: Air jet striking disk

Find: Manometer deflection; Force to hold disk

Solution:

Basic equations: Hydrostatic pressure, Bernoulli, and momentum flux in x direction

$$\frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{constant} \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV\!\!\!\!-\, + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible 3) No friction 4) Flow along streamline 5) Uniform flow 6) Horizontal flow ($g_x = 0$)

Applying Bernoulli between jet exit and stagnation point

$$\frac{p}{\rho_{air}} + \frac{V^2}{2} = \frac{p_0}{\rho_{air}} + 0 \qquad p_0 - p = \frac{1}{2}\cdot\rho_{air}\cdot V^2$$

But from hydrostatics $\quad p_0 - p = SG\cdot\rho\cdot g\cdot\Delta h \quad$ so $\quad \Delta h = \frac{\frac{1}{2}\cdot\rho_{air}\cdot V^2}{SG\cdot\rho\cdot g} = \frac{\rho_{air}\cdot V^2}{2\cdot SG\cdot\rho\cdot g}$

$$\Delta h = 0.002377\cdot\frac{slug}{ft^3}\times\left(225\cdot\frac{ft}{s}\right)^2\times\frac{1}{2\cdot 1.75}\times\frac{ft^3}{1.94\cdot slug}\times\frac{s^2}{32.2\cdot ft} \qquad \Delta h = 0.55\cdot ft \qquad \Delta h = 6.6\cdot in$$

For x momentum $\quad R_x = V\cdot\left(-\rho_{air}\cdot A\cdot V\right) = -\rho_{air}\cdot V^2\cdot\frac{\pi\cdot D^2}{4}$

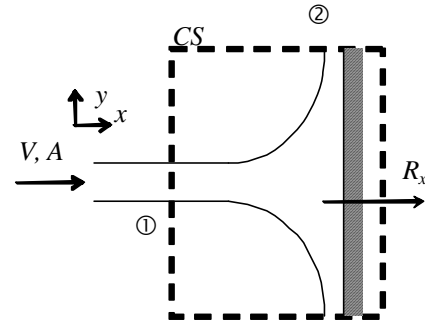
$$R_x = -0.002377\cdot\frac{slug}{ft^3}\times\left(225\cdot\frac{ft}{s}\right)^2\times\frac{\pi\cdot\left(\frac{0.5}{12}\cdot ft\right)^2}{4}\times\frac{lbf\cdot s^2}{slug\,ft} \qquad R_x = -0.164\cdot lbf$$

The force of the jet on the plate is then $\quad F = -R_x \qquad F = 0.164\cdot lbf$

Problem *4.107

[2]

*4.107 Students are planning a mock battle with water hoses on a campus lawn. The engineering students know that in order to have a greater impact on the adversary, it is advantageous to adjust the hose nozzle to create a narrower jet. How do they know this? Explain in terms of the force generated by a horizontal water jet impacting on a fixed vertical plane.

If 650 N is the maximum force that human skin can tolerate over a small area without damage, what is the maximum safe water flow (in liters per minute) that can be supplied to each hose when the minimum exit diameter of the nozzles is 6 mm?

Given: Water jet striking surface

Find: Force on surface

Solution:

Basic equations: Momentum flux in x direction $F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure throughout 4) Uniform flow

Hence $R_x = u_1\cdot(-\rho\cdot u_1\cdot A_1) = -\rho\cdot V^2\cdot A = -\rho\cdot\left(\frac{Q}{A}\right)^2\cdot A = -\frac{\rho\cdot Q^2}{A} = -\frac{4\cdot\rho\cdot Q^2}{\pi\cdot D^2}$ where Q is the flow rate

The force of the jet on the surface is then $F = -R_x = \frac{4\cdot\rho\cdot Q^2}{\pi\cdot D^2}$

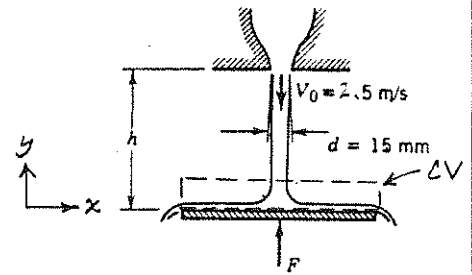
For a fixed flow rate Q, the force of a jet varies as $\frac{1}{D^2}$: A smaller diameter leads to a larger force. This is because as the diameter decreases the speed increases, and the impact force varies as the square of the speed, but linearly with area

For a force of F = 650 N

$$Q = \sqrt{\frac{\pi\cdot D^2\cdot F}{4\cdot\rho}} \qquad Q = \sqrt{\frac{\pi}{4}\times\left(\frac{6}{1000}\cdot m\right)^2\times 650\cdot N\times\frac{m^3}{1000\cdot kg}\times\frac{kg\cdot m}{s^2\cdot N}}\times\frac{1\cdot L}{10^{-3}\cdot m^3}\times\frac{60\cdot s}{1\cdot min} \qquad Q = 257\cdot\frac{L}{min}$$

Problem *4.108

Given: Jet flowing downward, striking horizontal disk, as shown.

Find: (a) Velocity in jet at h.
(b) Expression for force to hold disk.
(c) Evaluate for $h = 3.0\,m$.

Solution: Apply Bernoulli and momentum equations. Use CV shown.

Basic equations:

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant} \qquad (p = 0 \text{ by } (5))$$

$$F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{CV} w\rho\, d\forall + \int_{CS} w\rho \vec{V}\cdot d\vec{A} \qquad (F_{B_z}=0\ (6),\ \tfrac{\partial}{\partial t}=0\ (1))$$

Assumptions:
(1) Steady flow
(2) Incompressible flow
(3) Flow along a streamline
(4) Frictionless flow
(5) Atmospheric pressure along jet
(6) Neglect water on plate; $F_{B_z} = 0$
(7) Uniform flow at each section

The Bernoulli equation becomes

$$\frac{V_0^2}{2} + gh = \frac{V^2}{2} + g(0) \quad \text{or} \quad V^2 = V_0^2 + 2gh\,; \quad V = \sqrt{V_0^2 + 2gh} \qquad V$$

From the momentum equation

$$R_z = w_1\{-\rho V A\} + w_2\{+\rho V_0 A_0\} = +\rho V^2 A$$

$$w_1 = -V \qquad w_2 = 0$$

But from continuity, $\dot{m} = \rho V_0 A_0 = \rho V A$. Thus $VA = V_0 A_0$, and

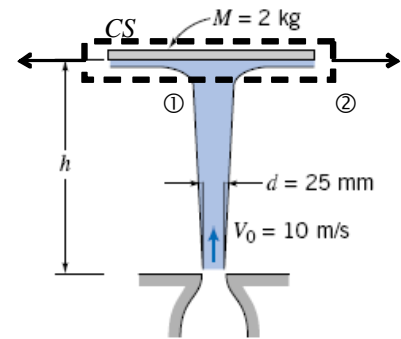
$$R_z = \rho V_0 A_0 V = \rho V_0 A_0 \sqrt{V_0^2 + 2gh} \qquad R_z$$

At $h = 3.0\,m$,

$$R_z = 999\,\frac{kg}{m^3} \times 2.5\,\frac{m}{s} \times \frac{\pi}{4}(0.015)^2 m^2 \left[(2.5)^2\frac{m^2}{s^2} + 2 \times 9.81\,\frac{m}{s^2} \times 3.0\,m\right]^{1/2} \frac{N\cdot s^2}{kg\cdot m}$$

$$R_z = 3.56\ N \ (\underline{\text{upward}} \text{ force}) \qquad R_z$$

Problem *4.109

***4.109** A 2-kg disk is constrained horizontally but is free to move vertically. The disk is struck from below by a vertical jet of water. The speed and diameter of the water jet are 10 m/s and 25 mm at the nozzle exit. Obtain a general expression for the speed of the water jet as a function of height, h. Find the height to which the disk will rise and remain stationary.

Given: Water jet striking disk

Find: Expression for speed of jet as function of height; Height for stationary disk

Solution:

Basic equations: Bernoulli; Momentum flux in z direction

$$\frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{constant} \qquad F_z = F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{CV} w\,\rho\,dV\!\!\!\!- + \int_{CS} w\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow

The Bernoulli equation becomes $\frac{V_0^2}{2} + g\cdot 0 = \frac{V^2}{2} + g\cdot h \qquad V^2 = V_0^2 - 2\cdot g\cdot h \qquad V = \sqrt{V_0^2 - 2\cdot g\cdot h}$

Hence $-M\cdot g = w_1\cdot\left(-\rho\cdot w_1\cdot A_1\right) = -\rho\cdot V^2\cdot A$

But from continuity $\rho\cdot V_0\cdot A_0 = \rho\cdot V\cdot A$ so $V\cdot A = V_0\cdot A_0$

Hence we get $M\cdot g = \rho\cdot V\cdot V\cdot A = \rho\cdot V_0\cdot A_0\cdot\sqrt{V_0^2 - 2\cdot g\cdot h}$

Solving for h
$$h = \frac{1}{2\cdot g}\cdot\left[V_0^2 - \left(\frac{M\cdot g}{\rho\cdot V_0\cdot A_0}\right)^2\right]$$

$$h = \frac{1}{2}\times\frac{s^2}{9.81\cdot m}\times\left[\left(10\cdot\frac{m}{s}\right)^2 - \left[2\cdot kg\times\frac{9.81\cdot m}{s^2}\times\frac{m^3}{1000\cdot kg}\times\frac{s}{10\cdot m}\times\frac{4}{\pi\cdot\left(\frac{25}{1000}\cdot m\right)^2}\right]^2\right]$$

$$h = 4.28\,m$$

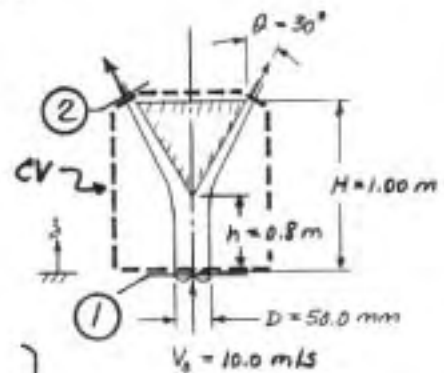
Given: Water jet supporting conical object, as shown.

Find: (a) Combined mass of cone and water, M, supported.
(b) Estimate mass of water in CV.

Solution: Apply continuity, Bernoulli, and momentum equations using CV shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$ (first term $= 0(1)$)

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 \quad (p_1, p_2: (6))$$

$$F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{CV} w \rho \, d\forall + \int_{CS} w \rho \vec{V} \cdot d\vec{A} \quad (F_{S_z} = 0(6), \text{ first term } = 0(1))$$

Assumptions:
(1) Steady flow
(2) No friction
(3) Flow along a streamline
(4) Incompressible flow
} required for Bernoulli
(5) Uniform flow at each cross-section
(6) $F_{S_z} = 0$ since p_{atm} acts everywhere

Then

$$0 = \{-|\rho V_1 A_1|\} + \{+|\rho V_2 A_2|\} \quad \text{so} \quad V_1 A_1 = V_2 A_2$$

From Bernoulli

$$\frac{V_1^2}{2} + g z_1 = \frac{V_2^2}{2} + g z_2 = \frac{V_0^2}{2} = \frac{V_2^2}{2} + gH \; ; \quad V_2^2 = V_0^2 - 2gH$$

From momentum

$$F_{B_z} = \int_{CS} w \rho \vec{V} \cdot d\vec{A} = -Mg = w_1\{-|\rho V_1 A_1|\} + w_2\{+|\rho V_2 A_2|\}$$

$$w_1 = V_0 \qquad w_2 = V_2 \cos\theta$$

or

$$-Mg = -V_0 \rho V_1 A_1 + V_2 \cos\theta \, \rho V_2 A_2 = \rho V_0 A_1 (V_2 \cos\theta - V_0)$$

so

$$M = \frac{(V_0 - V_2 \cos\theta) \rho V_0 A_1}{g}$$

From Bernoulli

$$V_2 = (V_0^2 - 2gH)^{1/2} = \left[(10)^2 \frac{m^2}{s^2} - 2 \times 9.81 \frac{m}{s^2} \times 1 \, m\right]^{1/2} = 8.97 \; m/s$$

Substituting

$$M = \left(10.0 \frac{m}{s} - 8.97 \frac{m}{s} \times \cos 30°\right) 999 \frac{kg}{m^3} \times 10 \frac{m}{s} \times \frac{\pi}{4}(0.050)^2 m^2 \times \frac{s^2}{9.81 \, m}$$

$M = 4.46$ kg ← (total mass in CV: water + object)

M

To find mass of water in CV, we have 3 options:

(1) assume area of jet is constant

$$M = \rho \forall \simeq \rho A_1 H = 999 \frac{kg}{m^3} \times \frac{\pi}{4}(0.05)^2 m^2 \times 1 m = 1.96 \text{ kg}$$

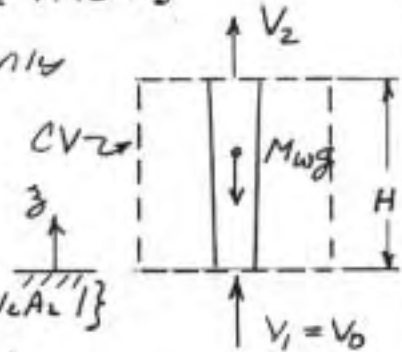
(2) Use a CV that encloses the free jet only

Continuity $V_1 A_1 = V_2 A_2$

Bernoulli $V_2 = (V_1^2 - 2gH)^{1/2}$

Momentum $-M_w g = u_1\{-|\rho V_1 A_1|\} + u_2\{+|\rho V_2 A_2|\}$

$u_1 = V_1 = V_0 \quad u_2 = V_2$

CV, z, V_2, $M_w g$, H, $V_1 = V_0$

Substituting in momentum

$$-M_w g = V_0(-\rho V_0 A_1) + V_2(+\rho V_0 A_1) = \rho V_0 A_1 (V_2 - V_0)$$

$$M_w = \frac{\rho V_0 A_1 (V_0 - V_2)}{g}$$

$$= 999 \frac{kg}{m^3} \times 10 \frac{m}{s} \times \frac{\pi}{4}(0.05)^2 m^2 (10 - 8.97)\frac{m}{s} \times \frac{s^2}{9.81 m}$$

$$M_w = 2.06 \text{ kg}$$

M_w

(3) Evaluate the area at each cross-section using Bernoulli and continuity, then integrate to find $\forall$.

$$VA = V_0 A_1 = (V_0^2 - 2gz)^{1/2} A = V_0 A_1 \quad \text{so} \quad A = \frac{V_0 A_1}{(V_0^2 - 2gz)^{1/2}}$$

$$\forall = \int_0^H A\,dz = \int_0^H \frac{V_0 A_1}{(V_0^2 - 2gz)^{1/2}}\,dz = A_1 \int_0^H \frac{V_0^2}{2g} \frac{1}{\left(1 - \frac{2gz}{V_0^2}\right)^{1/2}}\, d\left(\frac{2gz}{V_0^2}\right)$$

This can be integrated. Let $u = 1 - 2gz/V_0^2$, so $\int = \int \frac{-du}{u^{1/2}}$

Then $$\forall = A_1 \frac{V_0^2}{2g}\left[-2\left(1 - \frac{2gz}{V_0^2}\right)^{1/2}\right]_{z=0}^{z=h} = \frac{A_1}{g}\left[V_0^2 - V_0(V_0^2 - 2gh)^{1/2}\right]$$

and $$M_w = \rho \forall = \frac{\rho A_1 V_0 (V_0 - V_2)}{g} = 2.06 \text{ kg (same as (2) above)}$$

Thus the mass of the cone is $M_c = M - M_w = 2.40$ kg.

M_c

{Note: If V_0 were smaller or H larger, V_2 would differ more from V_0 and the jet area would increase significantly. Option (2) would still give the correct result with little effort.}

Given: Stream of air at standard conditions strikes a curved vane. Stagnation tube with water-filled manometer in exit plane.

Find: (a) Speed of air leaving nozzle.
(b) Horizontal component of force exerted on vane by jet.
(c) Comment on each assumption used to solve this problem.

Solution: Apply the definition of stagnation pressure and the x component of the momentum equation.

By definition $p_0 = p + \frac{1}{2}\rho_{air} V^2$

From fluid statics, $p_0 - p = \rho_{water}\, g\, \Delta h$

Combining, $\rho_{water}\, g\, \Delta h = \frac{1}{2}\rho_{air} V^2$ or $V = \sqrt{\dfrac{2\rho_{water}\, g\, \Delta h}{\rho_{air}}}$

$$V = \left[2 \times 1.94 \frac{slug}{ft^3} \times 32.2 \frac{ft}{s^2} \times 7\ in. \times \frac{ft^3}{0.00238\ slug} \times \frac{ft}{12\ in.}\right]^{\frac{1}{2}} = 175\ ft/s$$

V

The momentum equation is

$$F_{Sx} + \overset{=0(2)}{F_{Bx}} = \overset{=0(3)}{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall} + \int_{CS} u\, \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) No net pressure force
(2) $F_{Bx} = 0$
(3) Steady flow
(4) Uniform flow
(5) Constant speed on vane

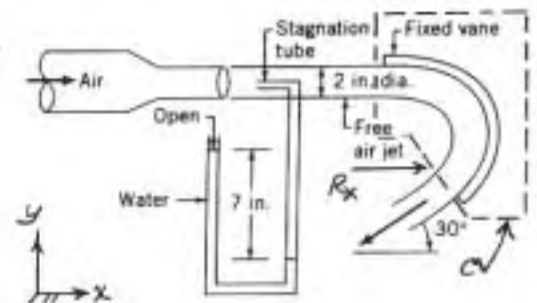
Then

$$R_x = u_1\{-\rho V A\} + u_2\{\rho V A\} = -\rho V^2 A(1 + \cos\theta)$$

$$u_1 = V \qquad u_2 = -V\cos\theta$$

$$R_x = -0.00238 \frac{slug}{ft^3} \times (175)^2 \frac{ft^2}{s^2} \times \frac{\pi}{4}\left(\frac{2}{12}\right)^2 ft^2 (1 + \cos 30°) = -2.97\ lbf$$

Force of air on vane is $K_x = -R_x = +2.97\ lbf$ (to right)

K_x

Comments on each assumption used to solve this problem:

- Frictionless flow in the nozzle is a good assumption.
- Incompressible flow is a good assumption for this low-speed flow.
- No horizontal component of body force is exact.
- No net pressure force on the control volume is exact.
- Frictionless flow along the vane is not realistic; air flow along the vane would be slowed by friction, reducing the momentum flux at the exit.

Problem *4.112

[2]

*4.112 A Venturi meter installed along a water pipe consists of a convergent section, a constant-area throat, and a divergent section. The pipe diameter is $D = 100$ mm and the throat diameter is $d = 40$ mm. Find the net fluid force acting on the convergent section if the water pressure in the pipe is 600 kPa (gage) and the average velocity is 5 m/s. For this analysis neglect viscous effects.

Given: Data on flow and venturi geometry

Find: Force on convergent section

Solution:

The given data is $\rho = 999 \cdot \frac{kg}{m^3}$ $\quad D = 0.1 \cdot m$ $\quad d = 0.04 \cdot m$ $\quad p_1 = 600 \cdot kPa$ $\quad V_1 = 5 \cdot \frac{m}{s}$

Then $A_1 = \frac{\pi \cdot D^2}{4}$ $\quad A_1 = 0.00785\, m^2$ $\quad A_2 = \frac{\pi}{4} \cdot d^2$ $\quad A_2 = 0.00126\, m^2$

$Q = V_1 \cdot A_1$ $\quad Q = 0.0393 \frac{m^3}{s}$ $\quad V_2 = \frac{Q}{A_2}$ $\quad V_2 = 31.3 \frac{m}{s}$

Governing equations:

Bernoulli equation $\quad \frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{const}$ $\qquad$ (4.24)

Momentum $\quad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t} \int_{CV} u \, \rho \, d\forall + \int_{CS} u \, \rho \vec{V} \cdot d\vec{A}$ $\qquad$ (4.18a)

Applying Bernoulli between inlet and throat $\quad \frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_2}{\rho} + \frac{V_2^2}{2}$

Solving for p_2 $\quad p_2 = p_1 + \frac{\rho}{2} \cdot \left(V_1^2 - V_2^2\right)$ $\quad p_2 = 600 \cdot kPa + 999 \cdot \frac{kg}{m^3} \times \left(5^2 - 31.3^2\right) \cdot \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} \times \frac{kN}{1000 \cdot N}$ $\quad p_2 = 125 \cdot kPa$

Applying the horizontal component of momentum

$-F + p_1 \cdot A_2 - p_2 \cdot A_2 = V_1 \cdot \left(-\rho \cdot V_1 \cdot A_1\right) + V_2 \cdot \left(\rho \cdot V_2 \cdot A_2\right)$ $\qquad$ or $\qquad$ $F = p_1 \cdot A_1 - p_2 \cdot A_2 + \rho \cdot \left(V_1^2 \cdot A_1 - V_2^2 \cdot A_2\right)$

$$F = 600 \cdot \frac{kN}{m^2} \times 0.00785 \cdot m^2 - 125 \cdot \frac{kN}{m^2} \times 0.00126 \cdot m^2 + 999 \cdot \frac{kg}{m^3} \times \left[\left(5 \cdot \frac{m}{s}\right)^2 \cdot 0.00785 \cdot m^2 - \left(31.3 \cdot \frac{m}{s}\right)^2 \cdot 0.00126 \cdot m^2\right] \cdot \frac{N \cdot s^2}{kg \times m}$$

$F = 3.52 \cdot kN$

Problem *4.113

Given: Plane nozzle discharging water steadily, striking an inclined plate.

Neglect friction in nozzle and along plate surface.

Find: (a) Minimum gage pressure at nozzle inlet.
(b) Magnitude and direction of force exerted by water stream on plate.
(c) Sketch pressure distribution on plate. Explain why the pressure distribution is shaped as you show it.

Solution: Apply continuity, Bernoulli, and momentum equations using the CV and coordinates shown.

Basic equations: $V_1 A_1 = V_2 A_2 \qquad \frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2$

$$R_y + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\rho \, d\forall + \int_{CS} v \rho \vec{V}\cdot d\vec{A}$$

(F_{B_y} = 0 (6); $\frac{\partial}{\partial t}$ = 0 (3))

Assumptions: (1) Frictionless flow
(2) Incompressible flow
(3) Steady flow
(4) Flow along a streamline
(5) Uniform flow at each section

Then from continuity $V_1 = \frac{A_2}{A_1}V_2 = \frac{w}{W}V_2 = \frac{12.7\text{ mm}}{51.8\text{ mm}} \times 12.2\frac{m}{s} = 2.99\text{ m/s}$

From Bernoulli $p_{1g} = \frac{\rho}{2}(V_2^2 - V_1^2) - \rho g(z_1 - z_2) + p_{2g}^{\;=0} \quad ; \; z_1 - z_2 = h$

$$p_{1g} = \left[\frac{1}{2}\times 999\frac{kg}{m^3}\times[(12.2)^2 - (2.99)^2]\frac{m^2}{s^2} - 999\frac{kg}{m^3}\times 9.81\frac{m}{s^2}\times 0.15\,m\right]\frac{N\cdot s^2}{kg\cdot m} = 68.4\text{ kPa (gage)} \quad p_{1g}$$

Calculate V_3 in the absense of the plate using Bernoulli ($p_2 = p_3$)

$$V_3 = \sqrt{V_2^2 + 2gH} = \left[(12.2)^2\frac{m^2}{s^2} + 2\times 9.81\frac{m}{s^2}\times 4.85\,m\right]^{\frac{1}{2}} = 15.6\text{ m/s}$$

From momentum: $R_x = 0$ since there is no friction on the plate surface.

Assumptions: (6) Neglect mass of plate and of water on plate.
(7) Atmospheric pressure acts on entire CV; $F_{S_y} = R_y$

Then $R_y = v_3\{-\dot{m}_3\} + v_4^{\,0}\{+\dot{m}_4\} + v_5^{\,0}\{+\dot{m}_5\} = V_3\cos\theta\,\rho Q$, since $v_3 = -V_3\cos\theta$

$$R_y = 15.6\frac{m}{s}\times\cos 30°\times 999\frac{kg}{m^3}\times 0.0155\frac{m^3}{s}\times\frac{N\cdot s^2}{kg\cdot m} = 209\text{ N}; \; K_y = -R_y = -209\text{ N} \qquad K_y$$

Pressure is maximum at stagnation, minimum (p_{atm}) at ④ and ⑤.
Pressure at ⓐ is higher than at ⓑ because of streamline curvature.

Given: Egyptian water clock. Surface level drops at rate, $\dot{s}$ = constant.

Find: (a) Expression for $r(h)$.
(b) Volume needed for n hours' operation.

Solution: Apply conservation of mass and the Bernoulli equation.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

Assumptions: (1) Quasi-steady flow; $\frac{\partial}{\partial t}$ small
(2) Incompressible flow
(3) Uniform flow at each cross section
(4) Flow along a streamline
(5) No friction
(6) $\rho_{air} \ll \rho_{H_2O}$

Writing Bernoulli from the liquid surface to the jet exit,

$$\frac{p_{atm}}{\rho} + \frac{\dot{s}^2}{2} + gh = \frac{p_{atm}}{\rho} + \frac{V^2}{2} + g(0);$$

For $\dot{s} \ll V$, then $V = \sqrt{2gh}$.

For the CV,

$$0 = \frac{\partial}{\partial t}\int_{\forall_{air}} \rho_{air}\, d\forall + \frac{\partial}{\partial t}\int_{\forall_{H_2O}} \rho_{H_2O}\, d\forall + \{-|\rho_{air} V_1 A_1|\} + \{|\rho_{H_2O} VA|\}$$

(terms with ρ_{air} crossed out: $\ll \rho_{H_2O}$ (6))

or

$$0 = \rho \frac{d\forall}{dt} + \rho VA = \rho \pi r^2 \frac{dh}{dt} + \rho\sqrt{2gh}\, A = 0$$

But h decreases, so $\frac{dh}{dt} = -\dot{s}$. Thus

$$\pi r^2 \dot{s} = \sqrt{2gh}\, A \quad \text{or} \quad r = \sqrt[4]{2g}\sqrt{\frac{A}{\pi \dot{s}}}\, h^{1/4}$$

r

For n hours' operation, $H = n\dot{s}$, and

$$\forall = \int_0^H \pi r^2\, dh = \int_0^{n\dot{s}} \sqrt{2gh}\,\frac{A}{\dot{s}}\, dh = \frac{2A}{3\dot{s}}\sqrt{2g}\,(n\dot{s})^{3/2}$$

or

$$\forall = \frac{2A\sqrt{2g}\, n^{3/2}\dot{s}^{1/2}}{3}$$

$\forall$

Check dimensions:

$$[\forall] = L^3 = [A\sqrt{g}\, n^{3/2}\dot{s}^{1/2}] = L^2 \frac{L^{1/2}}{t} t^{3/2} \frac{L^{1/2}}{t^{1/2}} = L^3 \checkmark$$

Problem *4.115

Given: Low-speed jet of incompressible liquid moving upward from nozzle.

Find: Expressions for $V(z)$, $A(z)$.
Location where $V = 0$.

Solution: Apply continuity and momentum equation using CV shown.

V+dV
A+dA
CV
dz
V, A
z
V_0, A_0

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} \qquad \left(\frac{\partial}{\partial t} = 0\ (1)\right)$$

$$F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{CV} w \rho \, d\forall + \int_{CS} w \rho \vec{V} \cdot d\vec{A} \qquad \left(F_{S_z} = 0\ (4,5),\ \frac{\partial}{\partial t} = 0\ (1)\right)$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) p_{atm} acts everywhere } $F_{S_z} = 0$
(5) No friction

Then

$$0 = \int_{CS} \vec{V} \cdot d\vec{A} = \{-VA\} + \{+(V+dV)(A+dA)\}; \quad VA = V_0 A_0 = \text{constant}$$

From momentum,

$$-\rho g \left(A + \frac{dA}{2}\right) dz = V\{-\rho VA\} + (V+dV)\{+\rho(V+dV)(A+dA)\} = \rho VA\,dV$$

since $dVdA \ll dA$. Also, since $dA\,dz \ll dz$, the left side is $-\rho g A\,dz$.
Thus

$$-\rho g A\,dz = \rho VA\,dV \quad \text{or} \quad V\,dV = -g\,dz$$

Integrating from V_0 at $z_0 = 0$ to V at z,

$$\int_{V_0}^{V} V\,dV = \frac{V^2}{2}\Big]_{V_0}^{V} = \frac{V^2}{2} - \frac{V_0^2}{2} = \int_{z_0}^{z} -g\,dz = -g(z - z_0) = -gz$$

Thus $V^2 = V_0^2 - 2gz$ or $V(z) = \sqrt{V_0^2 - 2gz}$ ← $V(z)$

Since $VA = V_0 A_0$, then $A = A_0 \frac{V_0}{V}$

$$A(z) = A_0 \frac{V_0}{\sqrt{V_0^2 - 2gz}} = \frac{A_0}{\sqrt{1 - 2gz/V_0^2}}$$ ← $A(z)$

Solving for z at $V = 0$,

$$z = \frac{V_0^2}{2g}$$ ← z

Problem *4.116

Given: Low-speed jet of incompressible liquid moving downward from nozzle.

Find: Expressions for $V(z)$, $A(z)$.
Location where $A = A_0/2$.

Solution: Apply continuity and momentum equations using CV shown.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} \qquad \left(\frac{\partial}{\partial t} = 0(1)\right)$$

$$F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{CV} w \rho \, d\forall + \int_{CS} w \rho \vec{V} \cdot d\vec{A} \qquad \left(F_{S_z} = 0(4,5), \ \frac{\partial}{\partial t} = 0(1)\right)$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) p_{atm} acts everywhere } $F_{S_z} = 0$
(5) No friction

Then

$$0 = \int_{CS} \vec{V} \cdot d\vec{A} = \{-VA\} + \{+(V+dV)(A+dA)\} \ ; \ VA = V_0 A_0 = \text{constant}$$

From momentum,

$$\rho g \left(A + \frac{dA}{2}\right) dz = V\{-\rho VA\} + (V+dV)\{-\rho (V+dV)(A+dA)\} = \rho VA \, dV$$

since $dV dA \ll dA$. Also, since $dA\,dz \ll dz$, the left side is $\rho g A dz$.

Thus

$$\rho g A dz = \rho V A dV \quad \text{or} \quad V dV = g dz$$

Integrating from V_0 at $z_0 = 0$ to V at z,

$$\int_{V_0}^{V} V dV = \frac{V^2}{2}\Big]_{V_0}^{V} = \frac{V^2}{2} - \frac{V_0^2}{2} = \int_{z_0}^{z} g \, dz = g(z - z_0) = gz$$

Thus

$$V^2 = V_0^2 + 2gz \quad \text{or} \quad V(z) = \sqrt{V_0^2 + 2gz}$$

$V(z)$

Since $VA = V_0 A_0$, $A = A_0 \frac{V_0}{V}$

$$A(z) = A_0 \frac{V_0}{\sqrt{V_0^2 + 2gz}} = \frac{A_0}{\sqrt{1 + 2gz/V_0^2}}$$

$A(z)$

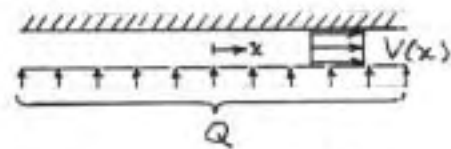
Solving for z,

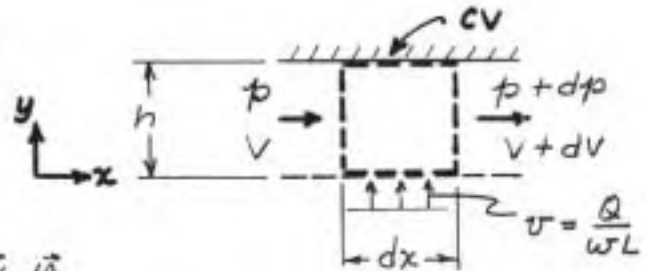
$$z = \frac{V_0^2}{2g}\left[\left(\frac{A_0}{A}\right)^2 - 1\right] \ ; \ \text{for } \frac{A}{A_0} = \frac{1}{2}, \ \frac{A_0}{A} = 2, \text{ and } z_{1/2} = \frac{3V_0^2}{2g}$$

$z_{1/2}$

Problem *4.117

Given: Uniform flow in narrow gap between parallel plates, as shown.

Fluid in gap has only horizontal motion.

Find: Expression for $p(x)$.

Solution: Apply continuity and x component of momentum equation.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} \quad (=0(1))$$

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A} \quad (F_{Bx}=0(5),\ \tfrac{\partial}{\partial t}=0(1))$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) Neglect friction
(5) $F_{Bx} = 0$

Then

$$0 = \int_{CS} \vec{V} \cdot d\vec{A} = \{-Vwh\} + \left\{-\frac{Q}{wL} w\,dx\right\} + \{(V+dV)wh\}; \quad wh\,dV = \frac{Q}{L} dx$$

$$V = \frac{Q}{wh}\frac{x}{L} + c; \quad c = 0 \text{ since } V(0) = 0; \quad V(x) = \frac{Q}{wh}\frac{x}{L}$$

From momentum,

$$p wh - (p+dp) wh = u_x \{-\rho V wh\} + u_{dx}\left\{-\rho \frac{Q}{wh} w\,dx\right\} + u_{x+dx}\{+\rho(V+dV)wh\}$$

$$u_x = V \qquad u_{dx} = 0 \qquad u_{x+dx} = V + dV$$

From continuity, $(V+dV)wh = Vwh + Q\frac{dx}{L}$, so

$$-dp\,wh = -\rho V^2 wh + 0 + (V+dV)\left(Vwh + Q\frac{dx}{L}\right)\rho$$

$$= -\rho V^2 wh + \rho V^2 wh + \rho V wh\,dV + V\rho Q\frac{dx}{L} + \rho Q\,dV\frac{dx}{L}$$

Neglecting products of differentials ($dV dx \ll dx$), and with $dV = \frac{Q}{wh}\frac{dx}{L}$

$$-dp = \rho V dV + \frac{V\rho Q}{wh}\frac{dx}{L} = \rho V \frac{Q}{wh}\frac{dx}{L} + \frac{V\rho Q}{wh}\frac{dx}{L} = 2\rho \frac{Q}{wh}\frac{x}{L}\frac{Q}{wh}\frac{dx}{L}$$

$$-dp = 2\rho\left(\frac{Q}{whL}\right)^2 x\,dx \qquad p(x) = -\rho\left(\frac{Q}{whL}\right)^2 x^2 + C$$

If $p(0) = p_0$, then $p(x) = p_0 - \rho\left(\frac{Q}{wh}\right)^2\left(\frac{x}{L}\right)^2$

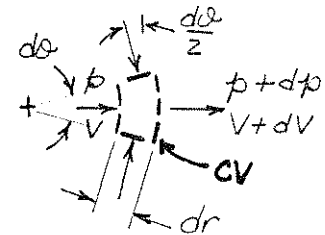
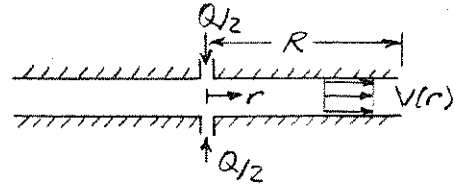
$p(x)$

Problem *4.118

Given: Uniform flow in narrow gap between parallel disks, as shown.

Liquid in gap has only radial motion.

Find: Expression for $p(r)$; plot

Solution: Apply continuity and momentum equations to the differential CV shown.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} \qquad \left(\frac{\partial}{\partial t}\int_{CV}\rho\,d\forall = 0\ (1)\right)$$

$$F_{S_r} + F_{B_r} = \frac{\partial}{\partial t}\int_{CV} V_r \rho\, d\forall + \int_{CS} V_r \rho \vec{V}\cdot d\vec{A} \qquad \left(F_{B_r} = 0\ (5),\ \frac{\partial}{\partial t}\int_{CV} V_r\rho\,d\forall = 0\ (1)\right)$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) Neglect friction
(5) $F_{B_r} = 0$
(6) No flow in θ direction
(7) $\sin\frac{d\theta}{2} = \frac{d\theta}{2}$

Then

$$0 = \int_{CS} \vec{V}\cdot d\vec{A} = \{-\rho V h r d\theta\} + \{\rho(V+dV)h(r+dr)d\theta\}\ ;\ Vr = \text{constant}$$

For $r = R$, $Q = V_R\, 2\pi r h$, so $V_R = Q/2\pi R h$

From momentum,

$$p\,hr\,d\theta + 2\left(p + \frac{dp}{2}\right)h\,dr \sin\frac{d\theta}{2} - (p+dp)h(r+dr)d\theta$$

$$= V\{-\rho V h r d\theta\} + (V+dV)\{\rho(V+dV)h(r+dr)d\theta\}$$

$$p\,hr\,d\theta + p\,h\,dr\,d\theta + \tfrac{1}{2}dp\,h\,dr\,d\theta - (pr + p\,dr + r\,dp + dr\,dp)h\,d\theta$$

$$= dV(\rho V h r\, d\theta) \qquad \{\text{Note terms in braces are equal.}\}$$

Assuming products of differentials are much smaller than single differentials,

$$-r\,dp\,h\,d\theta = dV(\rho V h r\, d\theta) \quad \text{or} \quad dp = -\rho V dV$$

Integrating, $p(r) - p(R) = -\rho\frac{V^2}{2} + \rho\frac{V_R^2}{2}$ or $p(r) - p_{atm} = \frac{\rho}{2}(V_R^2 - V^2)$
$= \rho\frac{V_R^2}{2}\left[1 - \left(\frac{V}{V_R}\right)^2\right]$

Since $V_R = \frac{Q}{2\pi R h}$, and Vr = constant, $\frac{V}{V_R} = \frac{R}{r}$, so

$$p(r) - p_{atm} = \frac{\rho}{2}\left(\frac{Q}{2\pi R h}\right)^2\left[1 - \left(\frac{R}{r}\right)^2\right]$$

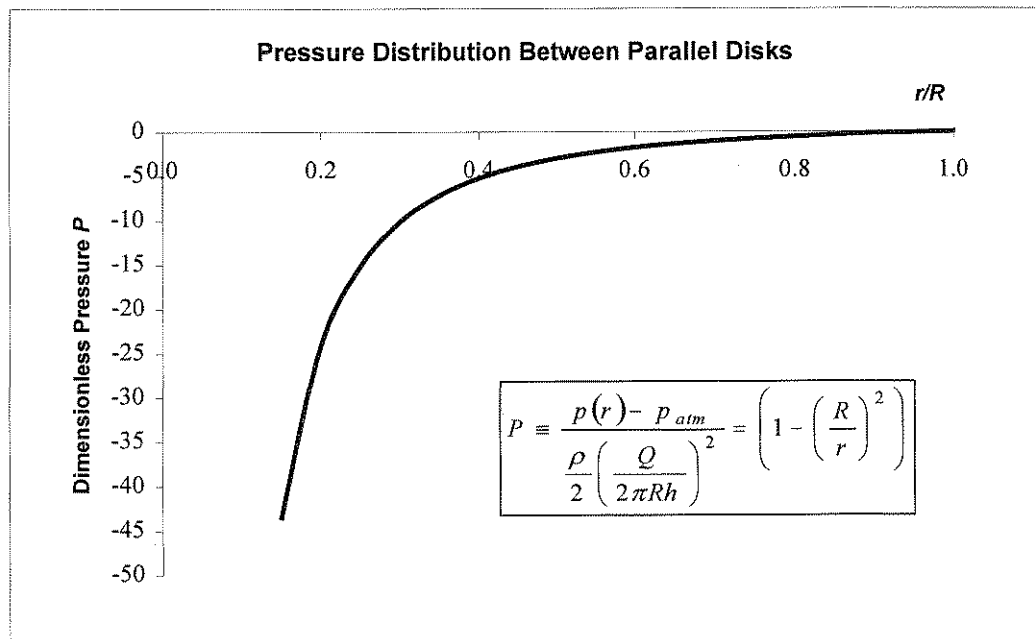
$p(r)$

Note since $r < R$, that $p(r) < p_{atm}$ between the disks.

Problem *4.118

The pressure distribution is computed and plotted in *Excel*:

r/R	*P*
0.15	-43.4
0.20	-24.0
0.25	-15.0
0.30	-10.1
0.35	-7.16
0.40	-5.25
0.45	-3.94
0.50	-3.00
0.55	-2.31
0.60	-1.78
0.65	-1.37
0.70	-1.04
0.75	-0.78
0.80	-0.563
0.85	-0.384
0.90	-0.235
0.95	-0.108
1.00	0.000

Problem *4.119

Given: Narrow gap between parallel disks filled with liquid.

At $t = 0^+$, upper disk begins to move downward at V_0.

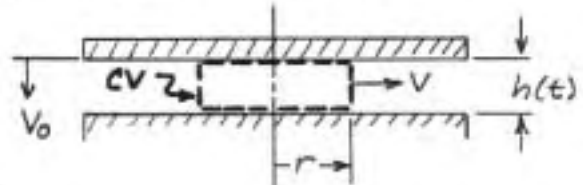
Neglect viscous effects; flow uniform in horizontal direction.

Find: Expression for velocity field, $V(r)$. Note flow is not steady.

Solution: Apply continuity, using the deformable CV shown.

Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, dV + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

CV V V_0 $h(t)$ r

Assumptions: (1) Incompressible flow
(2) Uniform flow at each cross section

Then

$$0 = \frac{\partial}{\partial t}\int_{CV} dV + \int_{CS} \vec{V} \cdot d\vec{A} = \frac{\partial}{\partial t}\int_{CV} dV + V\,2\pi r h$$

But

$$\int_{CV} dV = \pi r^2 h, \quad \text{so} \quad \frac{\partial}{\partial t}\int_{CV} dV = \frac{\partial}{\partial t}(\pi r^2 h) = \pi r^2 \frac{dh}{dt}$$

Thus

$$0 = \pi r^2 \frac{dh}{dt} + V\,2\pi r h = \pi r^2(-V_0) + V\,2\pi r h$$

so

$$V(r) = V_0 \frac{r}{2h}$$

$V(r)$

If V_0 is constant, so $h = h_0 - V_0 t$, and

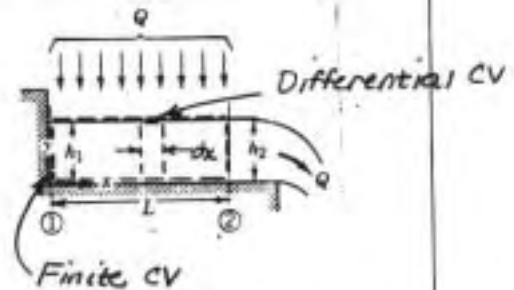
$$V(r,t) = \frac{V_0 r}{2(h_0 - V_0 t)} \quad \text{for } t < \frac{h_0}{V_0}$$

$V(r,t)$

Given: Liquid falling vertically into short, horizontal, rectangular open channel. Neglect viscous effects.

Find: (a) Expression for h_1 in terms of h_2, Q, and b.
(b) Sketch surface profile, $h(x)$.

<u>Solution</u>: Apply continuity and momentum equations to (i) finite CV, and (ii) differential CV, as shown.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} \quad (=0(1))$$

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A} \quad (F_{B_x}=0(6),\ \tfrac{\partial}{\partial t}=0(1))$$

Differential CV

Finite CV

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) Hydrostatic pressure distribution; $F_p(h) = \rho g b \frac{h^2}{2}$
(5) No friction on bed
(6) Horizontal bed; $F_{B_x} = 0$

Then for finite CV shown,

$$0 = \int_{CS} \vec{V}\cdot d\vec{A} = -Q + V_2 b h_2 \; ; \quad V_2 = \frac{Q}{bh_2}$$

From momentum

$$\rho g b \frac{h_1^2}{2} - \rho g b \frac{h_2^2}{2} = u_1\{0\} + u_2\{+\rho Q\} + u_3\{-\rho Q\}$$

$$u_2 = V_2 \qquad u_3 = 0$$

$$\frac{\rho g b}{2}(h_1^2 - h_2^2) = V_2 \rho Q = \frac{Q}{bh_2}\rho Q = \frac{\rho Q^2}{bh_2} \; ; \quad h_1 = \sqrt{h_2^2 + \frac{2Q^2}{g b^2 h_2}} \qquad h_1$$

For differential CV shown,

$$0 = \int_{CS} \vec{V}\cdot d\vec{A} = \{-Vbh\} + \left\{-\frac{Q}{bL}\, b\,dx\right\} + \{+(V+dV)\,b\,(h+dh)\}$$

$$0 = -\frac{Q}{L}dx + b(h\,dV + V\,dh) = -\frac{Q}{L}dx + b\,d(hV) \; ; \quad \frac{d(hV)}{dx} = \frac{Q}{L}$$

From momentum,

$$\rho g b \frac{h^2}{2} - \rho g b \frac{(h+dh)^2}{2} = V\{-\rho V b h\} + 0\left\{-\rho\frac{Q}{L}dx\right\} + (V+dV)\{+\rho(V+dV)b(h+dh)\}$$

Using continuity,

$$\frac{\rho g b}{2}(-2h\,dh + dh\,dh) = -\rho V^2 bh + (V+dV)\left\{\rho V b h + \rho\frac{Q}{L}dx\right\} \quad (dh\,dh \ll dh)$$

Cont'd. →

$$-\rho g b h dh = -\rho \not{V}^2 bh + \rho \not{V}^2 bh + \rho V bh dV + \frac{\rho Q}{L} V dx + \frac{\rho Q}{L} dV \not{dx}$$ ($\ll dx$)

or

$$-g h dh = V h dV + \frac{Q}{bL} V dx$$

From continuity, $VhdV = -V^2 dh + \frac{Q}{bL} V dx$, so

$$-g h dh = -V^2 dh + \frac{2Q}{bL} V dx$$

Solving,

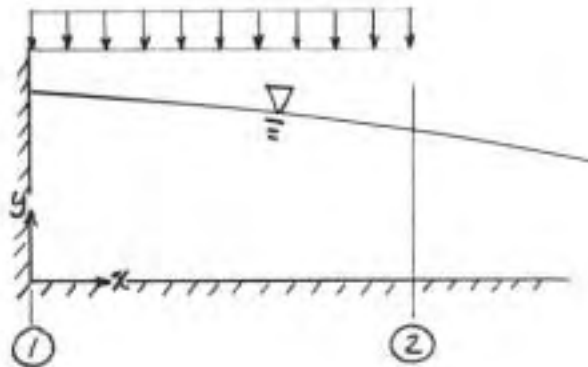
$$\frac{dh}{dx}(V^2 - gh) = \frac{2Q}{bL} V$$

$$\frac{dh}{dx} = \frac{2QV}{bL(V^2-gh)} = \frac{2QV}{bLgh(V^2/gh - 1)}$$

From finite CV analysis, $h_1 > h_2$, so $\frac{dh}{dx} < 0$. Thus $V^2/gh < 1$. As x increases, $V \uparrow$ and $h \downarrow$. Therefore

$$\frac{V^2}{gh} \uparrow, \quad \frac{V}{h} \uparrow, \quad \text{and} \quad \left|\frac{dh}{dx}\right| \uparrow.$$

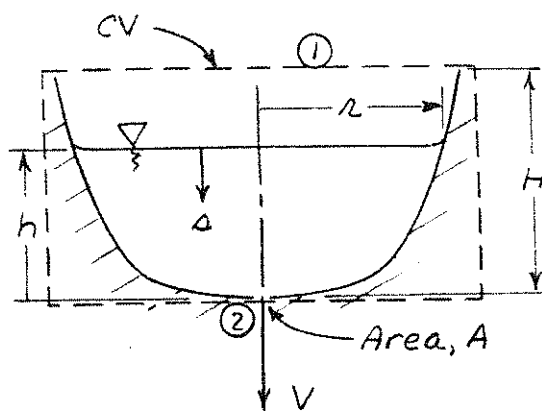
Sketch:

Problem *4.121

Open-Ended Problem Statement: Design a clepsydra (Egyptian water clock) — a vessel from which water drains by gravity through a hole in the bottom and time is indicated by the level of the remaining water. Specify the dimensions of the vessel and the size of the drain hole; indicate the amount of water needed to fill the vessel, and at what interval it must be filled. Plot the vessel shape. (This is an open-ended problem when choosing dimensions for a specific application.)

Discussion: The original Egyptian water clock was an open water-filled vessel with an orifice in the bottom. The vessel shape was designed so that the water level dropped at a constant rate during use.

Water leaves the orifice at higher speed when the water level within the vessel is high, and at lower speed when the water level within the vessel is low. The size of the orifice is constant. Thus the instantaneous volume flow rate depends on the water level in the vessel.

The rate at which the water level falls in the vessel depends on the volume flow rate and the area of the water surface. The surface area at each water level must be chosen so that the water level within the vessel decreases at a constant rate. The continuity and Bernoulli equations can be applied to determine the required vessel shape so that the water surface level drops at a constant rate.

Use the CV and notation shown (Problem 4.97):

Solution: Basic equations are

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, dV + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

Assumptions:
(1) Quasi-steady flow
(2) Incompressible flow
(3) Uniform flow at each cross-section
(4) Flow along a streamline
(5) No friction
(6) $\rho_{air} \ll \rho_{H_2O}$

Writing Bernoulli from the liquid surface to the jet exit,

$$\frac{p_{atm}}{\rho} + \frac{\Delta^2}{2} + gh = \frac{p_{atm}}{\rho} + \frac{V^2}{2} + g(0)$$

For $\Delta \ll V$, then $V = \sqrt{2gh}$

For the CV,

$$0 = \frac{\partial}{\partial t}\int_{V_{air}} \rho_{air} \, dV + \frac{\partial}{\partial t}\int_{V_{H_2O}} \rho_{H_2O} \, dV + \left\{-|\rho_{air} V_1 A_1|\right\} + \left\{|\rho_{H_2O} V A|\right\}$$

(air terms: $\ll \rho_{H_2O}$ (6))

or $0 = \rho \frac{d\forall}{dt} + \rho V A = \rho \pi r^2 \frac{dh}{dt} + \rho \sqrt{2gh} A$

But h decreases, so $\frac{dh}{dt} = -s$. Thus

$$\pi r^2 s = \sqrt{2gh} A \quad \text{or} \quad r = \sqrt[4]{2g} \sqrt{\frac{A}{\pi s}} h^{1/4}$$

r

For n hours operation, $H = ns$, and

$$\forall = \int_0^H \pi r^2 dh = \int_0^{ns} \sqrt{2gh} \frac{A}{s} dh = \frac{2A}{3s} \sqrt{2g} (ns)^{3/2}$$

or

$$\forall = \frac{2A\sqrt{2g}\, n^{3/2}}{3} s^{1/2}$$

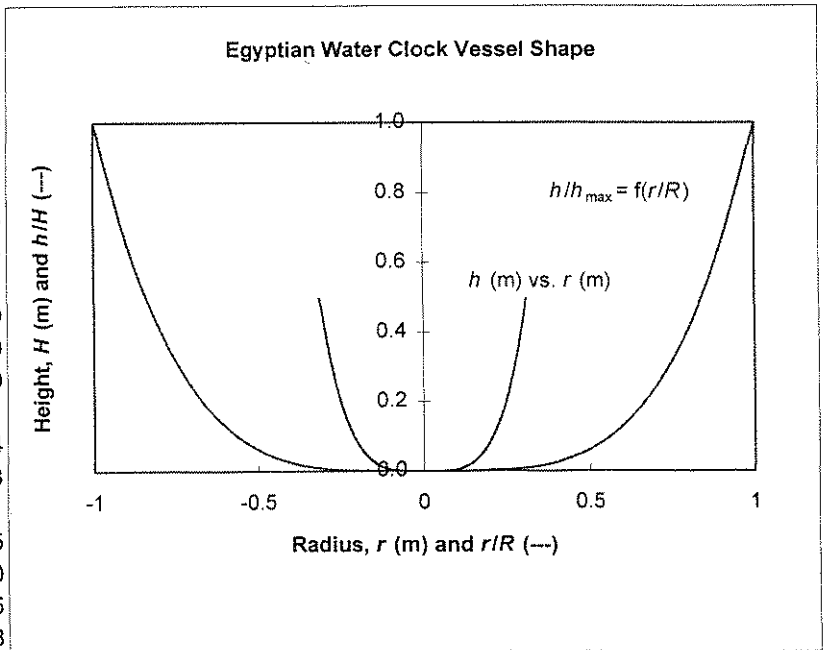
$\forall$

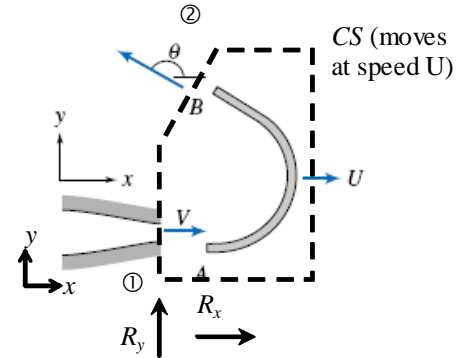
Evaluating and plotting:

Input Parameters:

Maximum water height:	H =	0.5	m
Number of hours' duration:	n =	24	hr

Dimensionless Shape		Actual Shape	
r/R	***h/H***	***r* (m)**	***h* (m)**
-1	1.00	-0.309	0.500
-0.9	0.656	-0.278	0.328
-0.8	0.410	-0.247	0.205
-0.7	0.240	-0.216	0.120
-0.6	0.130	-0.185	0.065
-0.5	0.063	-0.155	0.031
-0.4	0.026	-0.124	0.013
-0.3	0.008	-0.093	0.004
-0.2	0.002	-0.062	0.001
-0.1	0.000	-0.031	0.000
0	0	0	0
0.1	0.000	0.031	0.000
0.2	0.002	0.062	0.001
0.3	0.008	0.093	0.004
0.4	0.026	0.124	0.013
0.5	0.063	0.155	0.031
0.6	0.130	0.185	0.065
0.7	0.240	0.216	0.120
0.8	0.410	0.247	0.205
0.9	0.656	0.278	0.328
1	1.000	0.309	0.500

Egyptian Water Clock Vessel Shape

Problem *4.122

4.122 A jet of water is directed against a vane, which could be a blade in a turbine or in any other piece of hydraulic machinery. The water leaves the stationary 40-mm diameter nozzle with a speed of 25 m/s and enters the vane tangent to the surface at A. The inside surface of the vane at B makes angle $\theta = 150°$ with the x direction. Compute the force that must be applied to maintain the vane speed constant at $U = 5$ m/s.

Given: Water jet striking moving vane

Find: Force needed to hold vane to speed U = 5 m/s

Solution:

Basic equations: Momentum flux in x and y directions

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

$$F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\,\rho\,dV + \int_{CS} v\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow 5) Jet relative velocity is constant

Then $R_x = u_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) + u_2\cdot\left(\rho\cdot V_2\cdot A_2\right) = -(V-U)\cdot[\rho\cdot(V-U)\cdot A] + (V-U)\cdot\cos(\theta)\cdot[\rho\cdot(V-U)\cdot A]$

$R_x = \rho(V-U)^2\cdot A\cdot(\cos(\theta)-1)$ $\quad A = \frac{\pi}{4}\cdot\left(\frac{40}{1000}\cdot m\right)^2$ $\quad A = 1.26\times10^{-3}\,m^2$

Using given data

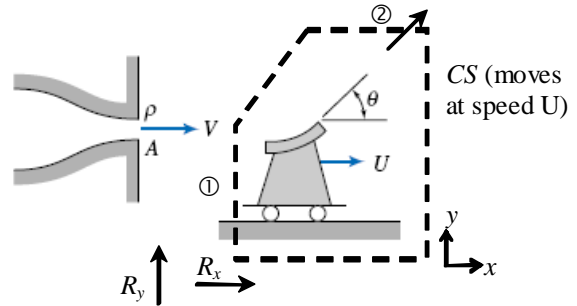
$$R_x = 1000\cdot\frac{kg}{m^3}\times\left[(25-5)\cdot\frac{m}{s}\right]^2\times1.26\times10^{-3}\cdot m^2\times(\cos(150\cdot deg)-1)\times\frac{N\cdot s^2}{kg\cdot m} \qquad R_x = -940\,N$$

Then $R_y = v_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) + v_2\cdot\left(\rho\cdot V_2\cdot A_2\right) = -0 + (V-U)\cdot\sin(\theta)\cdot[\rho\cdot(V-U)\cdot A]$

$R_y = \rho(V-U)^2\cdot A\cdot\sin(\theta)$ $\quad R_y = 1000\cdot\frac{kg}{m^3}\times\left[(25-5)\cdot\frac{m}{s}\right]^2\times1.26\times10^{-3}\cdot m^2\times\sin(150\cdot deg)\times\frac{N\cdot s^2}{kg\cdot m}$ $\quad R_y = 252\,N$

Hence the force required is 940 N to the left and 252 N upwards to maintain motion at 5 m/s

Problem 4.123

4.123 Water from a stationary nozzle impinges on a moving vane with turning angle $\theta = 120°$. The vane moves away from the nozzle with constant speed, $U = 10$ m/s, and receives a jet that leaves the nozzle with speed $V = 30$ m/s. The nozzle has an exit area of 0.004 m^2. Find the force that must be applied to maintain the vane speed constant.

Given: Water jet striking moving vane

Find: Force needed to hold vane to speed U = 10 m/s

Solution:

Basic equations: Momentum flux in x and y directions

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

$$F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\,\rho\,dV + \int_{CS} v\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow 5) Jet relative velocity is constant

Then
$$R_x = u_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) + u_2\cdot\left(\rho\cdot V_2\cdot A_2\right) = -(V-U)\cdot[\rho\cdot(V-U)\cdot A] + (V-U)\cdot\cos(\theta)\cdot[\rho\cdot(V-U)\cdot A]$$

$$R_x = \rho(V-U)^2\cdot A\cdot(\cos(\theta)-1)$$

Using given data

$$R_x = 1000\cdot\frac{kg}{m^3}\times\left[(30-10)\cdot\frac{m}{s}\right]^2\times 0.004\cdot m^2\times(\cos(120\cdot deg)-1)\times\frac{N\cdot s^2}{kg\cdot m}$$
$$R_x = -2400\,N$$

Then
$$R_y = v_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) + v_2\cdot\left(\rho\cdot V_2\cdot A_2\right) = -0 + (V-U)\cdot\sin(\theta)\cdot[\rho\cdot(V-U)\cdot A]$$

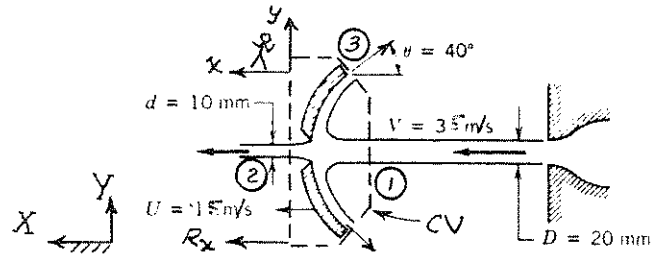
$$R_y = \rho(V-U)^2\cdot A\cdot\sin(\theta) \qquad R_y = 1000\cdot\frac{kg}{m^3}\times\left[(30-10)\cdot\frac{m}{s}\right]^2\times 0.004\cdot m^2\times\sin(120\cdot deg)\times\frac{N\cdot s^2}{kg\cdot m}$$
$$R_y = 1386\,N$$

Hence the force required is 2400 N to the left and 1390 N upwards to maintain motion at 10 m/s

Problem 4.124

Given: Circular dish and jet moving as shown.

Find: Force required to maintain dish motion.

Solution: Apply continuity and x momentum equation to CV moving with dish as shown.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}_{xyz} \cdot d\vec{A} \quad (=0(1))$$

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho \, d\forall + \int_{CS} u_{xyz} \rho \vec{V} \cdot d\vec{A} \quad (F_{B_x}=0(3),\ \tfrac{\partial}{\partial t}=0(1))$$

Assumptions: (1) Steady flow w.r.t. CV
(2) No pressure forces on CV
(3) Horizontal; $F_{B_x} = 0$
(4) Uniform flow at each section
(5) No change in speed of jet relative to vane
(6) Incompressible flow

Then

$$0 = \int_{CS} \vec{V}_{xyz} \cdot d\vec{A} = (V-U)\left(-\frac{\pi D^2}{4} + \frac{\pi d^2}{4} + A_{3,4}\right)$$

$$A_{3,4} = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}\left[(0.020)^2 - (0.010)^2\right] m^2 = 2.36 \times 10^{-4}\, m^2$$

From the momentum equation

$$R_x = u_1\left\{-\rho(V-U)\frac{\pi D^2}{4}\right\} + u_2\left\{+\rho(V-U)\frac{\pi d^2}{4}\right\} + u_3\left\{+\rho(V-U)A_{3,4}\right\}$$

$$u_1 = V - U \qquad u_2 = V - U \qquad u_3 = -(V-U)\cos 40°$$

$$R_x = -\rho(V-U)^2\frac{\pi D^2}{4} + \rho(V-U)^2\frac{\pi d^2}{4} - \rho(V-U)^2\frac{\pi}{4}(D^2-d^2)\cos 40°$$

$$= -\rho(V-U)^2\frac{\pi}{4}(D^2-d^2)(1+\cos 40°)$$

$$= -999\,\frac{kg}{m^3} \times (35-15)^2\frac{m^2}{s^2} \times 2.36\times 10^{-4}\, m^2 (1+\cos 40°) \times \frac{N\cdot s^2}{kg\cdot m}$$

$R_x = -167$ N (force must be applied to right) — R_x

{ Note: $R_y = Mg$, since there is no net momentum flux in the y-direction. }

Problem 4.125

[2]

4.125 A jet boat takes in water at a constant volumetric rate Q through side vents and ejects it at a high jet speed V_j at the rear. A variable-area exit orifice controls the jet speed. The drag on the boat is given by $F_{drag} \approx kV^2$, where V is the boat speed. Find an expression for the steady speed V. If a jet speed $V_j = 25$ m/s produces a boat speed of 10 m/s, what jet speed will be required to double the boat speed?

Given: Data on jet boat

Find: Formula for boat speed; jet speed to double boat speed

CV in boat coordinates

Solution:

Governing equation:

Momentum
$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V}_{xyz}\,\rho\, d\forall + \int_{CS} \vec{V}_{xyz}\,\rho\,\vec{V}_{xyz} \cdot d\vec{A} \quad (4.26)$$

Applying the horizontal component of momentum

$$F_{drag} = V\cdot(-\rho\cdot Q) + V_j\cdot(\rho\cdot Q)$$ or, with $F_{drag} = k\cdot V^2$ $\qquad k\cdot V^2 = \rho\cdot Q\cdot V_j - \rho\cdot Q\cdot V$

$$k\cdot V^2 + \rho\cdot Q\cdot V - \rho\cdot Q\cdot V_j = 0$$

Solving for V
$$V = -\frac{\rho\cdot Q}{2\cdot k} + \sqrt{\left(\frac{\rho\cdot Q}{2\cdot k}\right)^2 + \frac{\rho\cdot Q\cdot V_j}{k}}$$

Let
$$\alpha = \frac{\rho\cdot Q}{2\cdot k}$$

$$V = -\alpha + \sqrt{\alpha^2 + 2\cdot\alpha\cdot V_j}$$

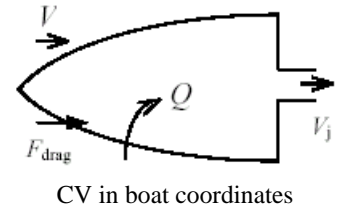
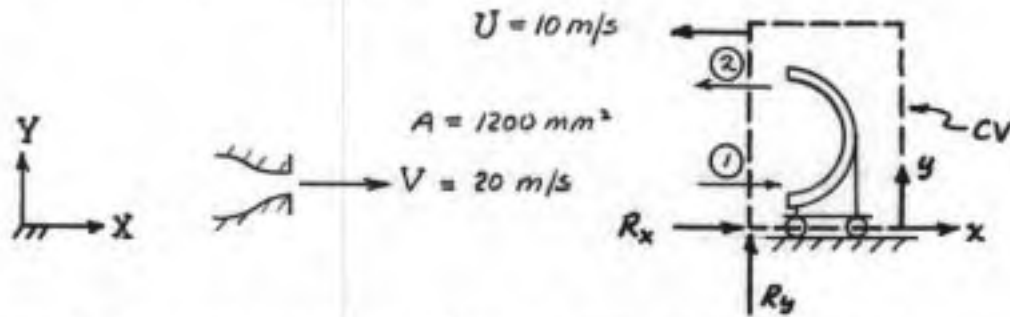
We can use given data at V = 10 m/s to find α $\qquad V = 10\cdot\frac{m}{s} \qquad V_j = 25\cdot\frac{m}{s}$

$$10\cdot\frac{m}{s} = -\alpha + \sqrt{\alpha^2 + 2\cdot 25\cdot\frac{m}{s}\cdot\alpha} \qquad \alpha^2 + 50\cdot\alpha = (10+\alpha)^2 = 100 + 20\cdot\alpha + \alpha^2 \qquad \alpha = \frac{10}{3}\cdot\frac{m}{s}$$

Hence
$$V = -\frac{10}{3} + \sqrt{\frac{100}{9} + \frac{20}{3}\cdot V_j}$$

For V = 20 m/s
$$20 = -\frac{10}{3} + \sqrt{\frac{100}{9} + \frac{20}{3}\cdot V_j} \qquad \frac{100}{9} + \frac{20}{3}\cdot V_j = \frac{70}{3} \qquad V_j = 80\cdot\frac{m}{s}$$

Problem 4.126

[2]

Given: Jet of oil (SG = 0.8) striking moving vane.

Find: Force needed to maintain vane speed constant.

Solution: Apply x component of momentum equation to moving CV shown.

Basic equation:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

(F_{B_x} = 0 (2); $\frac{\partial}{\partial t}$ term = 0 (3))

Assumptions: (1) No net pressure force on CV; $F_{S_x} = R_x$
(2) $F_{B_x} = 0$
(3) Steady flow
(4) Flow uniform at each section
(5) Jet area and speed relative to vane are constant

The subscript xyz is a reminder that all velocities must be evaluated relative to the CV. Then

$$R_x = u_1\{-|\rho(V+U)A|\} + u_2\{|\rho(V+U)A|\}$$

$$u_1 = V + U \qquad\qquad u_2 = -(V+U)$$

and $R_x = -\rho(V+U)^2A - \rho(V+U)^2A = -2\rho(V+U)^2A = -2\,SG\,\rho_{H_2O}(V+U)^2A$

$$R_x = -2(0.8)\,999\,\frac{kg}{m^3}\,(20+10)^2\frac{m^2}{s^2}\times 1200\,mm^2 \times \frac{m^2}{10^6\,mm^2}\times\frac{N\cdot s^2}{kg\cdot m} = -1.73\text{ kN}$$

R_x ←

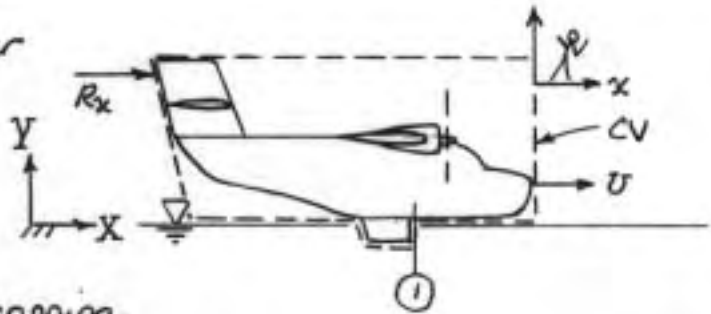
This force must be applied to the left on the vane.

{Note $R_y = mg$, since there are no vertical components of velocity.}

Problem 4.127

Given: Aircraft scooping water from lake:

1620 gal in 12 sec

Find: Added thrust needed to maintain steady aircraft speed during scooping.

Solution: Use CV moving with aircraft, as shown. Apply momentum.

Basic equation: $F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, dV + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$ ($F_{B_x} = 0$ (1), $\frac{\partial}{\partial t} \approx 0$ (2))

Assumptions: (1) Horizontal motion, so $F_{Bx} = 0$
(2) Neglect u_{xyz} within the CV
(3) Uniform flow at inlet cross-section
(4) Neglect hydrostatic pressure

Then

$$R_x = u_1\{-|\rho Q_1|\} = -U(-\rho Q) = +U\rho Q$$

$$u_1 = -U$$

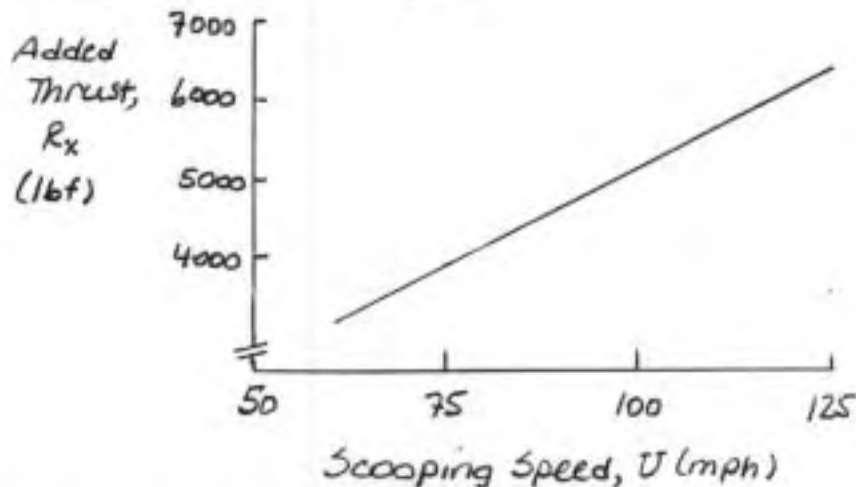
From data given

$$Q = \frac{\Delta V}{\Delta t} = \frac{1620\ \text{gal}}{12\ \text{sec}} \times \frac{\text{ft}^3}{7.48\ \text{gal}} = 18.0\ \text{ft}^3/\text{s}$$

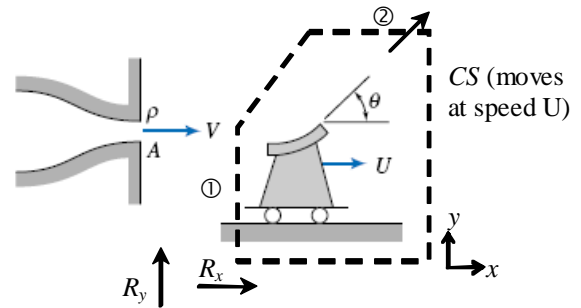
For an aircraft speed of $U = 75$ mph (110 ft/s)

$$R_x = 110\,\frac{\text{ft}}{\text{s}} \times 1.94\,\frac{\text{slug}}{\text{ft}^3} \times 18.0\,\frac{\text{ft}^3}{\text{s}} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} = 3{,}840\ \text{lbf}$$

R_x

For a range of aircraft speeds:

{Thus at 60 mph the added thrust is 3,070 lbf, while at 125 mph the added thrust is 6,400 lbf.}

Problem 4.128

4.128 Consider a single vane, with turning angle θ, moving horizontally at constant speed, U, under the influence of an impinging jet as in Problem 4.123. The absolute speed of the jet is V. Obtain general expressions for the resultant force and power that the vane could produce. Show that the power is maximized when $U = V/3$.

Given: Water jet striking moving vane

Find: Expressions for force and power; Show that maximum power is when U = V/3

Solution:

Basic equation: Momentum flux in x direction
$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow 5) Jet relative velocity is constant

Then
$$R_x = u_1\cdot(-\rho\cdot V_1\cdot A_1) + u_2\cdot(\rho\cdot V_2\cdot A_2) = -(V-U)\cdot[\rho\cdot(V-U)\cdot A] + (V-U)\cdot\cos(\theta)\cdot[\rho\cdot(V-U)\cdot A]$$

$$R_x = \rho(V-U)^2\cdot A\cdot(\cos(\theta) - 1)$$

This is force on vane; Force exerted by vane is equal and opposite
$$F_x = \rho\cdot(V-U)^2\cdot A\cdot(1-\cos(\theta))$$

The power produced is then
$$P = U\cdot F_x = \rho\cdot U\cdot(V-U)^2\cdot A\cdot(1-\cos(\theta))$$

To maximize power wrt to U
$$\frac{dP}{dU} = \rho\cdot(V-U)^2\cdot A\cdot(1-\cos(\theta)) + \rho\cdot(2)\cdot(-1)\cdot(V-U)\cdot U\cdot A\cdot(1-\cos(\theta)) = 0$$

Hence
$$V - U - 2\cdot U = V - 3\cdot U = 0 \qquad U = \frac{V}{3}$$ for maximum power

Note that there is a vertical force, but it generates no power

Given: Circular dish with $D = 0.15$ m and jet as shown.

Find: (a) Thickness of jet sheet at $R = 75$ mm.
(b) Horizontal force required to maintain dish motion.

Solution: Apply the momentum equation to a CV moving with the dish, as shown.

Basic equation:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

($F_{B_x} = 0(2)$, $\frac{\partial}{\partial t} = 0(3)$)

Assumptions: (1) No pressure forces
(2) Horizontal; $F_{B_x} = 0$
(3) Steady flow w.r.t. CV
(4) Uniform flow at each section
(5) Use relative velocities
(6) No change in relative velocity on the dish

Then

$$R_x = u_1\{-\rho(V-U)A\} + u_2\{+\rho(V-U)A\}$$

$$u_1 = V-U \qquad u_2 = -(V-U)\cos\theta$$

$$R_x = -\rho(V-U)^2A - \rho(V-U)^2A\cos\theta = -\rho(V-U)^2A(1+\cos\theta)$$

$$= -999\frac{kg}{m^3}\times(45-10)^2\frac{m^2}{s^2}\times\frac{\pi}{4}(0.050)^2m^2(1+\cos 40°)\frac{N\cdot s^2}{kg\cdot m}$$

$$R_x = -4.24 \text{ kN} \text{ (force must act to right)}$$

R_x

Apply conservation of mass to determine the jet sheet thickness:

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A}$

Using the above assumptions, then

$$0 = -\rho V_1A_1 + \rho V_2A_2$$

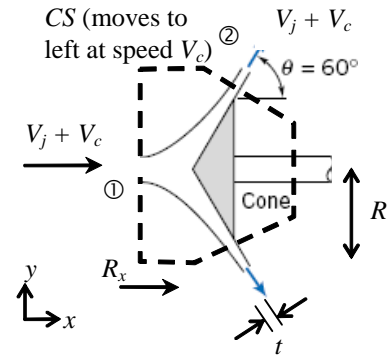
$$V_1 = V-U \;;\; V_2 = V-U \;;\; A_1 = \frac{\pi d^2}{4} \;;\; A_2 = 2\pi Rt$$

Therefore $A_1 = A_2 = \frac{\pi d^2}{4} = 2\pi Rt$, and $t = \frac{d^2}{8R}$

$$t = \frac{1}{8}\times(0.050)^2m^2\times\frac{1}{0.075\, m} = 4.17\times10^{-3}\, m \text{ or } 4.17 \text{ mm}$$

t

Problem 4.130

4.130 Water, in a 4-in. diameter jet with speed of 100 ft/s to the right, is deflected by a cone that moves to the left at 45 ft/s. Determine (a) the thickness of the jet sheet at a radius of 9 in. and (b) the external horizontal force needed to move the cone.

Given:
Water jet striking moving cone

Find:
Thickness of jet sheet; Force needed to move cone

Solution:

Basic equations: Mass conservation; Momentum flux in x direction

$$\frac{\partial}{\partial t}\int_{CV} \rho\, dV + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0 \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\, \rho\, dV + \int_{CS} u\, \rho \vec{V} \cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow 5) Jet relative velocity is constant

Then $\quad -\rho \cdot V_1 \cdot A_1 + \rho \cdot V_2 \cdot A_2 = 0 \qquad -\rho \cdot (V_j + V_c) \cdot \frac{\pi \cdot D_j^2}{4} + \rho \cdot (V_j + V_c) \cdot 2 \cdot \pi \cdot R \cdot t = 0$ (Refer to sketch)

Hence $\quad t = \frac{D_j^2}{8 \cdot R} \qquad t = \frac{1}{8} \times (4 \cdot \text{in})^2 \times \frac{1}{9 \cdot \text{in}} \qquad t = 0.222\,\text{in}$

Using relative velocities, x momentum is

$$R_x = u_1 \cdot (-\rho \cdot V_1 \cdot A_1) + u_2 \cdot (\rho \cdot V_2 \cdot A_2) = -(V_j + V_c) \cdot \left[\rho \cdot (V_j + V_c) \cdot A_j\right] + (V_j + V_c) \cdot \cos(\theta) \cdot \left[\rho \cdot (V_j + V_c) \cdot A_j\right]$$

$$R_x = \rho (V_j + V_c)^2 \cdot A_j \cdot (\cos(\theta) - 1)$$

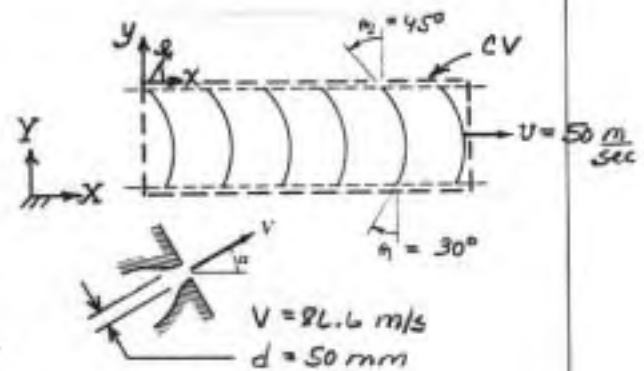
Using given data

$$R_x = 1.94 \cdot \frac{\text{slug}}{\text{ft}^3} \times \left[(100 + 45) \cdot \frac{\text{ft}}{\text{s}}\right]^2 \times \frac{\pi \cdot \left(\frac{4}{12} \cdot \text{ft}\right)^2}{4} \times (\cos(60 \cdot \text{deg}) - 1) \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}} \qquad R_x = -1780 \cdot \text{lbf}$$

Hence the force is 1780 lbf to the left; the upwards equals the weight

Given: Series of vanes struck by continuous jet, as shown.

Find: (a) Nozzle angle, α.
(b) Force to hold vane speed constant.

Solution: Apply momentum equation using CV moving with vanes, as shown.

Basic equation:

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

($F_{Bx} = 0$(2), $\frac{\partial}{\partial t} = 0$(3))

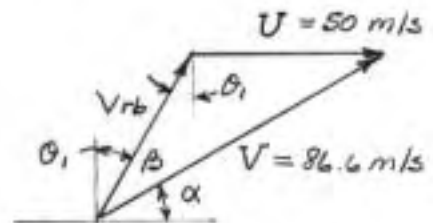
Assumptions: (1) No pressure forces
(2) Horizontal; $F_{Bx} = 0$
(3) Steady flow w.r.t. CV
(4) Uniform flow at each section
(5) No change in relative velocity on vane
(6) Flow enters and leaves tangent to vanes

The nozzle angle may be obtained from trigonometry. The inlet velocity relationship is shown in the sketch:

From the law of sines,

$$\frac{\sin\alpha}{V_{rb}} = \frac{\sin(90+\theta_1)}{V} = \frac{\sin\beta}{U}$$

$U = 50$ m/s, V_{rb}, θ_1, β, $V = 86.6$ m/s, α

$$\beta = \sin^{-1}\left[\frac{U}{V}\sin(90+\theta_1)\right] = \sin^{-1}\left[\frac{50}{86.6}\sin(120°)\right] = 30°$$

From the sketch, $90° = \alpha + \beta + \theta_1$, so $\alpha = 90° - \beta - \theta_1 = 90° - 30° - 30° = 30°$ $\qquad\alpha$

Also $V_{rb}\cos\theta_1 = V\sin\alpha$; $V_{rb} = V\frac{\sin\alpha}{\cos\theta_1} = 86.6\frac{m}{s}\times\frac{\sin 30°}{\cos 30°} = 50.0$ m/s

From momentum equation (note all of $\dot{m}$ flows across vanes)

$$R_x = u_1\{-\dot{m}\} + u_2\{\dot{m}\} = V_{rb}\sin\theta_1(-\dot{m}) - V_{rb}\sin\theta_2(\dot{m}) = V_{rb}\dot{m}(-\sin\theta_1 - \sin\theta_2)$$

$$u_1 = V_{rb}\sin\theta_1 \quad u_2 = -V_{rb}\sin\theta_2; \qquad R_y = \dot{m}V_{rb}(-\cos\theta_1 + \cos\theta_2)$$

Thus, since $\dot{m} = \rho Q$,

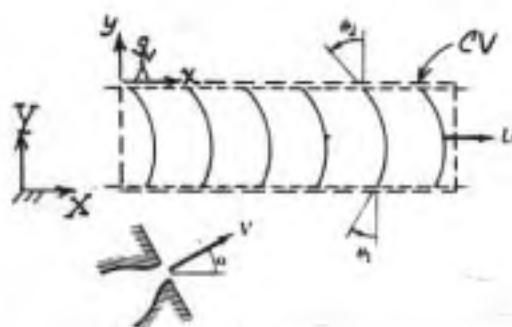
$$R_x = V_{rb}\rho Q(-\sin\theta_1 - \sin\theta_2)$$

$$= 50\frac{m}{s}\times 999\frac{kg}{m^3}\times 0.170\frac{m^3}{s}(-\sin 30° - \sin 45°)\times\frac{N\cdot s^2}{kg\cdot m}$$

$R_x = -10.3$ kN (to left) $\qquad R_x$

{Note: The net force on the CV in the y-direction is $R_y = -1.35$ kN.}

Given: Series of vanes struck by continuous jet, as shown.

Find: For $\alpha \approx 0$ ($\theta_1 \approx 90°$), vane speed, U, to maximize power produced by vane.

Solution: Apply momentum equation using CV moving with vanes, as shown.

Basic equation:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

($F_{B_x} = 0$ (2), $\frac{\partial}{\partial t}\int_{CV} = 0$ (3))

Assumptions:
1. No pressure forces
2. Horizontal; $F_{B_x} = 0$
3. Steady flow w.r.t. CV
4. Uniform flow at each section
5. No change in relative velocity on vane
6. Flow enters and leaves tangent to vanes

For $\alpha \approx 0$, $V_{rb} \approx V - U$; the momentum equation becomes

$$R_x = u_1\{-\dot{m}\} + u_2\{+\dot{m}\} = -\dot{m}(V-U) - \dot{m}(V-U)\sin\theta_2 = -\dot{m}(V-U)(1+\sin\theta_2)$$

$$u_1 \approx V_{rb} \approx V-U\,;\quad u_2 \approx -V_{rb}\sin\theta_2 \approx -(V-U)\sin\theta_2$$

The vane system produces force, $K_x = -R_x$, and power $\mathscr{P} = K_x U$. Thus

$$\mathscr{P} = K_x U = -R_x U = \dot{m}(V-U)U(1+\sin\theta_2) \qquad (1)$$

To find maximum power, set $\frac{d\mathscr{P}}{dU} = 0$

$$\frac{d\mathscr{P}}{dU} = \dot{m}(-1)U(1+\sin\theta_2) + \dot{m}(V-U)(1)(1+\sin\theta_2) = \dot{m}(V-2U)(1+\sin\theta_2)$$

Thus power is maximized when $V - 2U = 0$, or $U = \frac{V}{2}$ (for $\mathscr{P}_{max}$) $\qquad U$

{Note from Eq. 1 that $\theta_2 \to 90°$ increases power also.}

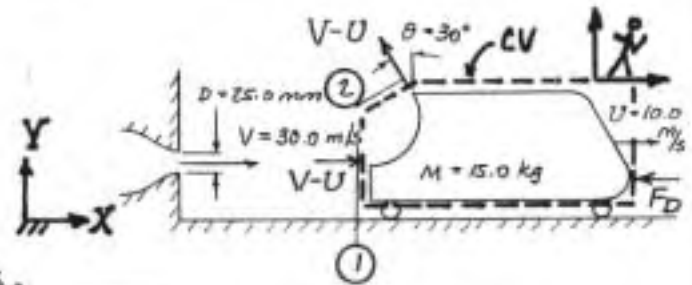
{Note also that $K_y = -R_y = -\dot{m}V_{rb}\cos\theta_2$ but this force does not produce power.}

Problem 4.133

Given: Cart propelled by steady water jet, as shown. Total resistance to motion is

$$F_D = kU^2$$

Where $k = 0.92 \frac{N \cdot s^2}{m^2}$

Find: Acceleration of cart at instant when $U = 10$ m/s.

Solution: Apply the momentum equation using CV and CS shown.

Basic equation: $F_{Sx} + F_{Bx} - \int_{CV} a_{rf_x} \rho \, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho \, d\forall + \int_{CS} u_{xyz} \rho \vec{V} \cdot d\vec{A}$ ($F_{Bx} = 0$ (2); $\frac{\partial}{\partial t} \approx 0$ (3))

Assumptions: (1) Only resistance is F_D; $F_{Sx} = -F_D = -kU^2$
(2) Horizontal; $F_{Bx} = 0$
(3) Neglect $\partial u / \partial t$ of mass of water in CV
(4) No change in speed w.r.to vane
(5) Uniform flow at each cross-section

Then

$$-kU^2 - a_{rf_x} M_{cv} = u_1 \{-\rho(V-U)A\} + u_2\{+\rho(V-U)A\}$$

Measure u w.r.to CV: $\quad u_1 = V - U \qquad u_2 = -(V-U)\sin\theta$

$$-kU^2 - a_{rf_x} M_{cv} = -\rho(V-U)^2 A - \rho(V-U)^2 A \sin\theta = -\rho(V-U)^2 A(1+\sin\theta)$$

so

$$a_{rf_x} = \frac{1}{M}\left[\rho(V-U)^2 A(1+\sin\theta) - kU^2\right]$$

$$= \frac{1}{15\,\text{kg}}\left[999\frac{\text{kg}}{\text{m}^3}(30-10)^2\frac{\text{m}^2}{\text{s}^2}\frac{\pi}{4}(0.025)^2\text{m}^2(1+\sin 30°) - 0.92\frac{\text{N}\cdot\text{s}^2}{\text{m}^2}\times(10)^2\frac{\text{m}^2}{\text{sec}^2}\times\frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right]$$

$$a_{rf_x} = 13.5 \text{ m/s}^2 \quad \text{(to right)}$$

a_{rf_x}

Problem 4.134

Given: Splitter dividing flow into two flat streams, as shown.

Find: (a) Mass flow rate ratio, $\dot{m}_2/\dot{m}_3$, so net vertical force is zero.
(b) Horizontal force need to maintain constant speed.

Solution: Apply x and y components of momentum to CV drawn with boundaries ⊥ to flows, as shown.

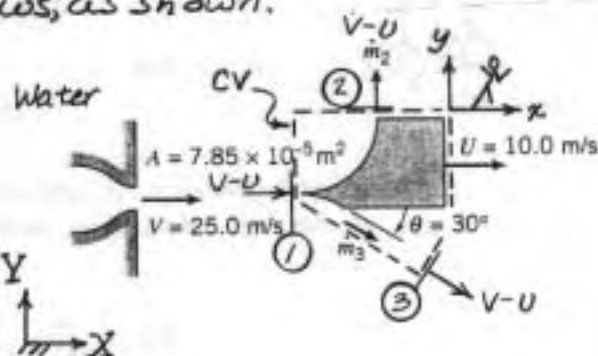
Basic equations:

$$F_{Sy} + F_{By} = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\, d\forall + \int_{CS} v_{xyz}\,\rho \vec{V}\cdot d\vec{A}$$

(with $F_{Sy} = 0(1)$, $F_{By} = 0(2)$, $\frac{\partial}{\partial t} = 0(3)$)

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall + \int_{CS} u_{xyz}\,\rho \vec{V}\cdot d\vec{A}$$

(with $F_{Bx} = 0$, $\frac{\partial}{\partial t} = 0(3)$)

Water; CV; $A = 7.85 \times 10^{-5}\ m^2$; $V - U$; $V = 25.0$ m/s; $U = 10.0$ m/s; $\theta = 30°$; $V-U$, $\dot{m}_2$; $\dot{m}_3$; $V-U$; ①, ②, ③; x, y; Y, X

Assumptions: (1) No pressure forces
(2) Neglect mass of water on vane
(3) Steady flow w.r.to vane
(4) Uniform flow at each section
(5) No change in speed w.r.to vane

Then

$$0 = \int_{CS} v \rho \vec{V}\cdot d\vec{A} = v_1\{-\dot{m}_1\} + v_2\{+\dot{m}_2\} + v_3\{+\dot{m}_3\}$$

Measure w.r.to CV: $v_1 = 0$ $\quad v_2 = V-U$ $\quad v_3 = -(V-U)\sin\theta$

so $\quad 0 = (V-U)\dot{m}_2 - (V-U)\sin\theta\, \dot{m}_3\ ; \quad \frac{\dot{m}_2}{\dot{m}_3} = \sin\theta = \frac{1}{2}$

$\dot{m}_2/\dot{m}_3$

and

$$F_{Sx} = \int_{CS} u \rho \vec{V}\cdot d\vec{A} = R_x = u_1\{-\dot{m}_1\} + u_2\{+\dot{m}_2\} + u_3\{+\dot{m}_3\}$$

Measure w.r.to CV: $u_1 = V-U$ $\quad u_2 = 0$ $\quad u_3 = (V-U)\cos\theta$

$$R_x = (V-U)(-\dot{m}_1) + (V-U)\cos\theta(\dot{m}_3) = (V-U)(\dot{m}_3\cos\theta - \dot{m}_1)$$

From continuity $\quad 0 = -\dot{m}_1 + \dot{m}_2 + \dot{m}_3 = -\dot{m}_1 + \frac{\dot{m}_3}{2} + \dot{m}_3\ ; \quad \dot{m}_3 = \frac{2}{3}\dot{m}_1$

$$R_x = (V-U)\left(\frac{2}{3}\dot{m}_1\cos\theta - \dot{m}_1\right) = (V-U)\dot{m}_1\left(\frac{2\cos\theta}{3} - 1\right)$$

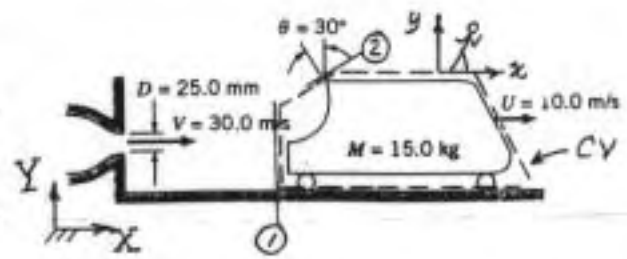
$$R_x = (25-10)\frac{m}{s} \times 999\frac{kg}{m^3} \times (25-10)\frac{m}{s} \times 7.85\times10^{-5} m^2 \left(\frac{2}{3}\cos 30° - 1\right)\frac{N\cdot s^2}{kg\cdot m}$$

$$R_x = -7.46\ N \quad (\text{to left})$$

R_x

{ Force must be applied to left to maintain vane speed constant; if R_x were zero, vane would accelerate. }

Problem 4.135

Given: Hydraulic catapult of Problem 4.133, rolling on level track with negligible resistance, speed U.

Find: Time required to accelerate from rest to $U = V/2$.

Solution: Apply x component of momentum equation to accelerating CV.

Basic equation:
$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x} \rho \, d\mathcal{V} = \overset{\approx 0(3)}{\frac{\partial}{\partial t}} \int_{CV} u_{xyz} \rho \, d\mathcal{V} + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) $F_{S_x} = 0$, since no pressure forces, no resistance
(2) $F_{B_x} = 0$, since horizontal
(3) Neglect mass of water on vane
(4) Uniform flow in jet
(5) No change in relative velocity on vane

Then
$$-a_{rf_x} M_{cv} = u_1 \{-\rho(V-U)A\} + u_2\{+\rho(V-U)A\} = -(1+\sin\theta)\rho(V-U)^2 A$$

$$u_1 = V - U \qquad u_2 = -(V-U)\sin\theta$$

so
$$\frac{dU}{dt} = \frac{\rho A(1+\sin\theta)}{M}(V-U)^2$$

To integrate, note since $V =$ constant, $d(V-U) = -dU$, so

$$-\int_0^{V/2} \frac{d(V-U)}{(V-U)^2} = \int_0^t \frac{\rho A(1+\sin\theta)}{M} dt$$

or
$$\left.\frac{1}{V-U}\right]_{U=0}^{U=V/2} = \frac{2}{V} - \frac{1}{V} = \frac{1}{V} = \frac{\rho A(1+\sin\theta)}{M} t$$

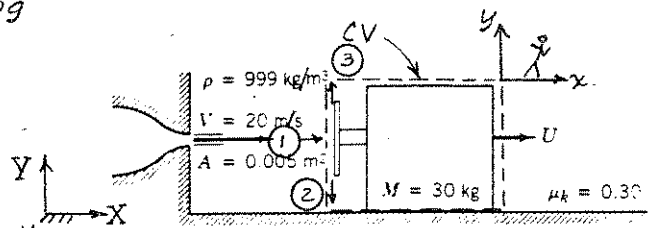
Thus
$$t = \frac{M}{\rho V A(1+\sin\theta)}$$

$$= 15.0\,\text{kg} \times \frac{\text{m}^3}{999\,\text{kg}} \times \frac{\text{s}}{30.0\,\text{m}} \times \frac{4}{\pi(0.025)^2\,\text{m}^2} \times \frac{1}{(1+\sin 30°)}$$

$$t = 0.680\ \text{s}$$

t

Problem 4.136

Given: Vane/slider assembly moving under influence of jet.

Find: Terminal speed.

Solution: Apply x momentum equation to linearly accelerating CV.

Basic equation:

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x} \rho \, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho \, d\forall + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

($F_{B_x} = 0$ (1); $\frac{\partial}{\partial t} \approx 0$ (2))

Assumptions: (1) Horizontal motion, so $F_{Bx} = 0$
(2) Neglect mass of liquid on vane, $u \approx 0$ on vane
(3) Uniform flow at each section
(4) Measure velocities relative to CV

Then

$$-Mg\mu_k - a_{rf_x} M = u_1\{-|\rho(V-U)A|\} + u_2\{+\dot{m}_2\} + u_3\{+\dot{m}_3\}$$

$$u_1 = V - U \qquad u_2 = 0 \qquad u_3 = 0$$

$$-Mg\mu_k - M\frac{dU}{dt} = -\rho(V-U)^2 A$$

or

$$\frac{dU}{dt} = \frac{\rho(V-U)^2 A}{M} - g\mu_k$$

At terminal speed, $dU/dt = 0$ and $U = U_t$, so

$$0 = \frac{\rho(V-U_t)^2 A}{M} - g\mu_k \quad \text{or} \quad V - U_t = \sqrt{\frac{Mg\mu_k}{\rho A}}$$

and

$$U_t = V - \sqrt{\frac{Mg\mu_k}{\rho A}}$$

$$= 20\,\frac{m}{s} - \left[30\,kg \times 9.81\,\frac{m}{s^2} \times 0.3 \times \frac{m^3}{999\,kg} \times \frac{1}{0.005\,m^2}\right]^{1/2}$$

$$U_t = 15.8\ m/s$$

U_t

Problem 4.137

Given: Cart propelled by a horizontal liquid jet of constant speed. Neglect resistance along horizontal track.

Initial mass is M_0.

Find: (a) A general expression for speed, U, as cart accelerates from rest.

(b) V for $U = 1.5$ m/s @ $t = 30$ s

Solution:

a) Apply x component of momentum equation using linearly accelerating CV shown.

Basic equation:

$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x} \rho \, d\forall = \overset{\approx 0(3)}{\frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho \, d\forall} + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) No resistance

(2) $F_{B_x} = 0$ since track is horizontal

(3) Neglect u_{xyz} within CV

(4) Uniform flow at jet exit

Then

$$-a_{rf_x} M = u\{|\rho V A|\} = -\rho V^2 A$$

$$u = -V$$

From continuity, $M = M_0 - \dot{m}t = M_0 - \rho V A t$. Using $a_{rf_x} = \frac{dU}{dt}$,

$$\frac{dU}{dt} = \frac{\rho V^2 A}{M_0 - \rho V A t}$$

Separating variables and integrating,

$$\int_0^U dU = U = \int_0^t \frac{\rho V^2 A \, dt}{M_0 - \rho V A t} = -V \ln(M_0 - \rho V A t)\Big|_0^t = V \ln\left(\frac{M_0}{M_0 - \rho V A t}\right)$$

or

$$\frac{U}{V} = \ln\left(\frac{M_0}{M_0 - \rho V A t}\right)$$

$\frac{U}{V}$

Check dimensions: $[\rho V A t] = \frac{M}{L^3}\frac{L}{t}L^2 t = M$ ✓

b) Using the given data in Excel (with Solver) the jet speed for $U = 1.5$ m/s @ $t = 30$ s is $V = 0.61$ m/s

Problem 4.138

4.138 For the vane/slider problem of Problem 4.136, find and plot expressions for the acceleration, speed, and position of the slider as a function of time.

Given:
Data on vane/slider

Find:
Formula for acceleration, speed, and position; plot

Solution:

The given data is $\rho = 999 \cdot \frac{kg}{m^3}$ $\quad M = 30 \cdot kg$ $\quad A = 0.005 \cdot m^2$ $\quad V = 20 \cdot \frac{m}{s}$ $\quad \mu_k = 0.3$

The equation of motion, from Problem 4.136, is $\frac{dU}{dt} = \frac{\rho \cdot (V-U)^2 \cdot A}{M} - g \cdot \mu_k$

The acceleration is thus $a = \frac{\rho \cdot (V-U)^2 \cdot A}{M} - g \cdot \mu_k$

Separating variables

$$\frac{dU}{\frac{\rho \cdot (V-U)^2 \cdot A}{M} - g \cdot \mu_k} = dt$$

Substitute $u = V - U$ $\quad dU = -du$

$$\frac{du}{\frac{\rho \cdot A \cdot u^2}{M} - g \cdot \mu_k} = -dt$$

$$\int \frac{1}{\left(\frac{\rho \cdot A \cdot u^2}{M} - g \cdot \mu_k\right)} du = -\sqrt{\frac{M}{g \cdot \mu_k \cdot \rho \cdot A}} \cdot \operatorname{atanh}\left(\sqrt{\frac{\rho \cdot A}{g \cdot \mu_k \cdot M}} \cdot u\right)$$

and u = V - U so

$$-\sqrt{\frac{M}{g \cdot \mu_k \cdot \rho \cdot A}} \cdot \operatorname{atanh}\left(\sqrt{\frac{\rho \cdot A}{g \cdot \mu_k \cdot M}} \cdot u\right) = -\sqrt{\frac{M}{g \cdot \mu_k \cdot \rho \cdot A}} \cdot \operatorname{atanh}\left[\sqrt{\frac{\rho \cdot A}{g \cdot \mu_k \cdot M}} \cdot (V-U)\right]$$

Using initial conditions

$$-\sqrt{\frac{M}{g \cdot \mu_k \cdot \rho \cdot A}} \cdot \operatorname{atanh}\left[\sqrt{\frac{\rho \cdot A}{g \cdot \mu_k \cdot M}} \cdot (V-U)\right] + \sqrt{\frac{M}{g \cdot \mu_k \cdot \rho \cdot A}} \cdot \operatorname{atanh}\left(\sqrt{\frac{\rho \cdot A}{g \cdot \mu_k \cdot M}} \cdot V\right) = -t$$

$$V - U = \sqrt{\frac{g \cdot \mu_k \cdot M}{\rho \cdot A}} \cdot \tanh\left(\sqrt{\frac{g \cdot \mu_k \cdot \rho \cdot A}{M}} \cdot t + \operatorname{atanh}\left(\sqrt{\frac{\rho \cdot A}{g \cdot \mu_k \cdot M}} \cdot V\right)\right)$$

$$U = V - \sqrt{\frac{g \cdot \mu_k \cdot M}{\rho \cdot A}} \cdot \tanh\left(\sqrt{\frac{g \cdot \mu_k \cdot \rho \cdot A}{M}} \cdot t + \operatorname{atanh}\left(\sqrt{\frac{\rho \cdot A}{g \cdot \mu_k \cdot M}} \cdot V\right)\right)$$

Note that $\operatorname{atanh}\left(\sqrt{\frac{\rho\cdot A}{g\cdot \mu_k\cdot M}}\cdot V\right) = 0.213 - \frac{\pi}{2}\cdot i$

which is complex and difficult to handle in *Excel*, so we use the identity $\operatorname{atanh}(x) = \operatorname{atanh}\left(\frac{1}{x}\right) - \frac{\pi}{2}\cdot i$ for $x > 1$

so
$$U = V - \sqrt{\frac{g\cdot \mu_k\cdot M}{\rho\cdot A}}\cdot \tanh\left(\sqrt{\frac{g\cdot \mu_k\cdot \rho\cdot A}{M}}\cdot t + \operatorname{atanh}\left(\frac{1}{\sqrt{\frac{\rho\cdot A}{g\cdot \mu_k\cdot M}}\cdot V}\right) - \frac{\pi}{2}\cdot i\right)$$

and finally the identity $\tanh\left(x - \frac{\pi}{2}\cdot i\right) = \frac{1}{\tanh(x)}$

to obtain
$$U = V - \frac{\sqrt{\frac{g\cdot \mu_k\cdot M}{\rho\cdot A}}}{\tanh\left(\sqrt{\frac{g\cdot \mu_k\cdot \rho\cdot A}{M}}\cdot t + \operatorname{atanh}\left(\sqrt{\frac{g\cdot \mu_k\cdot M}{\rho\cdot A}}\cdot \frac{1}{V}\right)\right)}$$

For the position x
$$\frac{dx}{dt} = V - \frac{\sqrt{\frac{g\cdot \mu_k\cdot M}{\rho\cdot A}}}{\tanh\left(\sqrt{\frac{g\cdot \mu_k\cdot \rho\cdot A}{M}}\cdot t + \operatorname{atanh}\left(\sqrt{\frac{g\cdot \mu_k\cdot M}{\rho\cdot A}}\cdot \frac{1}{V}\right)\right)}$$

This can be solved analytically, but is quite messy. Instead, in the corresponding *Excel* workbook, it is solved numerically using a simple Euler method. The complete set of equations is

$$U = V - \frac{\sqrt{\frac{g\cdot \mu_k\cdot M}{\rho\cdot A}}}{\tanh\left(\sqrt{\frac{g\cdot \mu_k\cdot \rho\cdot A}{M}}\cdot t + \operatorname{atanh}\left(\sqrt{\frac{g\cdot \mu_k\cdot M}{\rho\cdot A}}\cdot \frac{1}{V}\right)\right)}$$

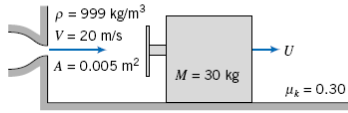
$$a = \frac{\rho\cdot (V - U)^2\cdot A}{M} - g\cdot \mu_k$$

$$x(n+1) = x(n) + \left(V - \frac{\sqrt{\frac{g\cdot \mu_k\cdot M}{\rho\cdot A}}}{\tanh\left(\sqrt{\frac{g\cdot \mu_k\cdot \rho\cdot A}{M}}\cdot t + \operatorname{atanh}\left(\sqrt{\frac{g\cdot \mu_k\cdot M}{\rho\cdot A}}\cdot \frac{1}{V}\right)\right)}\right)\cdot \Delta t$$

The plots are presented in the *Excel* workbook

Problem 4.138

[4]

4.138 For the vane/slider problem of Problem 4.136, find and plot expressions for the acceleration, speed, and position of the slider as a function of time.

Given: Data on vane/slider

Find: Formula for acceleration, speed, and position; plot

Solution:

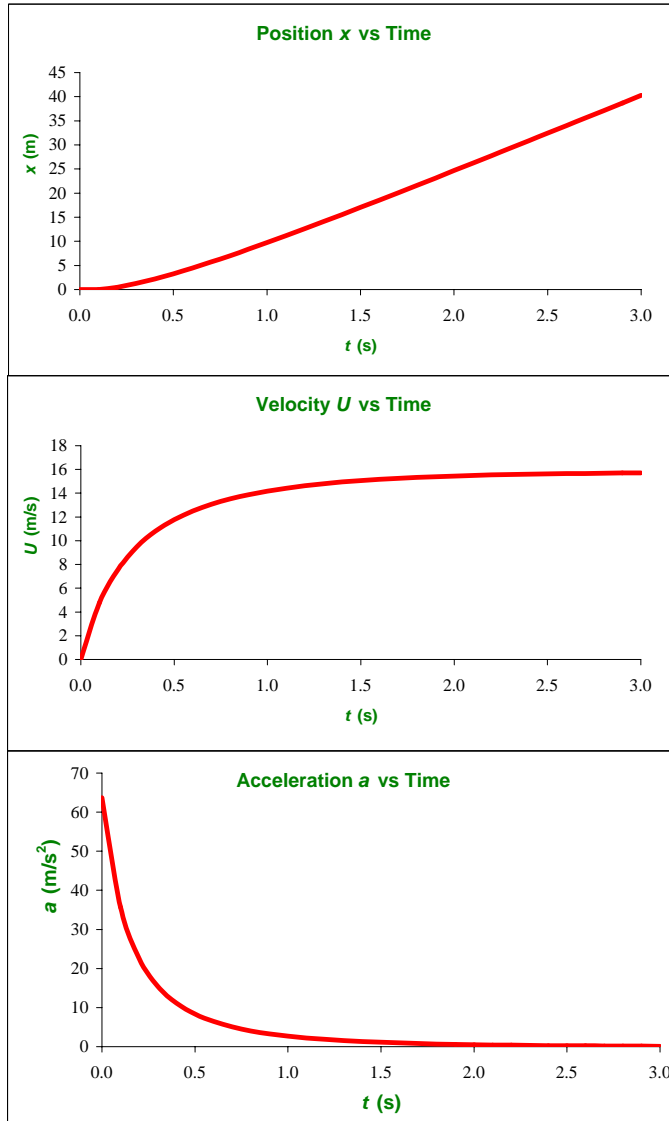
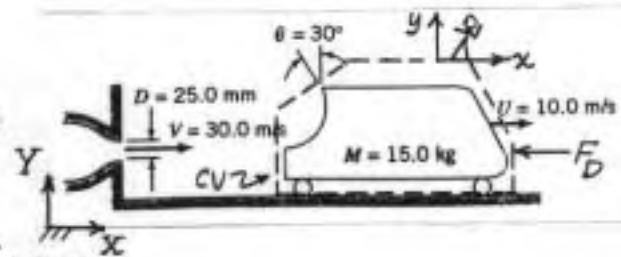
The equations are

$$U = V - \frac{\sqrt{\dfrac{g \cdot \mu_k \cdot M}{\rho \cdot A}}}{\tanh\left(\sqrt{\dfrac{g \cdot \mu_k \cdot \rho \cdot A}{M}} \cdot t + \operatorname{atanh}\left(\sqrt{\dfrac{g \cdot \mu_k \cdot M}{\rho \cdot A}} \cdot \dfrac{1}{V}\right)\right)}$$

$$a = \frac{\rho \cdot (V - U)^2 \cdot A}{M} - g \cdot \mu_k$$

$$x(n+1) = x(n) + \left(V - \frac{\sqrt{\dfrac{g \cdot \mu_k \cdot M}{\rho \cdot A}}}{\tanh\left(\sqrt{\dfrac{g \cdot \mu_k \cdot \rho \cdot A}{M}} \cdot t + \operatorname{atanh}\left(\sqrt{\dfrac{g \cdot \mu_k \cdot M}{\rho \cdot A}} \cdot \dfrac{1}{V}\right)\right)}\right) \cdot \Delta t$$

$\rho =$ 999 kg/m³
$\mu_k =$ 0.3
$A =$ 0.005 m²
$V =$ 20 m/s
$M =$ 30 kg
$\Delta t =$ 0.1 s

t (s)	*x* (m)	*U* (m/s)	*a* (m/s²)
0.0	0.0	0.0	63.7
0.1	0.0	4.8	35.7
0.2	0.5	7.6	22.6
0.3	1.2	9.5	15.5
0.4	2.2	10.8	11.2
0.5	3.3	11.8	8.4
0.6	4.4	12.5	6.4
0.7	5.7	13.1	5.1
0.8	7.0	13.5	4.0
0.9	8.4	13.9	3.3
1.0	9.7	14.2	2.7
1.1	11.2	14.4	2.2
1.2	12.6	14.6	1.9
1.3	14.1	14.8	1.6
1.4	15.5	14.9	1.3
1.5	17.0	15.1	1.1
1.6	18.5	15.2	0.9
1.7	20.1	15.3	0.8
1.8	21.6	15.3	0.7
1.9	23.1	15.4	0.6
2.0	24.7	15.4	0.5
2.1	26.2	15.5	0.4
2.2	27.8	15.5	0.4
2.3	29.3	15.6	0.3
2.4	30.9	15.6	0.3
2.5	32.4	15.6	0.2
2.6	34.0	15.6	0.2
2.7	35.6	15.7	0.2
2.8	37.1	15.7	0.2
2.9	38.7	15.7	0.1
3.0	40.3	15.7	0.1

Given: Hydraulic catapult of Problem 4.133 rolling on level track with resistance, $F_D = kU^2$, speed U, starting from rest at $t = 0$.

Find: (a) When acceleration is maximum
(b) Sketch of acceleration vs. time
(c) Value of θ to maximize acceleration, why?
(d) If U will ever reach V; explanation

Solution: Apply x component of momentum equation to accelerating CV

Basic equation:

$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x} \rho \, dV = \overset{\approx 0(3)}{\frac{\partial}{\partial t}} \int_{CV} u_{xyz} \rho \, dV + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) $F_{S_x} = -F_D = -kU^2$, where $k = 0.92 \ N \cdot s^2/m^2$
(2) $F_{B_x} = 0$, since horizontal
(3) Neglect mass of water on vane
(4) Uniform flow in jet
(5) No change in relative velocity on vane

Then

$$-kU^2 - a_{rf_x} M_{CV} = u_1\{-\rho(V-U)A\} + u_2\{+\rho(V-U)A\} = -(1+\sin\theta)\rho(V-U)^2 A$$

$$u_1 = V - U \qquad u_2 = -(V-U)\sin\theta$$

so

$$\frac{dU}{dt} = \frac{\rho A(1+\sin\theta)}{M}(V-U)^2 - kU^2/M \qquad (1)$$

(a) Acceleration is maximum at $t = 0$, when $U = 0$

(b) Acceleration vs. time will be

(c) From Eq. 1, dU/dt is maximum when $\theta = \pi/2$ and $\sin\theta = 1$

(d) From Eq. 1, $\frac{dU}{dt}$ will go to zero when $U < V$; this will be the terminal speed for the cart, U_t. From Eq. 1, $\frac{dU}{dt} = 0$ when

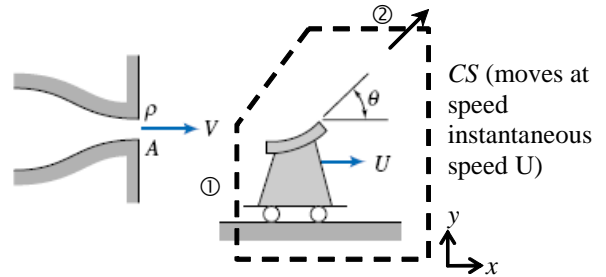
$$\rho A(1+\sin\theta)(V-U)^2 = kU^2$$

or

$$U = \frac{\left[\frac{\rho A(1+\sin\theta)}{k}\right]^{1/2}}{1 + \left[\frac{\rho A(1+\sin\theta)}{k}\right]^{1/2}} V = 0.472V$$

U will be asymptotic to V.

Problem 4.140

4.140 The acceleration of the vane/cart assembly of Problem 4.123 is to be controlled as it accelerates from rest by changing the vane angle, θ. A constant acceleration, $a = 1.5\ \mathrm{m/s^2}$, is desired. The water jet leaves the nozzle of area $A = 0.025\ \mathrm{m^2}$, with speed $V = 15$ m/s. The vane/cart assembly has a mass of 55 kg; neglect friction. Determine θ at $t = 5$ s. Plot $\theta(t)$ for the given constant acceleration over a suitable range of t.

Given: Water jet striking moving vane/cart assembly

Find: Angle θ at t = 5 s; Plot θ(t)

Solution:

Basic equation: Momentum flux in x direction for accelerating CV

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\,\rho\,dV\!\!\!/ = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,dV\!\!\!/ + \int_{CS} u_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: 1) cahnges in CV 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow 5) Constant jet relative velocity

Then $$-M\cdot a_{rfx} = u_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) + u_2\cdot\left(\rho\cdot V_2\cdot A_2\right) = -(V-U)\cdot[\rho\cdot(V-U)\cdot A] + (V-U)\cdot\cos(\theta)\cdot[\rho\cdot(V-U)\cdot A]$$

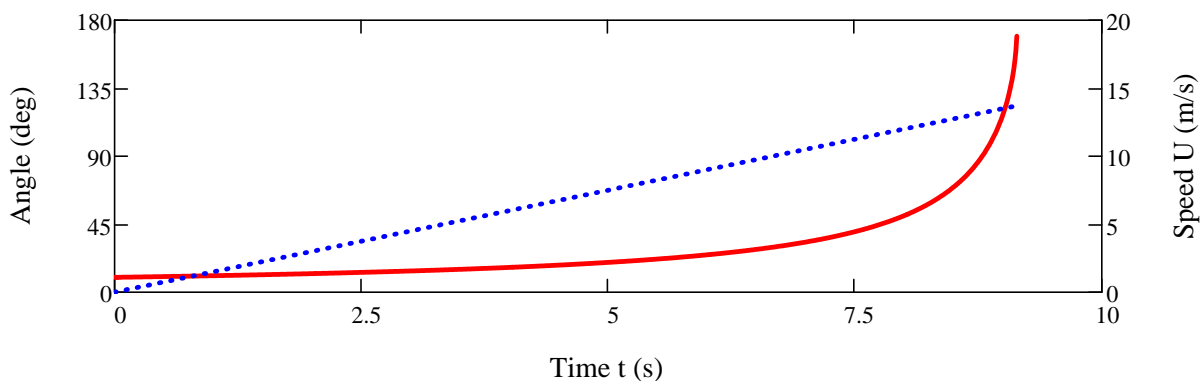
$$-M\cdot a_{rfx} = \rho(V-U)^2\cdot A\cdot(\cos(\theta)-1) \quad \text{or} \quad \cos(\theta) = 1 - \frac{M\cdot a_{rfx}}{\rho\cdot(V-U)^2\cdot A}$$

Since $a_{rfx} = \text{constant}$ then $U = a_{rfx}\cdot t$ $\quad \cos(\theta) = 1 - \dfrac{M\cdot a_{rfx}}{\rho\cdot\left(V - a_{rfx}\cdot t\right)^2\cdot A}$

$$\theta = \operatorname{acos}\left[1 - \frac{M\cdot a_{rfx}}{\rho\cdot\left(V-a_{rfx}\cdot t\right)^2\cdot A}\right]$$

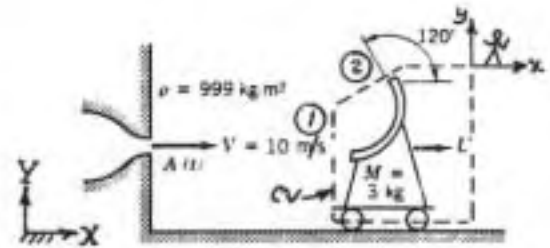
Using given data

$$\theta = \operatorname{acos}\left[1 - 55\cdot\mathrm{kg}\times 1.5\cdot\frac{\mathrm{m}}{\mathrm{s}^2}\times\frac{\mathrm{m}^3}{1000\cdot\mathrm{kg}}\times\frac{1}{\left(15\cdot\frac{\mathrm{m}}{\mathrm{s}} - 1.5\cdot\frac{\mathrm{m}}{\mathrm{s}^2}\times 5\cdot\mathrm{s}\right)^2}\times\frac{1}{0.025\cdot\mathrm{m}^2}\right] \qquad \theta = 19.7\,\mathrm{deg} \quad \text{at } t = 5\text{ s}$$

Angle (deg): 0, 45, 90, 135, 180
Speed U (m/s): 0, 5, 10, 15, 20
Time t (s): 0, 2.5, 5, 7.5, 10

The solution is only valid for θ up to 180º (when t = 9.14 s). This graph can be plotted in *Excel*

Given: Vaned cart rolling with negligible resistance.

$a_{rf_x} = 2\, m/s^2$ = constant

Jet area is $A(t)$, programmed.

Find: (a) Expression for $A(t)$ at cart.
(b) Sketch for $t \leq 4s$.
(c) Evaluate at $t = 2s$.

Solution: Apply x momentum to CV with linear acceleration.

Basic equation:

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall + \int_{CS} u_{xyz}\rho \vec{V}_{xyz}\cdot d\vec{A}$$

$F_{S_x} = 0(1)$, $F_{B_x} = 0(2)$, $\frac{\partial}{\partial t} \approx 0(3)$

Assumptions: (1) No resistance to motion
(2) Horizontal motion, so $F_{B_x} = 0$
(3) Neglect mass of liquid in CV
(4) Uniform flow at each section
(5) All velocities measured relative to CV
(6) No change in stream area or speed on vane

Then (with $a_{rf_x} = a$)

$$-aM = u_1\{-|\rho(V-U)A|\} + u_2\{+|\rho(V-U)A|\} = -\frac{3}{2}\rho(V-U)^2 A$$

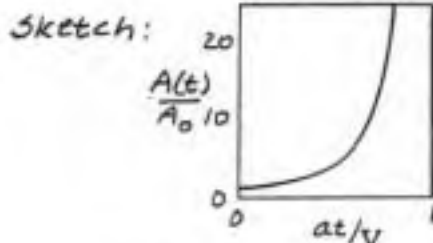
$$u_1 = V - U \qquad u_2 = (V-U)\cos 120° = -\frac{1}{2}(V-U)$$

Since a = constant, $U = at$, and

$$A = A(t) = \frac{2aM}{3\rho(V-at)^2}$$ A(t)

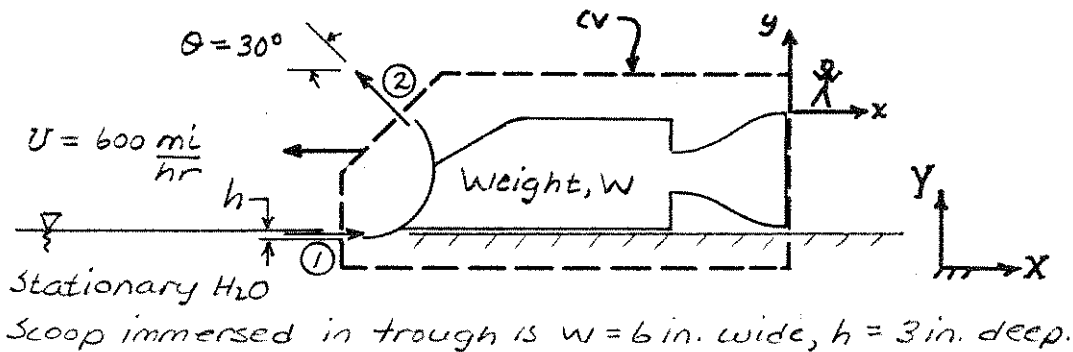
At $t = 0$, $A(0) = A_0 = \frac{2aM}{3\rho V^2}$. Thus $\frac{A}{A_0} = \frac{1}{(1-at/V)^2}$.

Sketch:

y-axis	$\frac{A(t)}{A_0}$: 0, 10, 20
x-axis	at/V: 0, 1

At $t = 2$ sec,

$$A = \frac{2}{3}\times 2\frac{m}{s^2}\times 3\,kg \times \frac{m^3}{999\,kg}\frac{1}{\left[10\frac{m}{s} - 2\frac{m}{s^2}\times 2s\right]^2}\times 10^6\frac{mm^2}{m^2} = 111\, mm^2$$ A(2)

Given: Rocket sled with water scoop brake. $W = 10,000$ lbf

Scoop immersed in trough is $w = 6$ in. wide, $h = 3$ in. deep.

Find: Time needed to decelerate to 20 mph. Plot: Speed vs. time.

Solution: Apply x component of momentum equation to linearly accelerating CV. Basic equation is

$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x} \rho \, dV = \overset{\approx 0(3)}{\frac{\partial}{\partial t}} \int_{CV} u_{xyz} \rho \, dV + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) $F_{S_x} = 0$
(2) $F_{B_x} = 0$
(3) Neglect u_{xyz} and its rate of change in CV
(4) Uniform flow at each section
(5) Speed of water relative to sled is constant

Then

$$-a_{rf_x} M = u_1\{-|\rho U w h|\} + u_2\{|\rho U w h|\} \; ; \; u_1 = U, \; u_2 = -U\cos\theta$$

$$-a_{rf_x} \frac{W}{g} = -\rho U^2 w h (1+\cos\theta), \text{ or } \quad a_{rf_x} = \frac{\rho g U^2 w h (1+\cos\theta)}{W}$$

Now $a_{rf_x} = -\frac{dU}{dt}$, because of coordinate choice. Thus

$$\frac{dU}{U^2} = -\frac{\gamma w h}{W}(1+\cos\theta)\, dt$$

and

$$\int_{U_i}^{U} \frac{dU}{U^2} = -\frac{1}{U} + \frac{1}{U_i} = -\frac{\gamma w h}{W}(1+\cos\theta)\, t \qquad (1)$$

Solving for t,

$$t = \left[\frac{1}{U} - \frac{1}{U_i}\right] \frac{W}{\gamma w h (1+\cos\theta)}$$

$$= \left[\frac{1}{20} - \frac{1}{600}\right] \frac{hr}{mi} \times \frac{mi}{5280\,ft} \times 3600\frac{s}{hr} \times \frac{ft^3}{62.4\,lbf} \times \frac{1}{6\,in.} \times \frac{1}{3\,in.} \times \frac{144\,in^2}{ft^2} \times \frac{10,000\,lbf}{1+\cos 30°}$$

$$t = 22.6 \text{ s} \qquad t$$

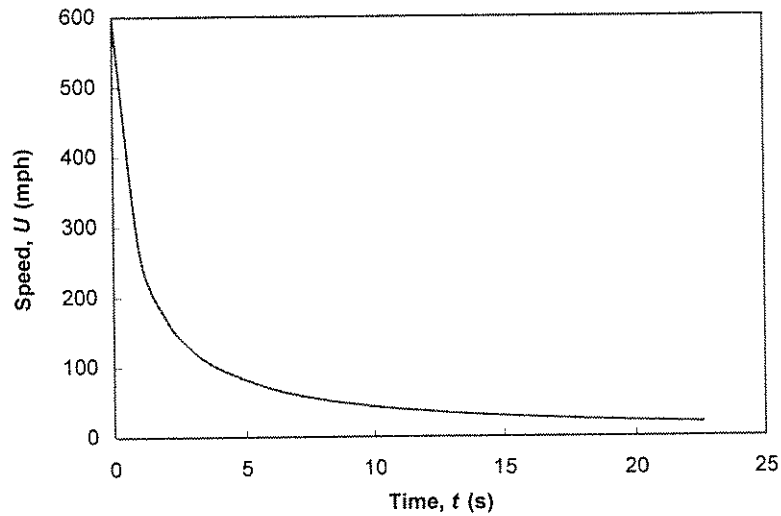
The plot is presented on the next page.

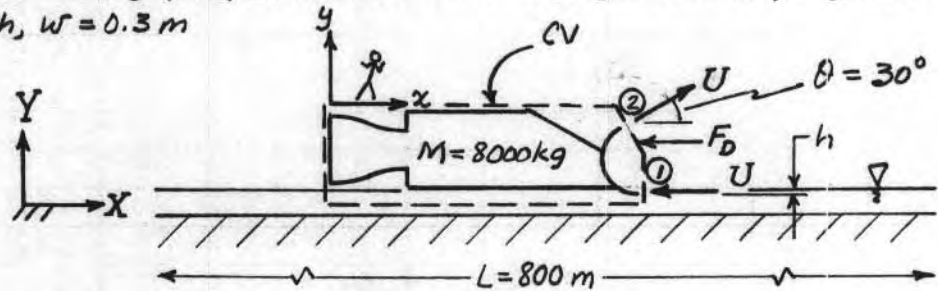
Solving Eq. 1 for U,

$$\frac{1}{U} = \frac{1}{U_i} + \frac{\gamma w h}{W}(1+\cos\theta)t = \frac{W + \gamma w h U_i (1+\cos\theta)t}{W U_i}$$

or $$U = \frac{W U_i}{W + \gamma w h U_i (1+\cos\theta)t} \qquad (2)$$

Plotting,

Problem 4.143

Given: Rocket sled slowed by scoop in water trough.

Aerodynamic drag proportional to U^2. At $U_0 = 300\ m/s$, $F_D = 90\ kN$.
Scoop width, $w = 0.3\ m$

Find: Depth of scoop immersion to slow to 100 m/s in trough length, L.

Solution: Apply x component of momentum equation using linearly accelerating CV shown.

Basic equation:
$$F_{Sx} + F_{Bx} - \int_{CV} a_{rfx}\,\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$
($F_{Bx} = 0$ (1); $\frac{\partial}{\partial t}\int_{CV} \approx 0$ (2))

Assumptions: (1) $F_{Bx} = 0$
(2) Neglect rate of change of u in CV
(3) Uniform flow at each section
(4) No change in relative speed of liquid crossing scoop

Then
$$-F_D - Ma_{rfx} = u_1\{-|\rho U w h|\} + u_2\{|\rho U w h|\}\ ;\ h = \text{scoop immersion}$$
$$u_1 = -U \qquad u_2 = U\cos\theta$$

But $F_D = kU^2$; $k = \frac{F_{D_0}}{U_0^2} = 90\ kN \times \frac{s^2}{(300)^2 m^2} \times \frac{10^3 N}{kN} \times \frac{kg\cdot m}{N\cdot s^2} = 1.00\ kg/m$
$$-kU^2 - M\frac{dU}{dt} = \rho U^2 w h(1+\cos\theta)\ ,\ \text{since } a_{rfx} = dU/dt,$$

Thus $-M\frac{dU}{dt} = [k + \rho w h(1+\cos\theta)]U^2 = -MU\frac{dU}{dX}$

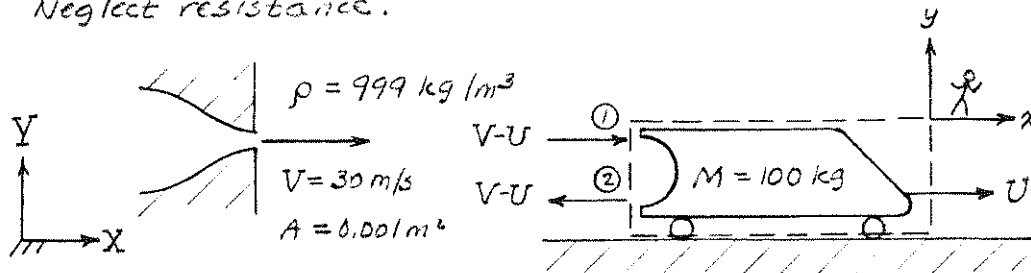
or $\frac{dU}{U} = -c\,dX$, where $c = \frac{k + \rho w h(1+\cos\theta)}{M}$

Integrating, $\ln\frac{U}{U_0} = -cX$, so $c = -\frac{1}{X}\ln\frac{U}{U_0}$
$$c = -\frac{1}{800\ m}\ln\left(\frac{100}{300}\right) = 1.37\times10^{-3}\ m^{-1}$$

Solving for h, $h = \frac{Mc - k}{\rho w(1+\cos\theta)}$
$$h = \left[8000\ kg \times \frac{1.37\times10^{-3}}{m} - 1.00\ \frac{kg}{m}\right]\frac{m^3}{999\ kg}\times\frac{1}{0.3\ m}\times\frac{1}{(1+\cos 30°)} = 0.0179\ m$$

$h = 17.9\ mm$ ← h

Problem 4.144

Given: Vehicle accelerated from rest by a hydraulic catapult.
Neglect resistance.

Find: Vehicle speed at $t = 5$ sec. Plot: Vehicle speed vs. time.

Solution: Apply x component of momentum equation using the linearly accelerating CV shown above.

Basic equation: $F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x} \rho \, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho \, d\forall + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$

($F_{S_x} = 0(1)$, $F_{B_x} = 0(2)$, $\frac{\partial}{\partial t} \approx 0(3)$)

Assumptions: (1) $F_{S_x} = 0$
(2) $F_{B_x} = 0$
(3) Neglect mass of liquid and rate of change of u in CV
(4) Uniform flow at each section
(5) Jet area and speed with respect to vehicle are constant

Then

$$-Ma_{rf_x} = -M\frac{dU}{dt} = u_1\{-|\rho(V-U)A|\} + u_2\{|\rho(V-U)A|\}$$

$$u_1 = V - U \qquad u_2 = -(V-U)$$

or

$$\frac{dU}{dt} = \frac{2\rho(V-U)^2 A}{M}$$

Note that $dU = -d(V-U)$, and separate variables to obtain

$$-\frac{d(V-U)}{(V-U)^2} = \frac{2\rho A}{M}dt$$

Integrate from $U = 0$ at $t = 0$ to U at t,

$$\int_{V-U=V}^{V-U} -\frac{d(V-U)}{(V-U)^2} = \frac{1}{V-U}\Big]_V^{V-U} = \frac{1}{V-U} - \frac{1}{V} = \frac{V-(V-U)}{V(V-U)} = \frac{U}{V(V-U)} = \frac{2\rho A}{M}t$$

Solving,

$$U = (V-U)\frac{2\rho VA}{M}t \quad \text{or} \quad U = V\left[\frac{\frac{2\rho VA}{M}t}{1+\frac{2\rho VA}{M}t}\right] \qquad (1)$$

For the given conditions at $t = 5$ s,

$$\frac{2\rho VA}{M}t = 2 \times 999\frac{kg}{m^3} \times 30\frac{m}{s} \times 0.001\,m^2 \times 5s \times \frac{1}{100\,kg} = 3.00$$

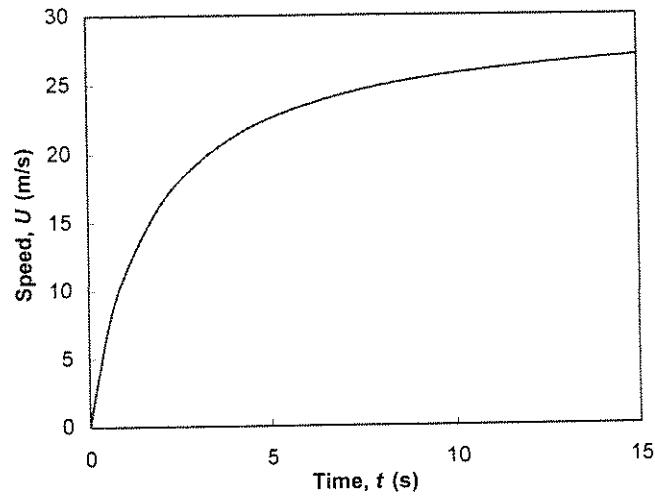
$$U = 30\frac{m}{s}\left[\frac{3.00}{1+3.00}\right] = 22.5\,m/s$$

U

The plot is on the next page.

Problem 4.144

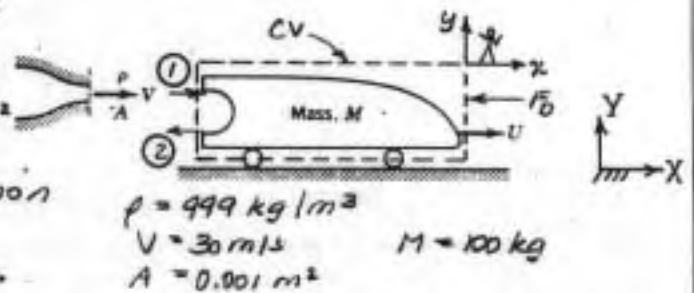
The speed vs. time plot is

Problem 4.145

Given: Cart accelerated from rest by hydraulic catapult.

$F_D = kU^2$; $k = 2.0\ N\cdot s^2/m^2$

CV, y, x, ρ, V, A, (1), (2), Mass, M, F_D, U, Y, X

Find: (a) Expression for acceleration in terms of speed, U.
(b) Evaluate at $U = 10\ m/s$.
(c) Fraction of U_t.

$\rho = 999\ kg/m^3$
$V = 30\ m/s$
$A = 0.001\ m^2$
$M = 100\ kg$

Solution: Apply x momentum for CV with linear acceleration.

Basic equation:

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall + \int_{CS} u_{xyz}\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) Horizontal, $F_{B_x} = 0$
(2) Neglect mass of liquid in CV (components of u cancel)
(3) Uniform flow at each section
(4) Measure all velocities relative to the CV
(5) No change in stream area or speed on vane

Then

$$-kU^2 - a_{rf_x}M = u_1\{-|\rho(V-U)A|\} + u_2\{+|\rho(V-U)A|\} = -2\rho(V-U)^2 A$$

$$u_1 = V-U \qquad u_2 = -(V-U)$$

or

$$a_{rf_x} = \frac{dU}{dt} = \frac{2\rho(V-U)^2 A - kU^2}{M}$$

$a(U)$

At $U = 10\ m/sec$

$$a_{rf_x} = \frac{2\times 999\frac{kg}{m^3}(30-10)^2\frac{m^2}{s^2}\times 0.001\,m^2 - 2.0\frac{N\cdot s^2}{m^2}\times(10)^2\frac{m^2}{s^2}\times\frac{kg\cdot m}{N\cdot s^2}}{100\,kg} = 5.99\ \frac{m}{s^2}$$

a_{rf_x}

At terminal speed, $a_{rf_x} = 0$. Then $2\rho(V-U_t)^2 A = kU_t^2$, or

$$V - U_t = U_t\sqrt{\frac{k}{2\rho A}}$$

solving,

$$U_t = \frac{V}{1+\sqrt{k/2\rho A}}$$

$$U_t = 30\frac{m}{s}\times\frac{1}{1+\left[\frac{1}{2}\times 2.0\frac{N\cdot s^2}{m^2}\times\frac{m^3}{999\,kg}\times\frac{1}{0.001\,m^2}\times\frac{kg\cdot m}{N\cdot s^2}\right]^{1/2}} = 15.0\ m/s$$

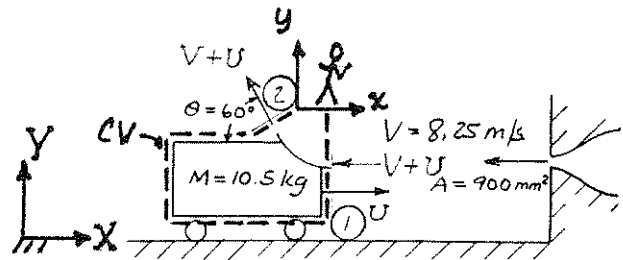
Finally,

$$\text{Fraction} = \frac{U}{U_t} = 10.0\frac{m}{s}\times\frac{s}{15.0\,m} = 0.667$$

Fraction

Given: Small vaned cart rolling on level track, struck by a water jet, as shown. At $t = 0$, $U_0 = 12.5$ m/sec. Neglect air resistance and rolling resistance.

Find: (a) Time and (b) distance needed to bring cart to rest, and (c) Plot of $U(t)$, $x(t)$.

Solution: Apply x component of momentum using cs and cv shown.

Basic equation:
$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x} \rho \, d\forall = \overset{\approx 0(3)}{\frac{\partial}{\partial t}} \int_{CV} u_{xyz} \rho \, d\forall + \int_{CS} u_{xyz} \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) No resistance; $F_{S_x} = 0$
(2) Horizontal; $F_{B_x} = 0$
(3) Neglect mass of water on vane; $\partial/\partial t \approx 0$
(4) No change in speed w.r.to vane
(5) Uniform flow at each cross-section

Then
$$-a_{rf_x} M_{CV} = u_1 \{-|\rho(V+U)A|\} + u_2 \{+|\rho(V+U)A|\}$$

$$a_{rf_x} = \frac{dU}{dt} \qquad u_1 = -(V+U) \qquad u_2 = -(V+U)\cos\theta \quad (\text{w.r.to CV})$$

So
$$-\frac{dU}{dt} M = \rho(V+U)^2 A - \rho(V+U)^2 A\cos\theta = \rho(V+U)^2 A(1-\cos\theta) \qquad (1)$$

Note V = constant, so $dU = d(V+U)$. Substituting

$$-\frac{d(V+U)}{(V+U)^2} = \frac{\rho A(1-\cos\theta)}{M} dt \qquad (2)$$

Integrate from U_0 at $t = 0$ to stop, when $U = 0$

$$\left.\frac{1}{V+U}\right]_{U=U_0}^{U=0} = \frac{1}{V} - \frac{1}{V+U_0} = \frac{V+U_0-V}{V(V+U_0)} = \frac{U_0}{V(V+U_0)} = \frac{\rho A(1-\cos\theta)t}{M}$$

Thus
$$t = \frac{U_0 M}{\rho(V+U_0) V A(1-\cos\theta)}$$

$$= 12.5 \frac{m}{sec} \times 10.5\,kg \times \frac{m^3}{999\,kg} \times \frac{sec}{(12.5+8.25)\,m} \times \frac{sec}{8.25\,m} \times \frac{1}{900\times10^{-6}\,m^2} \times \frac{1}{(1-\cos 60°)}$$

$t = 1.71$ sec (to stop) ← t

To find distance note $\frac{dU}{dt} = \frac{dU}{ds}\frac{ds}{dt} = \frac{dU}{ds} U = U\frac{dU}{ds}$; so from Eq. 1

$$-U\frac{dU}{ds} M = \rho(V+U)^2 A(1-\cos\theta)$$

Separating variables
$$\frac{U\,dU}{(V+U)^2} = -\frac{\rho A(1-\cos\theta)}{M} ds \qquad (3)$$

Equation 3 may be integrated. Using tables, and integrating from U_0 at $t = 0$ to stop (when $U = 0$),

$$\int_{U_0}^{0} \frac{U\,dU}{(V+U)^2} = \left[\ln(V+U) + \frac{V}{V+U}\right]_{U_0}^{0} = \ln\left(\frac{V}{V+U_0}\right) + \frac{V}{V} - \frac{V}{V+U_0} = -\frac{\rho A(1-\cos\theta)}{M}s$$

Simplifying and solving for s,

$$s = -\frac{M}{\rho A(1-\cos\theta)}\left(\ln\left(\frac{V}{V+U_0}\right) + 1 - \frac{V}{V+U_0}\right)$$

$$= -10.5\,kg \times \frac{m^3}{999\,kg} \times \frac{1}{900\times10^{-6}\,m^2} \times \frac{1}{(1-\cos 60°)}\left[\ln\left(\frac{8.25}{8.25+12.5}\right) + 1 - \frac{8.25}{8.25+12.5}\right]$$

$s = 7.47\ m$ (to stop) — s

From Eq. 2 the general solution is

$$\int_{U_0}^{U} -\frac{d(V+U)}{(V+U)^2} = \frac{1}{V+U}\Bigg]_{U_0}^{U} = \frac{1}{V+U} - \frac{1}{V+U_0} = \frac{(V+U_0)-(V+U)}{(V+U)(V+U_0)} = \frac{\rho A(1-\cos\theta)t}{M} = at$$

Thus $U_0 - U = a(V+U)(V+U_0)t = aV(V+U_0)t + aU(V+U_0)t$ {Let $b = V + U_0$}

Simplifying, $$U = \frac{U_0 - aVbt}{1+abt} \qquad (4)$$ — $U(t)$

Acceleration is found from Eq. 1

$$a_x = \frac{dU}{dt} = \frac{\rho A(1-\cos\theta)(V+U)^2}{M} = a(V+U)^2$$ — $a_x(U)$

Integrate Eq. 4 to get $X(t)$:

$$U = \frac{dX}{dt} = \frac{U_0 - abVt}{1+abt}$$

$$dX = \frac{U_0}{1+abt}dt - \frac{abVt}{1+abt}dt$$

Integrating

$$X = \frac{U_0}{ab}\ln(1+abt)\Big]_0^t - \frac{V}{ab}\int_0^t \frac{x}{1+x}dx = \left[\frac{U_0}{ab}\ln(1+abt) - \frac{V}{ab}(1+abt - \ln(1+abt))\right]_0^t$$

$$X = \frac{U_0}{ab}\ln(1+abt) - \frac{V}{ab}\left[abt - \ln(1+abt)\right]$$ — $X(t)$

Numerical values and plots are on the next page.

Problem 4.146

Acceleration, Velocity, and Position of Cart vs. Time:

Input Parameters:

$A =$	900	mm^2	9.00E-04	m^2
$M =$	10.5	kg		
$U_0 =$	12.5	m/s		
$V =$	8.25	m/s		
$\theta =$	60	degrees	1.047	rad
$\rho =$	999	kg/m^3		

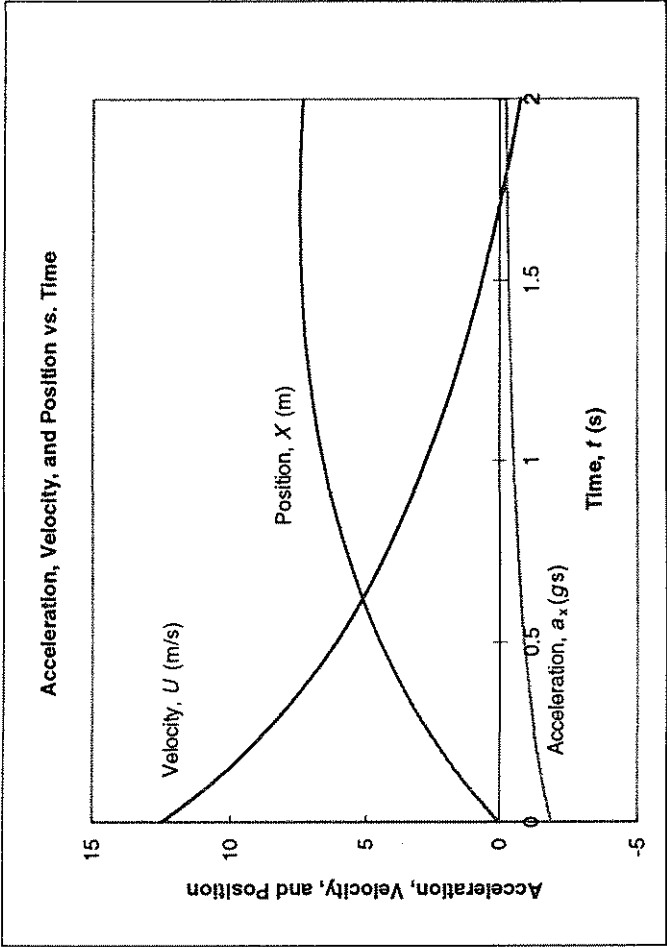
Calculated Parameters:

$a =$	0.0428	m^{-1}
$b =$	20.75	m/s

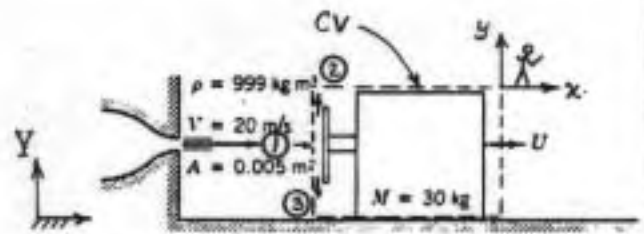
Calculated Results:

Time, t (s)	Velocity, U (m/s)	Accel., a_x (m/s)	Accel., a_x (gs)	Position, X (m)
0	12.5	-18.4	-1.88	0.00
0.1	10.8	-15.5	-1.58	1.16
0.2	9.37	-13.3	-1.35	2.17
0.3	8.13	-11.5	-1.17	3.04
0.4	7.06	-10.0	-1.02	3.80
0.5	6.12	-8.84	-0.901	4.46
0.6	5.29	-7.84	-0.800	5.03
0.7	4.54	-7.01	-0.714	5.52
0.8	3.88	-6.30	-0.642	5.94
0.9	3.28	-5.69	-0.580	6.30
1.0	2.74	-5.17	-0.527	6.60
1.1	2.24	-4.72	-0.481	6.85
1.2	1.79	-4.32	-0.440	7.05
1.3	1.38	-3.97	-0.405	7.21
1.4	0.998	-3.66	-0.373	7.33
1.5	0.646	-3.39	-0.345	7.41
1.6	0.319	-3.14	-0.320	7.46
1.7	0.0160	-2.93	-0.298	7.47
1.705	0.00000	-2.91	-0.297	7.47
1.8	-0.267	-2.73	-0.278	7.46
1.9	-0.530	-2.55	-0.260	7.42
2.0	-0.777	-2.39	-0.244	7.35

Acceleration, Velocity, and Position vs. Time

Velocity, U (m/s)
Position, X (m)
Acceleration, a_x (gs)
Acceleration, Velocity, and Position
Time, t (s)

Given: Vane/slider assembly moving under influence of jet.

$F_R = kU;\ k = 7.5\ N\cdot s/m$

Find: (a) Acceleration at instant when $U = 10\ m/s$.
(b) Terminal speed of slider.

Solution: Apply x momentum equation to linearly accelerating CV.

Basic equation:

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall + \int_{CS} u_{xyz}\rho \vec{V}_{xyz}\cdot d\vec{A}$$

($F_{B_x} = 0(1)$; $\frac{\partial}{\partial t} \approx 0(2)$)

Assumptions: (1) Horizontal, so $F_{B_x} = 0$
(2) Neglect mass of liquid on vane, $u \approx 0$ on vane
(3) Uniform flow at each section
(4) Measure velocities relative to CV

Then

$$-kU - a_{rf_x}M = u_1\{-|\rho(V-U)A|\} + u_2\{+\dot{m}_2\} + u_3\{+\dot{m}_3\}$$

$$u_1 = V - U \qquad u_2 = 0 \qquad u_3 = 0$$

$$-kU - M\frac{dU}{dt} = -\rho(V-U)^2 A$$

or

$$\frac{dU}{dt} = \frac{\rho(V-U)^2 A}{M} - \frac{kU}{M}$$

$$= 999\frac{kg}{m^3}(20-10)^2\frac{m^2}{s^2}\times 0.005\ m^2 \times \frac{1}{30\ kg} - 7.5\frac{N\cdot s}{m}\times 10\frac{m}{s}\times\frac{1}{30\ kg}\times\frac{kg\cdot m}{N\cdot s^2}$$

$$\frac{dU}{dt} = 14.2\ m/s^2 \quad (\text{at } U = 10\ m/s)$$

$\frac{dU}{dt}$

At terminal speed, $U = U_t$ and $dU/dt = 0$, so

$$0 = \frac{\rho(V-U)^2A}{M} - \frac{kU}{M} \quad \text{or} \quad V^2 - 2UV + U^2 - \frac{k}{\rho A}U = 0$$

$$U^2 - \left(2V + \frac{k}{\rho A}\right)U + V^2 = 0$$

$$U = \frac{2V + k/\rho A \pm\sqrt{(2V + k/\rho A)^2 - 4V^2}}{2} = V\left\{\left(1 + \frac{k}{2\rho VA}\right) \pm\sqrt{\left(1+\frac{k}{2\rho VA}\right)^2 - 1}\right\}$$

$$1 + \frac{k}{2\rho VA} = 1 + \frac{1}{2}\times 7.5\frac{N\cdot s}{m}\times\frac{m^3}{999\ kg}\times\frac{s}{20\ m}\times\frac{1}{0.005\ m^2}\times\frac{kg\cdot m}{N\cdot s^2} = 1.0375$$

$$U = V\left\{1.0375 \pm\sqrt{(1.0375)^2 - 1}\right\} = 0.761\ V = 0.761\times 20\frac{m}{s} = 15.2\ m/s$$

U_t

{The negative root was chosen so $U_t < V$, as required.}

Problem 4.148

4.148 For the vane/slider problem of Problem 4.147, find and plot expressions for the acceleration, speed, and position of the slider as functions of time. (Consider numerical integration.)

Given:

Data on vane/slider

Find:

Formula for acceleration, speed, and position; plot

Solution:

The given data is $\rho = 999\cdot\frac{kg}{m^3}$ $\quad M = 30\cdot kg$ $\quad A = 0.005\cdot m^2$ $\quad V = 20\cdot\frac{m}{s}$ $\quad k = 7.5\cdot\frac{N\cdot s}{m}$

The equation of motion, from Problem 4.147, is $\frac{dU}{dt} = \frac{\rho\cdot(V-U)^2\cdot A}{M} - \frac{k\cdot U}{M}$

The acceleration is thus $a = \frac{\rho\cdot(V-U)^2\cdot A}{M} - \frac{k\cdot U}{M}$

The differential equation for *U* can be solved analytically, but is quite messy. Instead we use a simple numerical method - Euler's method

$$U(n+1) = U(n) + \left[\frac{\rho\cdot(V-U)^2\cdot A}{M} - \frac{k\cdot U}{M}\right]\cdot\Delta t$$

where Δt is the time step

For the position x $\quad \frac{dx}{dt} = U$

so $\quad x(n+1) = x(n) + U\cdot\Delta t$

The final set of equations is

$$U(n+1) = U(n) + \left[\frac{\rho\cdot(V-U)^2\cdot A}{M} - \frac{k\cdot U}{M}\right]\cdot\Delta t$$

$$a = \frac{\rho\cdot(V-U)^2\cdot A}{M} - \frac{k\cdot U}{M}$$

$$x(n+1) = x(n) + U\cdot\Delta t$$

The results are plotted in the corresponding *Excel* workbook

Problem 4.148

4.148 For the vane/slider problem of Problem 4.147, find and plot expressions for the acceleration, speed, and position of the slider as functions of time. (Consider numerical integration.)

Given: Data on vane/slider

Find: Formula for acceleration, speed, and position; plot

Solution:

The final set of equations is

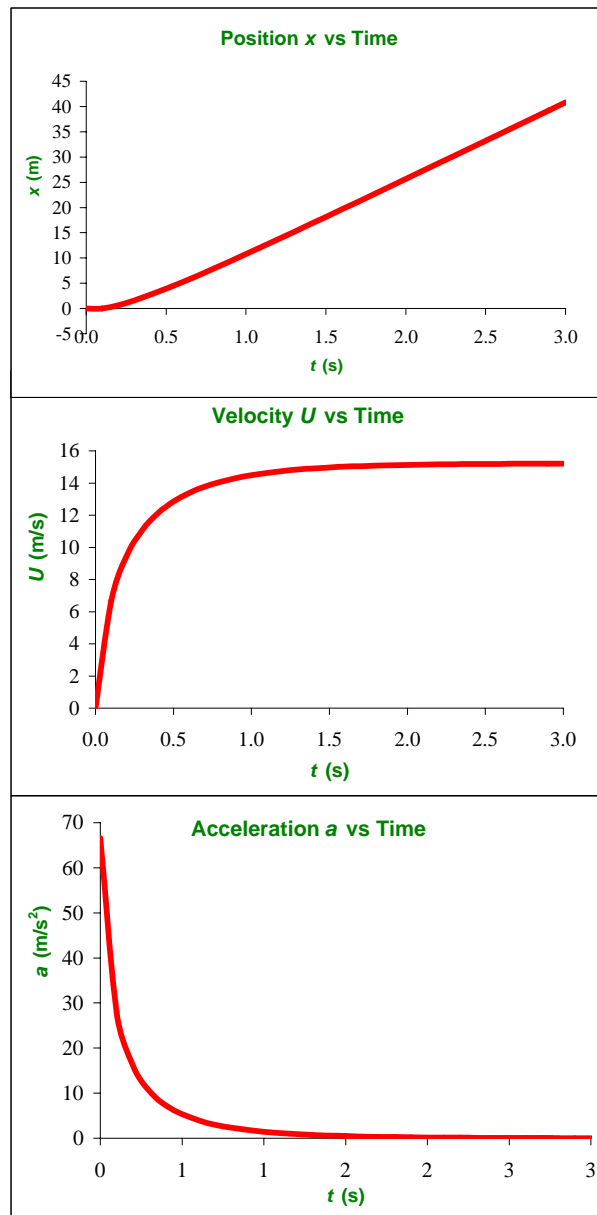
$$U(n+1) = U(n) + \left[\frac{\rho\cdot(V-U)^2\cdot A}{M} - \frac{k\cdot U}{M}\right]\cdot\Delta t$$

$$a = \frac{\rho\cdot(V-U)^2\cdot A}{M} - \frac{k\cdot U}{M}$$

$$x(n+1) = x(n) + U\cdot\Delta t$$

$\rho =$ 999 kg/m^3
$k =$ 7.5 N.s/m
$A =$ 0.005 m^2
$V =$ 20 m/s
$M =$ 30 kg
$\Delta t =$ 0.1 s

t (s)	*x* (m)	*U* (m/s)	*a* (m/s^2)
0.0	0.0	0.0	66.6
0.1	0.0	6.7	28.0
0.2	0.7	9.5	16.1
0.3	1.6	11.1	10.5
0.4	2.7	12.1	7.30
0.5	3.9	12.9	5.29
0.6	5.2	13.4	3.95
0.7	6.6	13.8	3.01
0.8	7.9	14.1	2.32
0.9	9.3	14.3	1.82
1.0	10.8	14.5	1.43
1.1	12.2	14.6	1.14
1.2	13.7	14.7	0.907
1.3	15.2	14.8	0.727
1.4	16.6	14.9	0.585
1.5	18.1	15.0	0.472
1.6	19.6	15.0	0.381
1.7	21.1	15.1	0.309
1.8	22.6	15.1	0.250
1.9	24.1	15.1	0.203
2.0	25.7	15.1	0.165
2.1	27.2	15.1	0.134
2.2	28.7	15.2	0.109
2.3	30.2	15.2	0.0889
2.4	31.7	15.2	0.0724
2.5	33.2	15.2	0.0590
2.6	34.8	15.2	0.0481
2.7	36.3	15.2	0.0392
2.8	37.8	15.2	0.0319
2.9	39.3	15.2	0.0260
3.0	40.8	15.2	0.0212

Problem 4.149

Given: Block and jet as shown.

Jet strikes block at $t > 0$.

Find: (a) Expression for acceleration.
(b) Time at which $U = 0$.

Solution: Apply x momentum equation to linearly accelerating CV.

Basic equation:

$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x} \rho \, d\forall = \overset{\approx 0(3)}{\frac{\partial}{\partial t}} \int_{CV} u_{xyz} \rho \, d\forall + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) No pressure or friction forces, so $F_{S_x} = 0$
(2) Horizontal, so $F_{B_x} = 0$
(3) Neglect mass of liquid in CV, $u \approx 0$ in CV
(4) Uniform flow at each section
(5) Measure velocities relative to CV

Then

$$-Ma_{rf_x} = -M\frac{dU}{dt} = u_1\{-|\rho(V+U)A|\} + u_2\{+\dot{m}_2\} + u_3\{+\dot{m}_3\}$$

$$u_1 = -(V+U) \qquad u_2 = 0 \qquad u_3 = 0$$

or

$$\frac{dU}{dt} = -\frac{\rho(V+U)^2 A}{M}$$

$\frac{dU}{dt}$

But, since V = constant, $dU = d(V+U)$, so

$$\frac{d(V+U)}{(V+U)^2} = -\frac{\rho A}{M} dt$$

Integrating from U_0 at $t = 0$ to $U = 0$ at t

$$\int_{V+U_0}^{V} \frac{d(V+U)}{(V+U)^2} = -\frac{1}{(V+U)}\Bigg]_{V+U_0}^{V} = -\frac{1}{V} + \frac{1}{V+U_0} = \frac{-U_0}{V(V+U_0)} = -\frac{\rho A t}{M}$$

Solving,

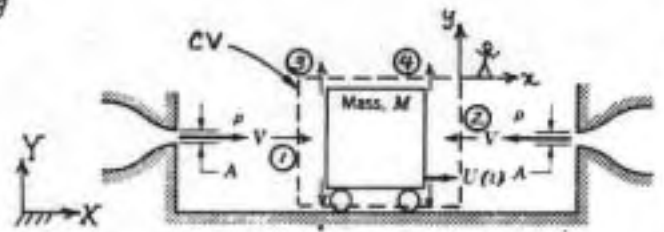
$$t = \frac{MU_0}{\rho VA(V+U_0)} = \frac{M}{\rho VA(1+V/U_0)}$$

t

Problem 4.150

Given: Block rolling between opposing jets, as shown.

Speed is U_0 at $t = 0$.

There is no resistance for $t > 0$.

Find: (a) Expression for acceleration, $a(t)$.
(b) Expression for speed, $U(t)$.

Solution: Apply x momentum to linearly accelerating CV.

Basic equation:

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x} \rho \, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho \, d\forall + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

($F_{S_x} = 0$ (1), $F_{B_x} = 0$ (2), $\frac{\partial}{\partial t} \approx 0$ (3))

Assumptions: (1) No pressure or friction forces, so $F_{S_x} = 0$
(2) Horizontal, so $F_{B_x} = 0$
(3) Neglect mass of liquid in CV; $u \approx 0$ in CV
(4) Uniform flow at each section
(5) Measure velocities relative to CV

Then

$$-a_{rf_x} M = -M\frac{dU}{dt} = u_1\{-|\rho(V-U)A|\} + u_2\{-|\rho(V+U)A|\} + u_3\{\dot{m}_3\} + u_4\{\dot{m}_4\}$$

$$u_1 = V - U \qquad u_2 = -(V+U) \qquad u_3 = 0 \qquad u_4 = 0$$

or

$$-M\frac{dU}{dt} = \rho A\left[-(V-U)^2 + (V+U)^2\right] = \rho A[4UV] = 4\rho VAU$$

Thus $\quad \dfrac{dU}{U} = -\dfrac{4\rho VA}{M}\,dt$

Integrating $\quad \displaystyle\int_{U_0}^{U}\frac{dU}{U} = \ln U\Big]_{U_0}^{U} = \ln\frac{U}{U_0} = -\frac{4\rho VA}{M}t$

or

$$U(t) = U_0 e^{-\frac{4\rho VA}{M}t}$$

$U(t)$

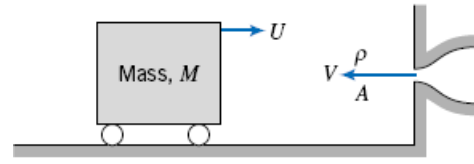
Also

$$a(t) = \frac{dU}{dt} = -\frac{4\rho VA}{M}U_0 e^{-\frac{4\rho VA}{M}t}$$

$a(t)$

Problem 4.151

[3]

4.151 Consider the diagram of Problem 4.149. If $M = 100$ kg, $\rho = 999$ kg/m^3, and $A = 0.01$ m^2, find the jet speed V required for the cart to be brought to rest after one second if the initial speed of the cart is $U_0 = 5$ m/s. For this condition, plot the speed U and position x of the cart as functions of time. What is the maximum value of x, and how long does the cart take to return to its initial position?

Given: Data on system

Find: Jet speed to stop cart after 1 s; plot speed & position; maximum x; time to return to origin

Solution:

The given data is $\rho = 999 \cdot \frac{kg}{m^3}$ $\quad M = 100 \cdot kg$ $\quad A = 0.01 \cdot m^2$ $\quad U_0 = 5 \cdot \frac{m}{s}$

The equation of motion, from Problem 4.149, is $\frac{dU}{dt} = -\frac{\rho \cdot (V+U)^2 \cdot A}{M}$

which leads to $\frac{d(V+U)}{(V+U)^2} = -\left(\frac{\rho \cdot A}{M} \cdot dt\right)$

Integrating and using the IC *U* = U_0 at *t* = 0 $\quad U = -V + \frac{V+U_0}{1 + \frac{\rho \cdot A \cdot (V+U_0)}{M} \cdot t}$

To find the jet speed *V* to stop the cart after 1 s, solve the above equation for *V*, with *U* = 0 and *t* = 1 s. (The equation becomes a quadratic in *V*). Instead we use *Excel*'s *Goal Seek* in the associated workbook

From *Excel* $\quad V = 5 \cdot \frac{m}{s}$

For the position *x* we need to integrate $\quad \frac{dx}{dt} = U = -V + \frac{V+U_0}{1 + \frac{\rho \cdot A \cdot (V+U_0)}{M} \cdot t}$

The result is $\quad x = -V \cdot t + \frac{M}{\rho \cdot A} \cdot \ln\left[1 + \frac{\rho \cdot A \cdot (V+U_0)}{M} \cdot t\right]$

This equation (or the one for *U* with *U* = 0) can be easily used to find the maximum value of *x* by differentiating, as well as the time for *x* to be zero again. Instead we use *Excel*'s *Goal Seek* and *Solver* in the associated workbook

From *Excel* $\quad x_{max} = 1.93 \cdot m$ $\quad t(x = 0) = 2.51 \cdot s$

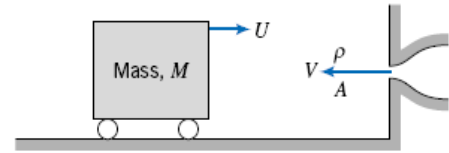
The complete set of equations is

$$U = -V + \frac{V+U_0}{1 + \frac{\rho \cdot A \cdot (V+U_0)}{M} \cdot t} \qquad x = -V \cdot t + \frac{M}{\rho \cdot A} \cdot \ln\left[1 + \frac{\rho \cdot A \cdot (V+U_0)}{M} \cdot t\right]$$

The plots are presented in the *Excel* workbook

Problem 4.151

[3]

4.151 Consider the diagram of Problem 4.149. If $M = 100$ kg, $\rho = 999$ kg/m^3, and $A = 0.01$ m^2, find the jet speed V required for the cart to be brought to rest after one second if the initial speed of the cart is $U_0 = 5$ m/s. For this condition, plot the speed U and position x of the cart as functions of time. What is the maximum value of x, and how long does the cart take to return to its initial position?

Given: Data on system

Find: Jet speed to stop cart after 1 s; plot speed & position; maximum x; time to return to origin

Solution:

The complete set of equations is

$$U = -V + \frac{V + U_0}{1 + \frac{\rho \cdot A \cdot (V + U_0)}{M} \cdot t} \qquad x = -V \cdot t + \frac{M}{\rho \cdot A} \cdot \ln\left[1 + \frac{\rho \cdot A \cdot (V + U_0)}{M} \cdot t\right]$$

M =	100	kg
ρ =	999	kg/m^3
A =	0.01	m^2
U_o =	5	m/s

t (s)	*x* (m)	*U* (m/s)
0.0	0.00	5.00
0.2	0.82	3.33
0.4	1.36	2.14
0.6	1.70	1.25
0.8	1.88	0.56
1.0	1.93	0.00
1.2	1.88	-0.45
1.4	1.75	-0.83
1.6	1.56	-1.15
1.8	1.30	-1.43
2.0	0.99	-1.67
2.2	0.63	-1.88
2.4	0.24	-2.06
2.6	-0.19	-2.22
2.8	-0.65	-2.37
3.0	-1.14	-2.50

To find V for $U = 0$ in 1 s, use *Goal Seek*

t (s)	*U* (m/s)	*V* (m/s)
1.0	0.00	5.00

To find the maximum x, use *Solver*

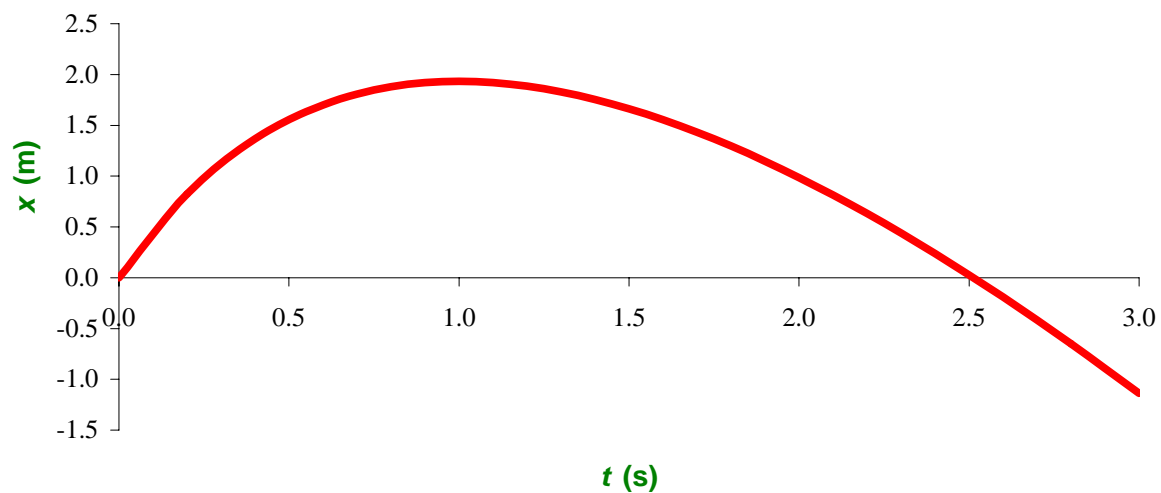
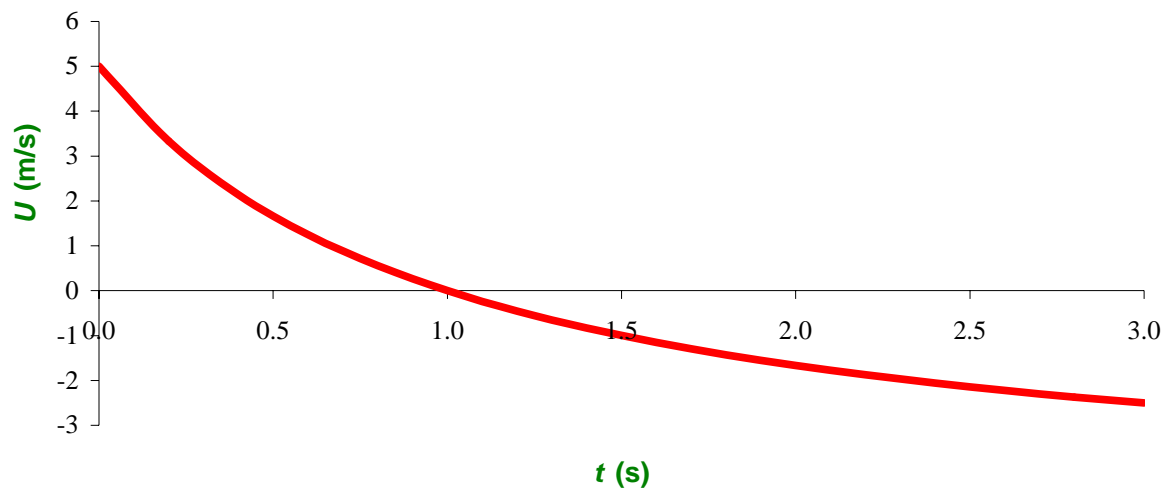
t (s)	*x* (m)
1.0	1.93

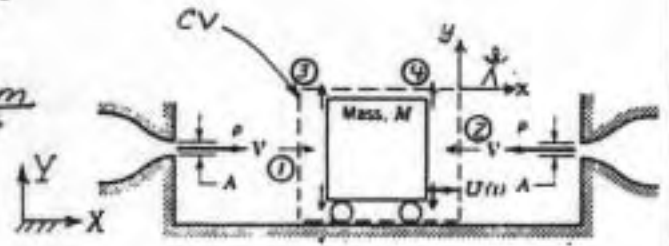
To find the time at which $x = 0$ use *Goal Seek*

t (s)	*x* (m)
2.51	0.00

Cart Position x vs Time
x (m)
2.5
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
t (s)

Cart Speed U vs Time
U (m/s)
6
5
4
3
2
1
0
-1
-2
-3
0.0
0.5
1.0
1.5
2.0
2.5
3.0
t (s)

Problem 4.152

Given: Block rolling between opposing jets, as shown

At $t = 0$, block moves at $U_0 = 10\,\frac{m}{s}$ starting from $X = 0$.

Find: (a) Time to reduce speed to $U = 0.5$ m/s.
(b) Position at that instant.

Solution: Apply x momentum equation to linearly accelerating CV.

Basic equation:

$$\overset{=0(1)}{F_{Sx}} + \overset{=0(2)}{F_{Bx}} - \int_{CV} a_{rf_x}\rho\,d\forall = \overset{\approx 0(3)}{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\,d\forall} + \int_{CS} u_{xyz}\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) No pressure or friction forces, so $F_{Sx} = 0$
(2) Horizontal, so $F_{Bx} = 0$
(3) Neglect mass of liquid in CV; $u \approx 0$ in CV
(4) Uniform flow at each section
(5) Measure velocities relative to CV

Then

$$-a_{rf_x}M = -M\frac{dU}{dt} = u_1\{-|\rho(V-U)A|\} + u_2\{-|\rho(V+U)A|\} + u_3\{\dot{m}_3\} + u_4\{\dot{m}_4\}$$

$$u_1 = V - U \qquad u_2 = -(V+U) \qquad u_3 = 0 \qquad u_4 = 0$$

or

$$-M\frac{dU}{dt} = \rho A\left[-(V-U)^2 + (V+U)^2\right] = \rho A\left[4UV\right] = 4\rho VAU$$

Thus

$$\frac{dU}{U} = -\frac{4\rho VA}{M}\,dt$$

Integrating, $$\int_{U_0}^{U}\frac{dU}{U} = \ln U\Big]_{U_0}^{U} = \ln\frac{U}{U_0} = -\frac{4\rho VA}{M}\,t \qquad (1)$$

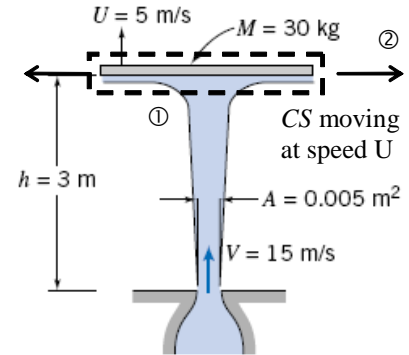
Thus $$t = -\frac{M}{4\rho VA}\ln\frac{U}{U_0} = -\frac{1}{4}\cdot\frac{M}{\rho VA}\ln\frac{0.5}{10} = 0.750\,\frac{M}{\rho VA}$$ ← t

From Eq. 1, $$U(t) = \frac{dX}{dt} = U_0 e^{-\frac{4\rho VA}{M}t}$$

Integrating, $$X = \int_0^X dX = \int_0^t U_0 e^{-\frac{4\rho VA}{M}t}\,dt = -\frac{MU_0}{4\rho VA}\,e^{-\frac{4\rho VA}{M}t}\Big]_0^t$$

$$X = \frac{MU_0}{4\rho VA}\left[1 - e^{-\frac{4\rho VA}{M}t}\right] = \frac{0.95}{4}\,\frac{MU_0}{\rho VA} = 0.238\,\frac{MU_0}{\rho VA}$$ ← X

Problem *4.153

*4.153 A vertical jet of water impinges on a horizontal disk as shown. The disk assembly mass is 30 kg. When the disk is 3 m above the nozzle exit, it is moving upward at $U = 5$ m/s. Compute the vertical acceleration of the disk at this instant.

Given: Water jet striking moving disk

Find: Acceleration of disk when at a height of 3 m

Solution:

Basic equations: Bernoulli; Momentum flux in z direction (treated as upwards) for linear accelerating CV

$$\frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{constant} \qquad F_{S_z} + F_{B_z} - \int_{CV} a_{rf_z}\,\rho\,dV\!\!\!\!- = \frac{\partial}{\partial t}\int_{CV} w_{xyz}\,\rho\,dV\!\!\!\!- + \int_{CS} w_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow (All in jet)

The Bernoulli equation becomes $\frac{V_0^2}{2} + g\cdot 0 = \frac{V_1^2}{2} + g\cdot(z - z_0)$ $\qquad V_1 = \sqrt{V_0^2 + 2\cdot g\cdot(z_0 - z)}$

$$V_1 = \sqrt{\left(15\cdot\frac{m}{s}\right)^2 + 2\times 9.81\cdot\frac{m}{s^2}\cdot(0-3)\cdot m} \qquad V_1 = 12.9\frac{m}{s}$$

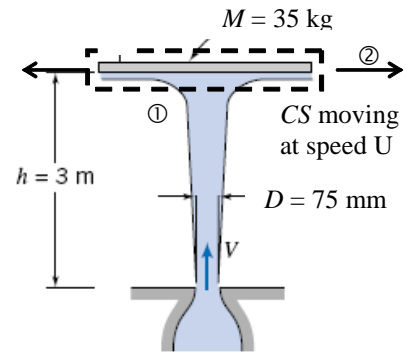
The momentum equation becomes

$$-W - M\cdot a_{rfz} = w_1\cdot(-\rho\cdot V_1\cdot A_1) + w_2\cdot(\rho\cdot V_2\cdot A_2) = (V_1 - U)\cdot\left[-\rho\cdot(V_1 - U)\cdot A_1\right] + 0$$

Hence $\quad a_{rfz} = \frac{\rho\cdot(V_1-U)^2\cdot A_1 - W}{M} = \frac{\rho\cdot(V_1-U)^2\cdot A_1}{M} - g = \frac{\rho\cdot(V_1-U)^2\cdot A_0\cdot\frac{V_0}{V_1}}{M} - g \qquad$ using $\quad V_1\cdot A_1 = V_0\cdot A_0$

$$a_{rfz} = 1000\cdot\frac{kg}{m^3}\times\left[(12.9-5)\cdot\frac{m}{s}\right]^2\times 0.005\cdot m^2\times\frac{15}{12.9}\times\frac{1}{30\cdot kg} - 9.81\cdot\frac{m}{s^2} \qquad a_{rfz} = 2.28\frac{m}{s^2}$$

Problem *4.154

*4.154 A vertical jet of water leaves a 75-mm diameter nozzle. The jet impinges on a horizontal disk (see Problem 4.153). The disk is constrained horizontally but is free to move vertically. The mass of the disk is 35 kg. Plot disk mass versus flow rate to determine the water flow rate required to suspend the disk 3 m above the jet exit plane.

Given: Water jet striking disk

Find: Plot mass versus flow rate to find flow rate for a steady height of 3 m

Solution:

Basic equations: Bernoulli; Momentum flux in z direction (treated as upwards)

$$\frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{constant} \qquad F_z = F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{CV} w\,\rho\,dV + \int_{CS} w\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow (All in jet)

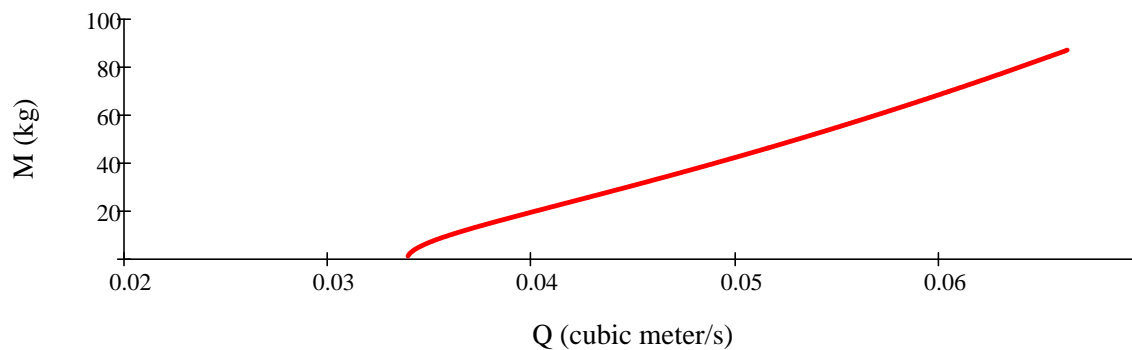
The Bernoulli equation becomes $\frac{V_0^2}{2} + g \cdot 0 = \frac{V_1^2}{2} + g \cdot h \qquad V_1 = \sqrt{V_0^2 - 2 \cdot g \cdot h}$

The momentum equation becomes

$$-M \cdot g = w_1 \cdot (-\rho \cdot V_1 \cdot A_1) + w_2 \cdot (\rho \cdot V_2 \cdot A_2) = V_1 \cdot (-\rho \cdot V_1 \cdot A_1) + 0$$

Hence $M = \frac{\rho \cdot V_1^2 \cdot A_1}{g}$ but from continuity $V_1 \cdot A_1 = V_0 \cdot A_0$

$$M = \frac{\rho \cdot V_1 \cdot V_0 \cdot A_0}{g} = \frac{\pi}{4} \cdot \frac{\rho \cdot V_0 \cdot D_0^2}{g} \cdot \sqrt{V_0^2 - 2 \cdot g \cdot h}$$ and also $Q = V_0 \cdot A_0$

This equation is difficult to solve for V_0 for a given M. Instead we plot first:

This graph can be parametrically plotted in *Excel*. The Goal Seek or Solver feature can be used to find Q when M = 35 kg

$$Q = 0.0469 \cdot \frac{m^3}{s}$$

Problem 4.155

Given: Rocket sled on horizontal track, slowed by retro-rocket.

Initial mass $M_0 = 1500\ kg$ Initial speed $U_0 = 500\ m/s$

Mass flow rate $\dot{m} = 7.75\ kg/s$ Exhaust speed $V_e = 2500\ m/s$

Firing time $t_{bo} = 20.0\ s$

Neglect aerodynamic drag and rolling resistance.

Find: (a) Algebraic expression for sled speed U as a function of t.

(b) Speed at end of retro-rocket firing.

Solution: Apply x-component of momentum equation to the linearly accelerating CV shown.

From continuity,

$$M_{cv} = M_0 - \dot{m}t, \quad t < t_{bo}$$

Basic equation:

$$F_{S_x}^{=0(1)} + F_{B_x}^{=0(2)} - \int_{CV} a_{rf_x}\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall^{\approx 0(3)} + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) No pressure, drag, or rolling resistance, so $F_{S_x} = 0$

(2) Horizontal motion, so $F_{B_x} = 0$

(3) Neglect unsteady effects within CV

(4) Uniform flow at nozzle exit plane

(5) $p_e = p_{atm}$

Then $\quad -a_{rf_x} M_{cv} = u_e\{+\dot{m}\} = +V_e\dot{m} \quad$ or $\quad \frac{dU}{dt} = -\frac{V_e\dot{m}}{M_{cv}} = -\frac{V_e\dot{m}}{M_0 - \dot{m}t}$

$u_e = V_e$

Thus $\quad dU = V_e\left(\frac{-\dot{m}\,dt}{M_0 - \dot{m}t}\right) \quad$ and $\quad U - U_0 = V_e \ln(M_0 - \dot{m}t)\Big]_0^t = V_e \ln\left(1 - \frac{\dot{m}t}{M_0}\right)$

$$U(t) = U_0 + V_e \ln\left(1 - \frac{\dot{m}t}{M_0}\right); \quad t < t_{bo}$$

$U(t)$

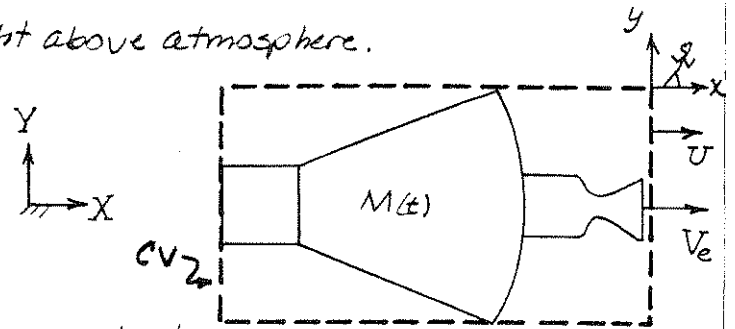
At t_{bo}, $\quad U(t_{bo}) = 500\ \frac{m}{s} + 2500\ \frac{m}{s} \times \ln\left(1 - 7.75\ \frac{kg}{s} \times 20.0\ s \times \frac{1}{1500\ kg}\right)$

$$U(t_{bo}) = 227\ m/s$$

$U(t_{bo})$

Given: Space capsule in level flight above atmosphere.

$U_0 = 8.0$ km/s
$M_0 = 1600$ kg

$\dot{m} = 8$ kg/s
$V_e = 3000$ m/s

Find: Time to reduce speed to $U = 5.00$ km/s.

Solution: Apply x component of momentum to CV with linear acceleration.

Basic equation:

$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x} \rho \, d\forall = \frac{\partial}{\partial t}\int_{CV} \overset{\approx 0(4)}{u_{xyz}} \rho \, d\forall + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) No resistance; $F_{S_x} = 0$
(2) Horizontal; $F_{B_x} = 0$
(3) Use velocities measured relative to CV
(4) Neglect velocity within CV
(5) Uniform flow at exit plane with negligible p_e (given)

From continuity,
$$\frac{dM}{dt} = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall = -\int_{CS} \rho \vec{V}_{xyz} \cdot d\vec{A} = -\dot{m}; \quad M(t) = M_0 - \dot{m}t$$

From momentum,
$$-a_{rf_x} M = -\frac{dU}{dt}(M_0 - \dot{m}t) = u_e\{+\dot{m}\} = V_e \dot{m}$$

Thus
$$\frac{dU}{dt} = -\frac{V_e \dot{m}}{M_0 - \dot{m}t} \qquad u_e = V_e$$

Integrating, $$U - U_0 = V_e \int_0^t \frac{-\dot{m}\,dt}{M_0 - \dot{m}t} = V_e \ln(M_0 - \dot{m}t)\Big]_0^t = V_e \ln\left(\frac{M_0 - \dot{m}t}{M_0}\right)$$

Solving for t,
$$\frac{M_0 - \dot{m}t}{M_0} = e^{\frac{U - U_0}{V_e}}; \quad M_0 - \dot{m}t = M_0 e^{U - U_0/V_e}$$

$$t = \frac{M_0}{\dot{m}}\left(1 - e^{U - U_0/V_e}\right)$$

$$= 1600\ \text{kg} \times \frac{\text{s}}{8\ \text{kg}}\left\{1 - e^{\left[(5.00 - 8.0)\frac{\text{km}}{\text{s}} \times \frac{\text{s}}{3000\ \text{m}} \times 1000\frac{\text{m}}{\text{km}}\right]}\right\}$$

$t = 126$ s

t

Given: Rocket sled accelerates from rest on a level track. Initial mass, $M_0 = 600$ kg, includes fuel - $M_f = 150$ kg. The rocket motor burns fuel at rate $\dot{m} = 15$ kg/s. Exhaust gases leave nozzle uniformly and axially at atmospheric pressure with $V_e = 2900$ m/s relative to the nozzle. Neglect air and rolling resistance.

Find: (a) Maximum speed reached by the sled.
(b) Maximum acceleration of sled during the run.

Plot: the sled speed and acceleration as functions of time.

Solution:

Apply the momentum equation to linearly accelerating CV shown.

Basic equation: $F_{S_x} + F_{B_x} - \int_{CV} a_{rf}\,\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$ (terms $=0$ by (1), (2), (3))

Assumptions: (1) no net-pressure forces ($p_e = p_{atm}$, given)
(2) horizontal motion, $F_{B_x} = 0$
(3) neglect $\partial/\partial t$ in CV
(4) uniform axial jet

From continuity, $M = M_0 - \dot{m}t$. Then

$$-a_{rf_x} M = -\frac{dU}{dt}(M_0 - \dot{m}t) = u_e\{\dot{m}\} = -V_e\dot{m} \quad \ldots (1)$$

Separating variables,

$$dU = V_e \frac{\dot{m}\,dt}{M_0 - \dot{m}t}$$

Integrating from $U=0$ at $t=0$ to U at t gives

$$U = -V_e \ln(M_0 - \dot{m}t)\Big]_0^t = -V_e \ln\frac{(M_0 - \dot{m}t)}{M_0} = V_e \ln\frac{M_0}{(M_0 - \dot{m}t)} \quad \ldots (2)$$

The speed is a maximum at burnout. At burnout $M_f = 0$ and $M = M_0 - \dot{m}t = 450$ kg

At burnout, $t = \frac{M_f)_{initial}}{\dot{m}_{fuel}} = 150\,\text{kg} \times \frac{s}{15\,\text{kg}} = 10\,\text{s}$

Then from Eq. 2

$$U_{max} = 2900\,\frac{m}{s} \ln\frac{600\,\text{kg}}{450\,\text{kg}} = 834\ \text{m/s} \quad \longleftarrow U_{max}$$

From Eq. 1 the acceleration is $\frac{dU}{dt} = \frac{\dot{m}V_e}{M_0 - \dot{m}t}$

The maximum acceleration occurs at the instant prior to burn out

$$\left.\frac{dU}{dt}\right)_{max} = 15\,\frac{kg}{s} \times 2900\,\frac{m}{s} \times \frac{1}{450\,kg} = 96.7\ \text{m/s}^2 \quad \longleftarrow \left.\frac{dU}{dt}\right)_{max}$$

Problem 4.157

The sled speed as a function of time is

$$U = V_e \ln \frac{M_0}{(M_0 - \dot{m}t)} \quad \text{for } 0 \le t \le 10\text{s}$$

$$U = \text{constant} = 834 \text{ m/s} \quad \text{for } t > 10 \quad \text{(neglecting resistance)}$$

The sled acceleration is given by

$$\frac{dU}{dt} = \frac{\dot{m} V_e}{(M_0 - \dot{m}t)} \quad \text{for } 0 \le t \le 10\text{s}$$

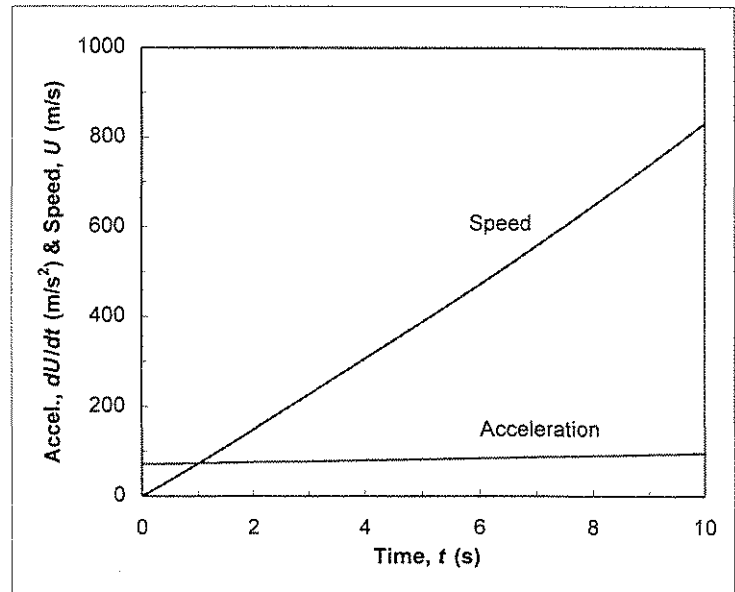
$$\frac{dU}{dt} = 0 \quad \text{for } t \ge 10\text{s}.$$

Acceleration and Velocity vs. Time for Rocket Sled:

Input Data:

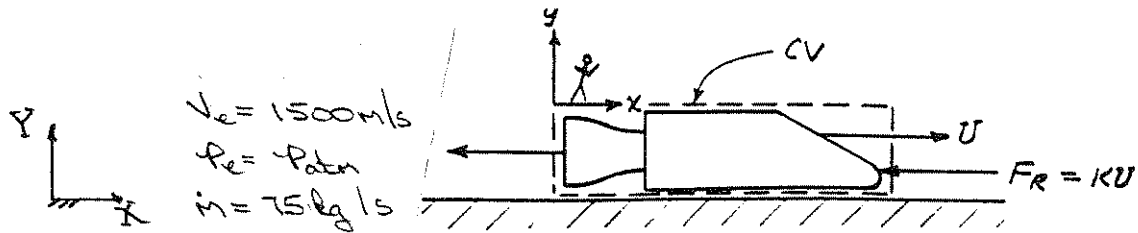
M_0 =	600	kg
m(dot) =	15	kg/s
V_e =	2900	m/s

Calculated Results:

Time, t (s)	Acceleration, dU/dt (m/s^2)	Velocity, U (m/s)
0	72.5	0
1	74.4	73.4
2	76.3	149
3	78.4	226
4	80.6	306
5	82.9	387
6	85.3	471
7	87.9	558
8	90.6	647
9	93.5	739
10	96.7	834

Problem 4.158

Given: Rocket sled with initial mass of 4 metric tons, including 1 ton of fuel. Motion resistance is given by kU where $k = 75$ N/m/s.

$V_e = 1500$ m/s
$p_e = p_{atm}$
$\dot{m} = 75$ kg/s

CV, U, $F_R = kU$

Find: Sled speed 10 s after starting from rest, & U_{max}

Plot: sled speed and acceleration as functions of time.

Solution:

Apply the x component of the momentum equation to linearly accelerating CV shown

Basic equation: $F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x} \rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho\, d\forall + \int_{CS} u_{xyz} (\rho \vec{V}_{xyz} \cdot d\vec{A})$ ($F_{B_x} = 0$ (2); $\frac{\partial}{\partial t} \approx 0$ (3))

Assumptions: (1) $p_e = p_{atm}$ (given) so $F_{S_x} = -F_R$
(2) $F_{B_x} = 0$
(3) neglect unsteady effects within CV
(4) uniform flow at exit plane

Then,

$$-F_R - a_{rf_x} M = u_e \{+|\dot{m}|\} = -V_e \dot{m} \qquad \{F_R = kU,\ u_e = -V_e\}$$

From continuity, $M = M_0 - \dot{m}t$. Substituting with $a_{rf_x} = \frac{dU}{dt}$

$$-kU - (M_0 - \dot{m}t)\frac{dU}{dt} = -V_e \dot{m}$$

$$\frac{dU}{dt} = \frac{V_e \dot{m} - kU}{M_0 - \dot{m}t} \quad \text{or} \quad \frac{dU}{V_e \dot{m} - kU} = \frac{dt}{M_0 - \dot{m}t}$$

Integrating, $-\frac{1}{k} \ln(V_e \dot{m} - kU)\Big]_0^U = -\frac{1}{\dot{m}} \ln(M_0 - \dot{m}t)\Big]_0^t$

and $\ln \frac{(V_e \dot{m} - kU)}{V_e \dot{m}} = \ln\left(1 - \frac{kU}{V_e \dot{m}}\right) = \frac{k}{\dot{m}} \ln \frac{(M_0 - \dot{m}t)}{M_0} = \frac{k}{\dot{m}} \ln\left(1 - \frac{\dot{m}t}{M_0}\right)$

Then $1 - \frac{kU}{V_e \dot{m}} = \left(1 - \frac{\dot{m}t}{M_0}\right)^{k/\dot{m}}$ and

$$U = \frac{V_e \dot{m}}{k}\left[1 - \left(1 - \frac{\dot{m}t}{M_0}\right)^{k/\dot{m}}\right] \qquad (1)$$

At $t = 10$ s

$$U = 1500\frac{m}{s} \times 75\frac{kg}{s} \times \frac{m}{75\, N \cdot s} \times \frac{N \cdot s^2}{kg \cdot m}\left[1 - \left(1 - 75\frac{kg}{s} \times 10\,s \times \frac{1}{4000\,kg}\right)^{75\frac{N \cdot s}{m} \times \frac{s}{75\,kg} \times \frac{kg \cdot m}{N \cdot s^2}}\right]$$

$U = 281$ m/s U

Note that all fuel would be expended at $t_{bo} = \frac{M_f}{\dot{M}} = 1000\,kg \times \frac{s}{75\,kg}$

i.e. at $t = 13.3\,s$

The sled speed as a function of time is then

$$U = \frac{V_e \dot{M}}{k}\left[1 - \left(1 - \frac{\dot{M}t}{M_0}\right)^{k/\dot{M}}\right] \quad \text{for } 0 \le t \le 13.3\,s$$

The speed reaches a maximum at $t = 13.3\,s$ and decays with time due to the motion resistance. $U_{max} = 375\,m/s$

The sled acceleration is given by

$$\frac{dU}{dt} = \frac{V_e \dot{M} - kU}{M_0 - \dot{M}t} \quad \text{for } 0 \le t \le 11.1\,s$$

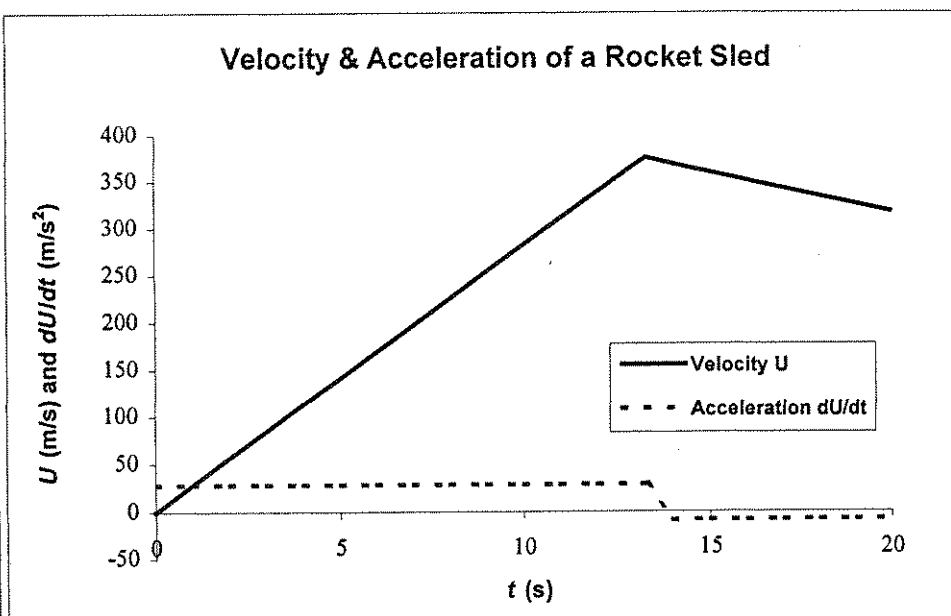
At $t \ge 13.3\,s$ $V_e = 0$ and

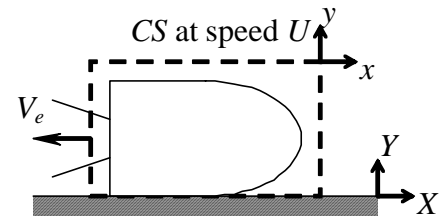
$$\frac{dU}{dt} = \frac{-kU}{M_0 - M_{fuel}}$$

Note that for $t > t_{bo} = 13.3\,s$, $\frac{dU}{dt} = -\frac{kU}{M_{bo}}$ and

$$\frac{dU}{U} = -\frac{k}{M_{bo}}dt \quad , \quad \ln\frac{U}{U_{bo}} = -\frac{k(t - t_{bo})}{M_{bo}}$$

and $U = U_{bo} e^{-k(t-t_{bo})/M_{bo}}$

t (s)	*U* (m/s)	*dU/dt* (m/s²)
0.0	0.0	28.1
1.0	28.1	28.1
2.0	56.3	28.1
3.0	84.4	28.1
4.0	113	28.1
5.0	141	28.1
6.0	169	28.1
7.0	197	28.1
8.0	225	28.1
9.0	253	28.1
10.0	281	28.1
11.0	309	28.1
12.0	338	28.1
13.2	371	28.1
13.3	375	28.1
14.0	369	-9.22
15.0	360	-8.99
16.0	351	-8.77
17.0	342	-8.55
18.0	334	-8.34
19.0	325	-8.14
20.0	317	-7.94

Problem 4.159

4.159 A rocket sled with initial mass of 900 kg is to be accelerated on a level track. The rocket motor burns fuel at constant rate $\dot{m} = 13.5$ kg/s. The rocket exhaust flow is uniform and axial. Gases leave the nozzle at 2750 m/s relative to the nozzle, and the pressure is atmospheric. Determine the minimum mass of rocket fuel needed to propel the sled to a speed of 265 m/s before burnout occurs. As a first approximation, neglect resistance forces.

Given: Data on rocket sled

Find: Minimum fuel to get to 265 m/s

Solution:

Basic equation: Momentum flux in x direction $F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\,\rho\, dV = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, dV + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$

Assumptions: 1) No resistance 2) $p_e = p_{atm}$ 3) Uniform flow 4) Use relative velocities

From continuity $\frac{dM}{dt} = m_{rate} = \text{constant}$ so $M = M_0 - m_{rate}\cdot t$ (Note: Software cannot render a dot!)

Hence from momentum $-a_{rfx}\cdot M = -\frac{dU}{dt}\cdot\left(M_0 - m_{rate}\cdot t\right) = u_e\cdot\left(\rho_e\cdot V_e\cdot A_e\right) = -V_e\cdot m_{rate}$

Separating variables $dU = \frac{V_e\cdot m_{rate}}{M_0 - m_{rate}\cdot t}\cdot dt$

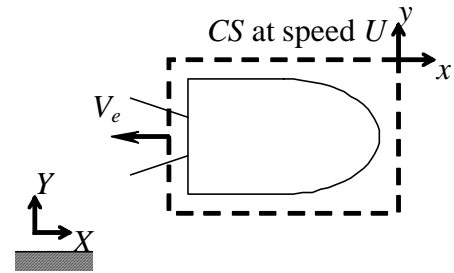
Integrating $U = V_e\cdot\ln\left(\frac{M_0}{M_0 - m_{rate}\cdot t}\right) = -V_e\cdot\ln\left(1 - \frac{m_{rate}\cdot t}{M_0}\right)$ or $t = \frac{M_0}{m_{rate}}\cdot\left(1 - e^{-\frac{U}{V_e}}\right)$

The mass of fuel consumed is $m_f = m_{rate}\cdot t = M_0\cdot\left(1 - e^{-\frac{U}{V_e}}\right)$

Hence $m_f = 900\cdot\text{kg}\times\left(1 - e^{-\frac{265}{2750}}\right)$ $m_f = 82.7\,\text{kg}$

Problem 4.160

[3]

4.160 A rocket motor is used to accelerate a kinetic energy weapon to a speed of 3500 mph in horizontal flight. The exit stream leaves the nozzle axially and at atmospheric pressure with a speed of 6000 mph relative to the rocket. The rocket motor ignites upon release of the weapon from an aircraft flying horizontally at $U_0 = 600$ mph. Neglecting air resistance, obtain an algebraic expression for the speed reached by the weapon in level flight. Determine the minimum fraction of the initial mass of the weapon that must be fuel to accomplish the desired acceleration.

Given: Data on rocket weapon

Find: Expression for speed of weapon; minimum fraction of mass that must be fuel

Solution:

Basic equation: Momentum flux in x direction $F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\,\rho\, dV = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, dV + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$

Assumptions: 1) No resistance 2) $p_e = p_{atm}$ 3) Uniform flow 4) Use relative velocities 5) Constant mass flow rate

From continuity $\frac{dM}{dt} = m_{rate} = \text{constant}$ so $M = M_0 - m_{rate}\cdot t$ (Note: Software cannot render a dot!)

Hence from momentum $-a_{rfx}\cdot M = -\frac{dU}{dt}\cdot\left(M_0 - m_{rate}\cdot t\right) = u_e\cdot\left(\rho_e\cdot V_e\cdot A_e\right) = -V_e\cdot m_{rate}$

Separating variables $dU = \frac{V_e\cdot m_{rate}}{M_0 - m_{rate}\cdot t}\cdot dt$

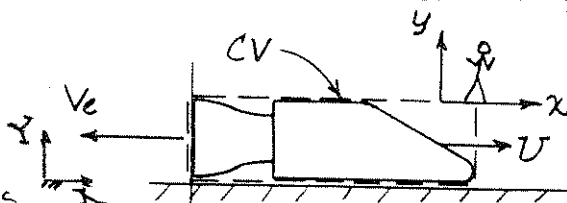
Integrating from $U = U_0$ at $t = 0$ to $U = U$ at $t = t$

$$U - U_0 = -V_e\cdot\left(\ln\left(M_0 - m_{rate}\cdot t\right) - \ln\left(M_0\right)\right) = -V_e\cdot\ln\left(1 - \frac{m_{rate}\cdot t}{M_0}\right)$$

$$U = U_0 - V_e\cdot\ln\left(1 - \frac{m_{rate}\cdot t}{M_0}\right)$$

Rearranging $$\text{MassFractionConsumed} = \frac{m_{rate}\cdot t}{M_0} = 1 - e^{-\frac{\left(U - U_0\right)}{V_e}} = 1 - e^{-\frac{(3500-600)}{6000}} = 0.383$$

Hence 38.3% of the mass must be fuel to accomplish the task. In reality, a much higher percentage would be needed due to drag effects

Given: Rocket sled moving on level track without resistance
Initial mass, $M_0 = 3000$ kg (includes $M_{fuel} = 1000$ kg)
$V_e = 2500$ m/s ; $p_e = p_{atm}$
Fuel consumption, $\dot{m} = 75$ kg/s

Find: Acceleration and speed of sled at ① $t = 10$ s.

Plot: sled speed and acceleration as functions of time.

Solution:

Apply x component of momentum to linearly accelerating CV; use continuity to find M(t)

Basic equations:
$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, dV + \int_{CS} \rho \vec{V}_{xyz} \cdot d\vec{A}$$
$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf}\, \rho\, dV = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\, \rho\, dV + \int_{CS} u_{xyz}\, (\rho \vec{V}_{xyz} \cdot d\vec{A})$$
($F_{S_x} = 0$ (1), $F_{B_x} = 0$ (2), $\frac{\partial}{\partial t} \approx 0$ (3))

Assumptions: (1) $F_{S_x} = 0$, no resistance (given)
(2) $F_{B_x} = 0$, horizontal
(3) neglect $\partial/\partial t$ inside CV
(4) uniform flow at nozzle exit
(5) $p_e = p_{atm}$ (given)

From continuity, $0 = \frac{\partial M}{\partial t} + \{+|\dot{m}|\} = \frac{dM}{dt} + \dot{m}$ or $dM = -\dot{m}\,dt$

Integrating, $\int_{M_0}^{M} dM = M - M_0 = \int_0^t -\dot{m}\,dt = -\dot{m}t$ or $M = M_0 - \dot{m}t$

From the momentum equation
$$-a_{rf_x} M = -a_{rf_x}(M_0 - \dot{m}t) = u_1\{+|\dot{m}|\} = -V_e \dot{m} \qquad \{u_1 = -V_e\}$$

Thus
$$a_{rf_x} = \frac{dU}{dt} = \frac{V_e \dot{m}}{(M_0 - \dot{m}t)} \qquad (1)$$

At $t = 10$ s
$$\frac{dU}{dt} = 2500\frac{m}{s} \times 75\frac{kg}{s} \times \frac{1}{3000\,kg - 75\frac{kg}{s} \times 10\,s} = 83.3\ m/s^2 \quad \leftarrow a_{rf_x}$$

From Eq. 1, $dU = V_e \frac{\dot{m}\,dt}{M_0 - \dot{m}t}$

Integrating from $U = 0$ at $t = 0$ to U at t gives
$$U = -V_e \ln(M_0 - \dot{m}t)\Big]_0^t = -V_e \ln\frac{(M_0 - \dot{m}t)}{M_0}$$
$$U = V_e \ln\frac{M_0}{(M_0 - \dot{m}t)} \qquad (2)$$

At $t = 10\text{ s}$

$$U = 2500\frac{m}{s}\ln\frac{3000\text{ kg}}{3000\text{ kg} - 75\frac{\text{kg}}{\text{s}}\times 10\text{ s}} = 719\text{ m/s} \qquad U.$$

Note that all fuel would be expended at $t_{bo} = \frac{M_f}{\dot{m}} = 1000\text{ kg}\cdot\frac{s}{75\text{ kg}}$

i.e. at $t_{b.o} = 13.3\text{ s}$.

The sled speed as a function of time is then

$$U = V_e \ln\frac{M_0}{(M_0 - \dot{m}t)} \quad \text{for } t \le 13.3\text{ s}$$

$$U = U_{max} = 1010\text{ m/s} \quad \text{for } t \ge 13.3\text{ s}$$

The sled acceleration is given by

$$\frac{dU}{dt} = \frac{\dot{m}V_e}{(M_0 - \dot{m}t)} \quad \text{for } 0 \le t \le 13.3\text{ s}$$

$$\frac{dU}{dt} = 0 \quad \text{for } t \ge 13.3\text{ s}$$

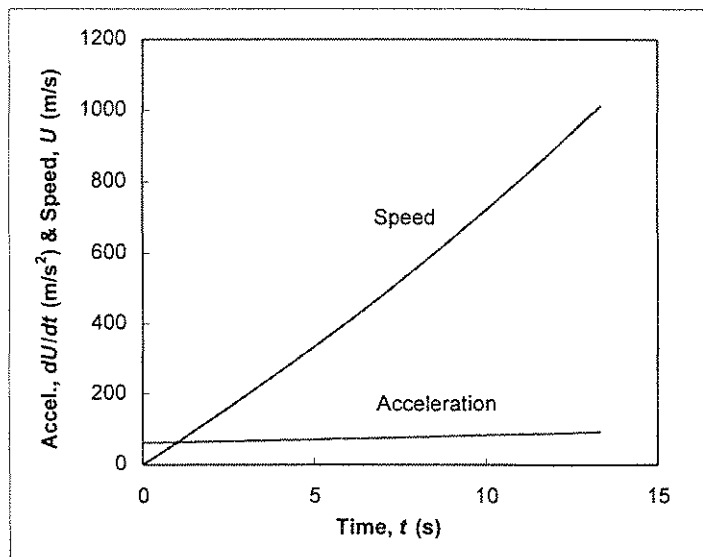
Acceleration and Speed vs. Time for Rocket Sled:

Input Data:

M_0 =	3000	kg
m(dot) =	75	kg/s
V_e =	2500	m/s

Calculated Results:

Time, t (s)	Acceleration, dU/dt (m/s^2)	Speed, U (m/s)
0	62.5	0
1	64.1	63.3
2	65.8	128
3	67.6	195
4	69.4	263
5	71.4	334
6	73.5	406
7	75.8	481
8	78.1	558
9	80.6	637
10	83.3	719
11	86.2	804
12	89.3	892
13	92.6	983
13.33	93.8	1014

Problem 4.162

Given: Rocket-propelled motorcycle, to jump, standing start, level.
Speed needed $U_j = 875$ km/hr Rocket exhaust speed $V_e = 2510$ m/s
Total mass $M_B = 375$ kg (without fuel)

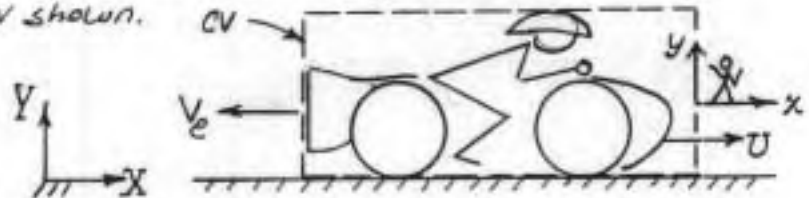
Find: Minimum fuel mass needed to reach V_j.

Solution: Apply x-component of momentum equation to linearly accelerating CV shown.

From continuity,

$$M_{cv} = M_0 - \dot{m}t$$

Basic equation:

$$\overset{\approx 0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x} \rho \, d\forall = \overset{\approx 0(3)}{\frac{\partial}{\partial t}} \int_{CV} u_{xyz} \rho \, d\forall + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) Neglect air and rolling resistance
(2) Level track, so $F_{B_x} = 0$
(3) Neglect unsteady effects within CV
(4) Uniform flow at nozzle exit plane
(5) $p_e = p_{atm}$

Then

$$-a_{rf_x} M_{cv} = u_e \{+\dot{m}\} = -V_e \dot{m} \quad \text{or} \quad \frac{dU}{dt} = \frac{V_e \dot{m}}{M_{cv}} = \frac{V_e \dot{m}}{M_0 - \dot{m}t}$$

$$u_e = -V_e$$

Separating variables and integrating,

$$dU = -V_e \left(\frac{-\dot{m}dt}{M_0 - \dot{m}t} \right) \quad \text{or} \quad U_j = -V_e \ln(M_0 - \dot{m}t)\Big|_0^t = V_e \ln\left(\frac{M_0}{M_0 - \dot{m}t}\right)$$

But $M_0 = M_B + M_F$ and $M_F = \dot{m}t$, so

$$\frac{U_j}{V_e} = \ln\left(\frac{M_B + M_F}{M_B}\right) = \ln\left(1 + \frac{M_F}{M_B}\right) \; ; \; 1 + \frac{M_F}{M_B} = e^{U_j/V_e} \; ; \; \frac{M_F}{M_B} = e^{U_j/V_e} - 1$$

Finally, $M_F = M_B\left(e^{U_j/V_e} - 1\right)$

$$M_F = 375 \text{ kg} \times \exp\left[875 \frac{\text{km}}{\text{hr}} \times \frac{\text{s}}{2510 \text{ m}} \times 1000 \frac{\text{m}}{\text{km}} \times \frac{\text{hr}}{3600 \text{ s}} - 1\right]$$

$$M_F = 38.1 \text{ kg}$$

M_F

The fuel mass required is about 10 percent of the mass of the motorcycle and rider.

Given: Home made rocket launched vertically from rest.

$M_0 = 20$ lbm, of which 15 lbm is fuel

$\dot{m} = 0.5$ lbm/s

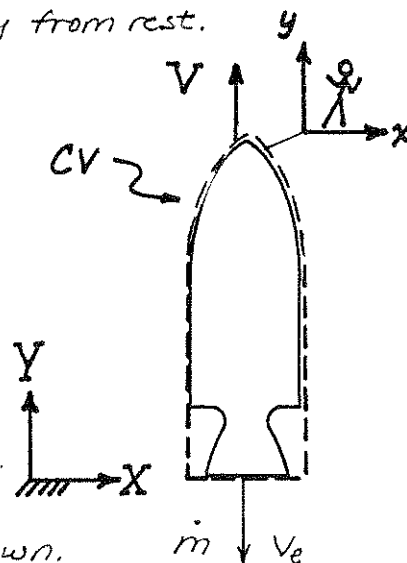
$V_e = 6500$ ft/s (relative to rocket)

$p_e = p_{atm}$

Neglect aerodynamic drag.

Find: (a) Speed at $t = 20$ s. (b) Height at $t = 20$ s.

Plot: Speed and height as functions of time.

Solution: Apply y-component of momentum equation to accelerating CV using CS shown.

Basic equation:

$$\cancelto{=0(1)}{F_{S_y}} + F_{B_y} - \int_{CV} a_{rf_y} \rho \, d\forall = \cancelto{\approx 0(2)}{\frac{\partial}{\partial t}} \int_{CV} v_{xyz} \rho \, d\forall + \int_{CS} v_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) Neglect air resistance; $p_e = p_{atm}$ (given)
(2) Neglect v_{xyz} and $\partial/\partial t$ within CV
(3) Uniform flow at nozzle exit section

Then

$$F_{B_y} - a_{rf_y} M = -Mg - M a_{rf_y} = v_e \{+\dot{m}\} = -V_e \dot{m}$$

and $\qquad v_e = -V_e$

$$a_{rf_y} = \frac{dV}{dt} = \frac{V_e \dot{m}}{M} - g$$

Introducing $M = M_0 - \dot{m}t$ and separating variables,

$$dV = \left(\frac{V_e \dot{m}}{M_0 - \dot{m}t} - g\right) dt$$

Integrating from rest at $t = 0$

$$V = \int_0^t \left(\frac{V_e \dot{m}}{M_0 - \dot{m}t} - g\right) dt = -V_e \ln(M_0 - \dot{m}t)\Big]_0^t - gt$$

or

$$V = V_e \ln\left(\frac{M_0}{M_0 - \dot{m}t}\right) - gt \qquad (1)$$

At $t = 20$ sec,

$$V = 6500 \frac{\text{ft}}{\text{s}} \ln\left(\frac{20 \text{ lbm}}{20 \text{ lbm} - 0.5 \frac{\text{lbm}}{\text{s}} \times 20 \text{ s}}\right) - 32.2 \frac{\text{ft}}{\text{s}^2} \times 20 \text{ s}$$

$$V(20 \text{ s}) = 3{,}860 \text{ ft/s} \qquad V_{20}$$

To find height, note $V = \frac{dY}{dt}$. Substitute into Eq. 1 to obtain

$$\frac{dY}{dt} = V_e \ln\left(\frac{M_0}{M_0 - \dot{m}t}\right) - gt = -V_e \ln\left(1 - \frac{\dot{m}t}{M_0}\right) - gt$$

Let $r = 1 - \frac{\dot{m}t}{M_0}$, and $dr = -\frac{\dot{m}}{M_0}dt$, then

$$dY = -V_e \ln r\, dt - gt\, dt = +\frac{V_e M_0}{\dot{m}} \ln r\, dr - gt\, dt$$

Integrating from $Y=0$ at $t=0$,

$$Y = \int_0^t \frac{V_e M_0}{\dot{m}} \ln r\, dr - \frac{1}{2}gt^2 = \frac{V_e M_0}{\dot{m}}\Big[r \ln r - r\Big]_0^t - \frac{1}{2}gt^2$$

$$= \frac{V_e M_0}{\dot{m}}\left\{\left(1 - \frac{\dot{m}t}{M_0}\right)\left[\ln\left(1 - \frac{\dot{m}t}{M_0}\right) - 1\right]\right\}\Bigg|_0^t - \frac{1}{2}gt^2$$

$$Y = \frac{V_e M_0}{\dot{m}}\left\{\left(1 - \frac{\dot{m}t}{M_0}\right)\left[\ln\left(1 - \frac{\dot{m}t}{M_0}\right) - 1\right] + 1\right\} - \frac{1}{2}gt^2$$

At $t = 20$ s,

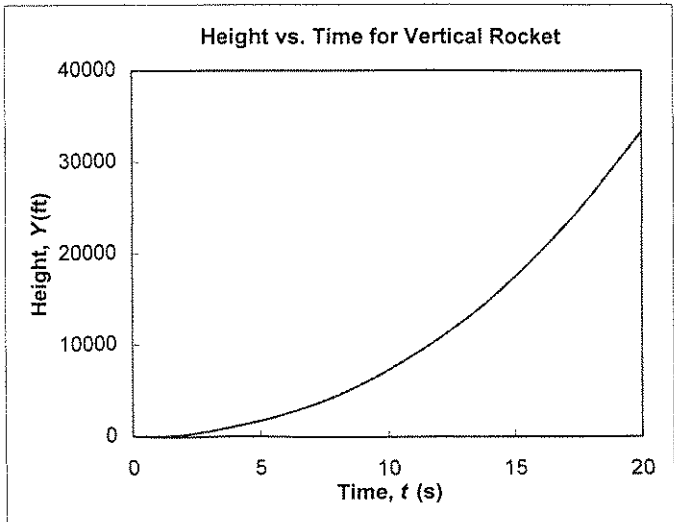
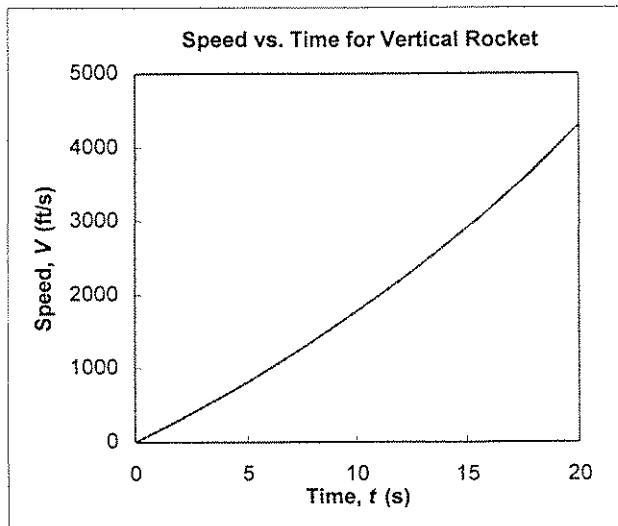
$$1 - \frac{\dot{m}t}{M_0} = 1 - 0.5\,\frac{\text{lbm}}{\text{s}} \times 20\,\text{s} \times \frac{1}{20\,\text{lbm}} = \frac{1}{2}$$

so

$$Y = 6500\,\frac{\text{ft}}{\text{s}} \times 20\,\text{lbm} \times \frac{\text{s}}{0.5\,\text{lbm}}\left\{\left(\frac{1}{2}\right)\left[\ln\left(\frac{1}{2}\right) - 1\right] + 1\right\} - \frac{1}{2} \times 32.2\,\frac{\text{ft}}{\text{s}^2}\,(20)^2\,\text{s}^2$$

$$Y = 33{,}500\ \text{ft}$$

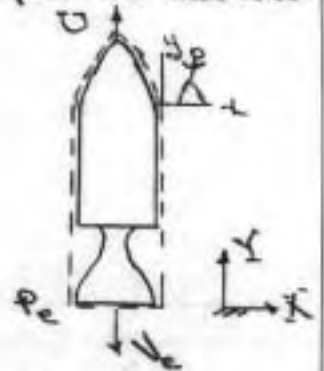
Y

Problem 4.164

Given: Liquid-fueled rocket launched from pad at sea level

$M_0 = 30{,}000$ kg $\quad \dot{m} = 2450$ kg/s

$V_e = 2270$ m/s $\quad p_e = 166$ kPa (abs)

Exit plane diameter, $D_e = 2.6$ m

U, x_e, y_e, p_e, V_e, X, Y

Find: acceleration at lift-off
expression for rocket speed, $U(t)$

Solution: Apply y component of momentum equation to CV with linear acceleration

Basic equation: $F_{S_y} + F_{B_y} - \int_{CV} a_{rf_y} \rho \, dV = \frac{\partial}{\partial t}\int_{CV} v_{xyz} \rho \, dV + \int_{CS} v_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$ (term ≈ 0 (2))

Assumptions: (1) F_{S_y} due to pressure, p_{atm} assumed constant, neglect air resistance
(2) neglect rate of change of momentum inside CV
(3) uniform flow at exit

Then, $\quad (-p_e - p_{atm})A_e - Mg - a_{rf_y} M = v_e\{+\dot{m}\} = -\dot{m}V_e$

Solving for a_{rf_y},

$$a_{rf_y} = \frac{dU}{dt} = \frac{1}{M}\left[\dot{m}V_e + (p_e - p_{atm})A_e\right] - g \quad \ldots(1)$$

$M = M(t)$. From conservation of mass $\frac{\partial}{\partial t}\int_{CV} \rho \, dV + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$

then $\frac{\partial}{\partial t}\int_{CV} \rho \, dV = \frac{dM}{dt} = -\int_{CS} \rho \vec{V} \cdot d\vec{A} = -\dot{m}_e$ (constant)

Hence $M(t) = M_0 - \dot{m}t$, and

$$a_{rf_y} = \frac{dU}{dt} = \frac{\dot{m}V_e}{M_0 - \dot{m}t} + \frac{(p_e - p_{atm})A_e}{M_0 - \dot{m}t} - g$$

$$U = \int_0^U dU = \int_0^t \frac{\dot{m}V_e}{M_0 - \dot{m}t}\,dt + \int_0^t \frac{(p_e - p_{atm})A_e}{M_0 - \dot{m}t}\,dt - \int_0^t g\,dt$$

$$U = -V_e \ln\left[\frac{M_0 - \dot{m}t}{M_0}\right] - \frac{(p_e - p_{atm})A_e}{\dot{m}} \ln\left[\frac{M_0 - \dot{m}t}{M_0}\right] - gt$$

$$U = -\left[V_e + \frac{(p_e - p_{atm})A_e}{\dot{m}}\right] \ln\left[\frac{M_0 - \dot{m}t}{M_0}\right] - gt \quad \longleftarrow \quad U(t)$$

At lift-off, $t = 0$, $M = M_0$

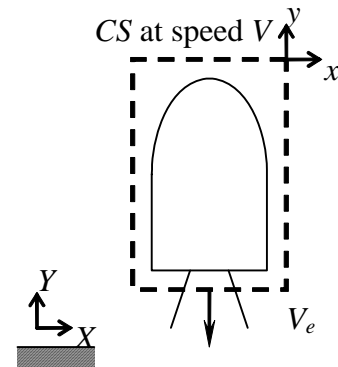
$$a_{rf_y} = \frac{1}{M}\left[\dot{m}V_e + (p_e - p_{atm})A_e\right] - g$$

$$= \frac{1}{3\times10^4\,\text{kg}}\left[2450\frac{\text{kg}}{\text{s}} \times 2270\frac{\text{m}}{\text{s}} + (166 - 101)10^3\frac{\text{N}}{\text{m}^2} \times \frac{\pi}{4}(2.6)^2\text{m}^2 \times \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right] - 9.81\frac{\text{m}}{\text{s}^2}$$

$$a_{rf_y} = 169 \text{ m/s}^2 \quad \longleftarrow \quad a_{rf_y}$$

Problem 4.165

[3]

4.165 Neglecting air resistance, what speed would a vertically directed rocket attain in 8 s if it starts from rest, has initial mass of 300 kg, burns 8 kg/s, and ejects gas at atmospheric pressure with a speed of 3000 m/s relative to the rocket? Plot the rocket speed as a function of time.

Given: Data on rocket

Find: Speed after 8 s; Plot of speed versus time

Solution:

Basic equation: Momentum flux in y direction
$$F_{S_y} + F_{B_y} - \int_{CV} a_{rf_y}\,\rho\, dV = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\, dV + \int_{CS} v_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: 1) No resistance 2) $p_e = p_{atm}$ 3) Uniform flow 4) Use relative velocities 5) Constant mass flow rate

From continuity $\frac{dM}{dt} = m_{rate} = \text{constant}$ so $M = M_0 - m_{rate}\cdot t$ (Note: Software cannot render a dot!)

Hence from momentum $-M\cdot g - a_{rfy}\cdot M = u_e\cdot\left(\rho_e\cdot V_e\cdot A_e\right) = -V_e\cdot m_{rate}$

Hence
$$a_{rfy} = \frac{dV}{dt} = \frac{V_e\cdot m_{rate}}{M} - g = \frac{V_e\cdot m_{rate}}{M_0 - m_{rate}\cdot t} - g \qquad (1)$$

Separating variables
$$dV = \left(\frac{V_e\cdot m_{rate}}{M_0 - m_{rate}\cdot t} - g\right)\cdot dt$$

Integrating from V = at t = 0 to V = V at t = t

$$V = -V_e\cdot\left(\ln\left(M_0 - m_{rate}\cdot t\right) - \ln\left(M_0\right)\right) - g\cdot t = -V_e\cdot\ln\left(1 - \frac{m_{rate}\cdot t}{M_0}\right) - g\cdot t$$

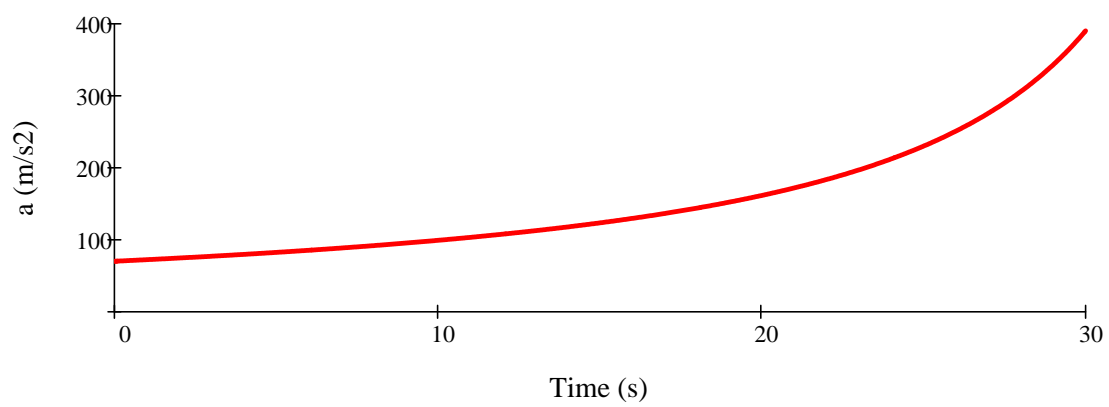
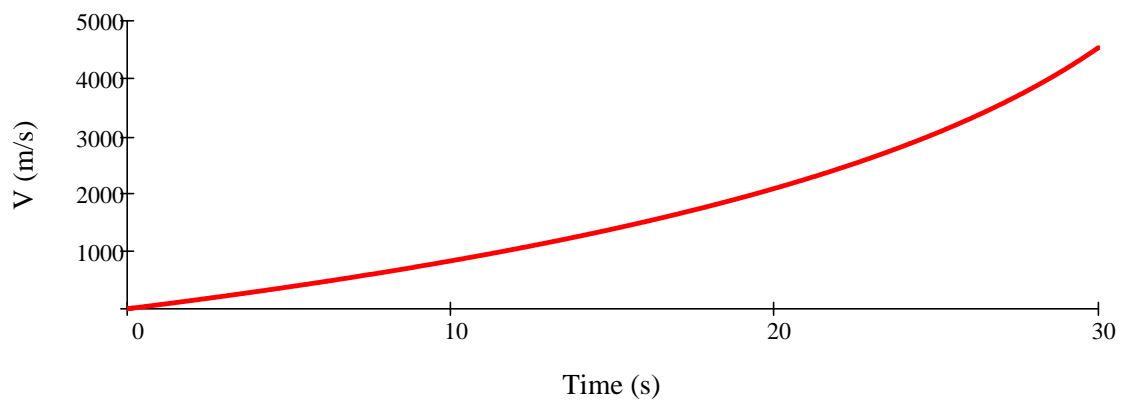
$$V = -V_e\cdot\ln\left(1 - \frac{m_{rate}\cdot t}{M_0}\right) - g\cdot t \qquad (2)$$

At t = 8 s
$$V = -3000\cdot\frac{m}{s}\cdot\ln\left(1 - 8\cdot\frac{kg}{s}\times\frac{1}{300\cdot kg}\times 8\cdot s\right) - 9.81\cdot\frac{m}{s^2}\times 8\cdot s \qquad V = 641\,\frac{m}{s}$$

The speed and acceleration as functions of time are plotted below. These are obtained from Eqs 2 and 1, respectively, and can be plotted in *Excel*

V (m/s)
5000
4000
3000
2000
1000
0
10
20
30
Time (s)

a (m/s2)
400
300
200
100
0
10
20
30
Time (s)

Problem 4.166

Open-Ended Problem Statement: Inflate a toy balloon with air and release it. Watch as the balloon darts about the room. Explain what causes the phenomena you see.

Discussion: Air blown into a balloon to inflate it must be compressed to overcome the skin's resistance to stretching. (Remember how hard it is to create enough pressure to "start" the inflation process!) After decreasing briefly, the required pressure seems to increase as inflation of the balloon continues.

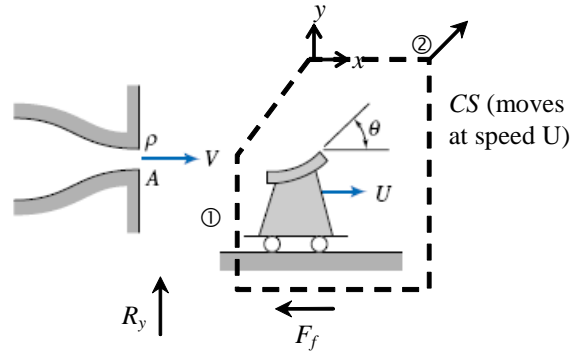
As the balloon is inflated, the skin stretches and stores energy. When the inflated balloon is released, the stored energy in the skin forces the compressed air out the open mouth of the balloon. The expansion of the compressed air to the lower surrounding atmospheric pressure creates a high-speed jet of air, which propels the relatively light balloon initially at a high speed.

The moving balloon is unstable because it has a poor aerodynamic shape. Therefore it darts about in a random pattern. The balloon keeps moving as long as it contains pressurized air to act as a propulsion jet. However, it is not long before the energy stored in the skin is exhausted and the air in the balloon is reduced to atmospheric pressure.

When the balloon reaches atmospheric pressure it is slowed by aerodynamic drag. Finally the empty, wrinkled balloon simply falls to the floor.

Some toys that use a balloon for propulsion are available. Most have stabilizing surfaces. It is instructive to study these toys carefully to understand how each works, and why each toy is shaped the way it is.

Problem 4.167

4.167 The vane/cart assembly of mass $M = 30$ kg, shown in Problem 4.123, is driven by a water jet. The water leaves the stationary nozzle of area $A = 0.02\ \text{m}^2$, with a speed of 20 m/s. The coefficient of kinetic friction between the assembly and the surface is 0.10. Plot the terminal speed of the assembly as a function of vane turning angle, θ, for $0 \le \theta \le \pi/2$. At what angle does the assembly begin to move if the coefficient of static friction is 0.15?

Given: Water jet striking moving vane

Find: Plot of terminal speed versus turning angle; angle to overcome static friction

Solution:

Basic equations: Momentum flux in x and y directions

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\,\rho\,dV = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,dV + \int_{CS} u_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

$$F_{S_y} + F_{B_y} - \int_{CV} a_{rf_y}\,\rho\,dV = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\,dV + \int_{CS} v_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: 1) Incompressible flow 2) Atmospheric pressure in jet 3) Uniform flow 4) Jet relative velocity is constant

Then $\quad -F_f - M\cdot a_{rfx} = u_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) + u_2\cdot\left(\rho\cdot V_2\cdot A_2\right) = -(V-U)\cdot[\rho\cdot(V-U)\cdot A] + (V-U)\cdot\cos(\theta)\cdot[\rho\cdot(V-U)\cdot A]$

$$a_{rfx} = \frac{\rho(V-U)^2\cdot A\cdot(1-\cos(\theta)) - F_f}{M} \qquad (1)$$

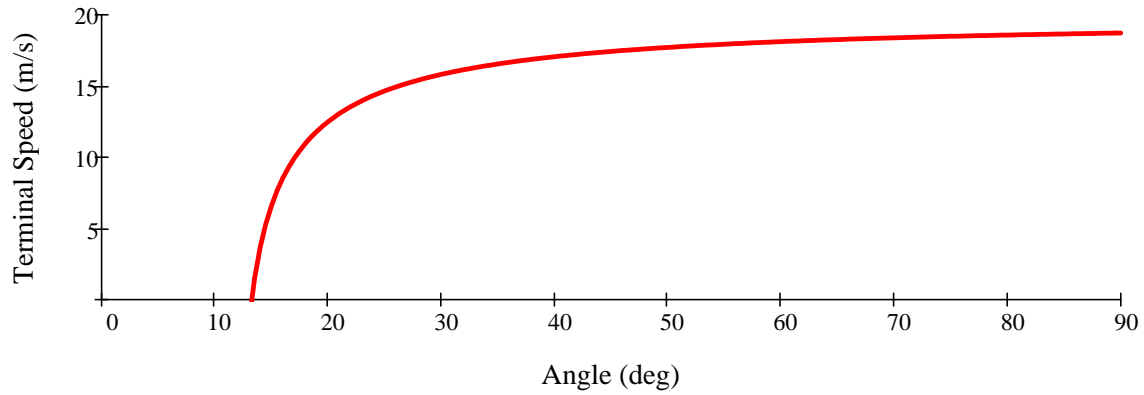
Also $\quad R_y - M\cdot g = v_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) + v_2\cdot\rho\cdot V_2\cdot A_2 = 0 + (V-U)\cdot\sin(\theta)\cdot[\rho\cdot(V-U)\cdot A]$

$$R_y = M\cdot g + \rho(V-U)^2\cdot A\cdot\sin(\theta)$$

At terminal speed $a_{rfx} = 0$ and $F_f = \mu_k R_y$. Hence in Eq 1

$$0 = \frac{\rho\cdot\left(V-U_t\right)^2\cdot A\cdot(1-\cos(\theta)) - \mu_k\cdot\left[M\cdot g + \rho\cdot\left(V-U_t\right)^2\cdot A\cdot\sin(\theta)\right]}{M} = \frac{\rho\cdot\left(V-U_t\right)^2\cdot A\cdot\left(1-\cos(\theta)-\mu_k\cdot\sin(\theta)\right)}{M} - \mu_k\cdot g$$

or $\quad V - U_t = \sqrt{\dfrac{\mu_k\cdot M\cdot g}{\rho\cdot A\cdot\left(1-\cos(\theta)-\mu_k\cdot\sin(\theta)\right)}} \qquad U_t = V - \sqrt{\dfrac{\mu_k\cdot M\cdot g}{\rho\cdot A\cdot\left(1-\cos(\theta)-\mu_k\cdot\sin(\theta)\right)}}$

The terminal speed as a function of angle is plotted below; it can be generated in *Excel*

For the static case $F_f = \mu_s \cdot R_y$ and $a_{rfx} = 0$ (the cart is about to move, but hasn't)

Substituting in Eq 1, with U = 0

$$0 = \frac{\rho \cdot V^2 \cdot A \cdot \left[1 - \cos(\theta) - \mu_s \cdot \left(\rho \cdot V^2 \cdot A \cdot \sin(\theta) + M \cdot g\right)\right.}{M}$$

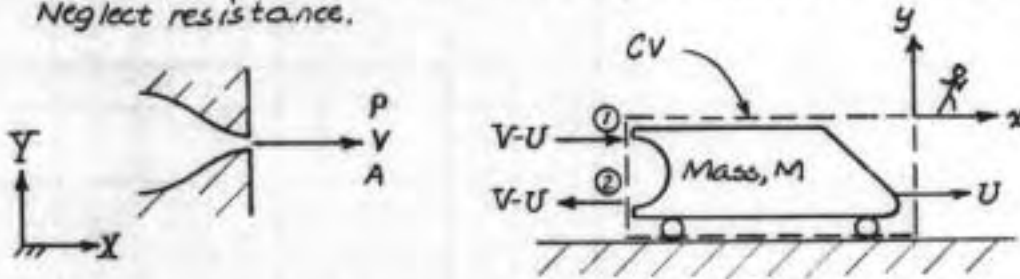
or

$$\cos(\theta) + \mu_s \cdot \sin(\theta) = 1 - \frac{\mu_s \cdot M \cdot g}{\rho \cdot V^2 \cdot A}$$

We need to solve this for θ! This can be done by hand or by using Excel's Goal Seek or Solver $\theta = 19\,\text{deg}$

Note that we need θ = 19º, but once started we can throttle back to about θ = 12.5º and still keep moving!

Problem 4.168

Given: Vehicle accelerated from rest by a hydraulic catapult. Neglect resistance.

Find: (a) Expression for acceleration at any time, t.
(b) Time required to reach $U = V/2$.

Solution: Apply x component of momentum equation using linearly accelerating CV shown above.

Basic equation: $$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x}\,\rho\, d\forall = \overset{\approx 0(3)}{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall} + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) $F_{S_x} = 0$
(2) $F_{B_x} = 0$
(3) Neglect mass of liquid and rate of change of u in CV
(4) Uniform flow at each section
(5) Jet area and speed with respect to vehicle are constant

Then

$$-Ma_{rf_x} = -M\frac{dU}{dt} = u_1\{-|\rho(V-U)A|\} + u_2\{|\rho(V-U)A|\}$$

$$u_1 = V-U \qquad u_2 = -(V-U)$$

or

$$a_{rf_x} = \frac{dU}{dt} = \frac{2\rho(V-U)^2A}{M} \quad ; \quad \frac{dU}{(V-U)^2} = \frac{2\rho A}{M}dt \quad ; \quad -\frac{d(V-U)}{(V-U)^2} = \frac{2\rho A}{M}dt$$

To obtain $a_{rf_x}(t)$, we must first find $U(t)$. Integrating from $U=0$ at $t=0$ to U at t,

$$\int_{V-U=V}^{V-U} -\frac{d(V-U)}{(V-U)^2} = \left.\frac{1}{V-U}\right]_V^{V-U} = \frac{1}{V-U} - \frac{1}{V} = \frac{V-(V-U)}{V(V-U)} = \frac{2\rho A}{M}t \quad ; \quad \frac{U}{V-U} = \frac{2\rho VA}{M}t$$

Solving,

$$U = (V-U)\frac{2\rho VA}{M}t \;, \quad U = V\frac{\frac{2\rho VA}{M}t}{1+\frac{2\rho VA}{M}t} \quad \text{and} \quad V-U = V\left[1 - \frac{\frac{2\rho VA}{M}t}{1+\frac{2\rho VA}{M}t}\right]$$

Substituting,

$$a_{rf_x} = \frac{2\rho V^2A}{M}\left[1 - \frac{\frac{2\rho VA}{M}t}{1+\frac{2\rho VA}{M}t}\right]^2 = \frac{2\rho V^2A}{M}\left[\frac{1}{1+\frac{2\rho VA}{M}t}\right]^2$$

$a_{rf_x}(t)$

The time to reach $U = V/2$ is

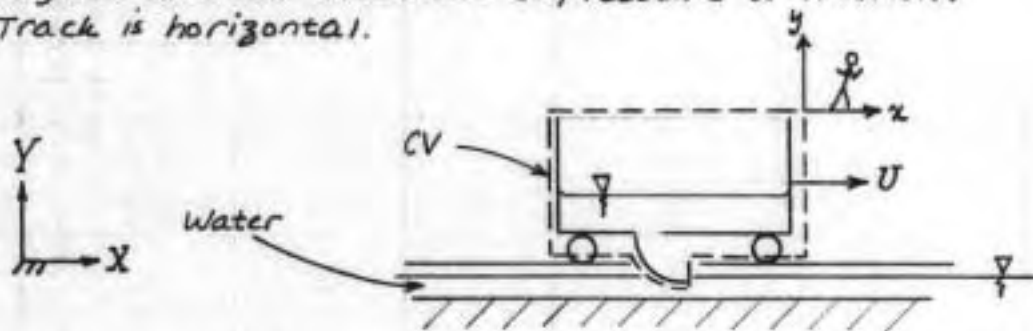
$$\frac{U}{V} = \frac{1}{2} = \frac{\frac{2\rho VA}{M}t}{1+\frac{2\rho VA}{M}t} \quad \text{or} \quad t = \frac{M}{2\rho VA}$$

$t(\frac{V}{2})$

Check: $\left[\frac{M}{\rho VA}\right] = M\frac{L^3}{M}\frac{t}{L}\frac{1}{L^2} = t$ ✓ ; $\left[\frac{\rho V^2 A}{M}\right] = \frac{M}{L^3}\frac{L^2}{t^2}L^2\frac{1}{M} = \frac{L}{t^2}$ ✓

Problem 4.169

[4]

Given: Moving tank slowed by lowering scoop into water trough.
Initial mass and speed are M_0 and U_0, respectively.
Neglect external forces due to pressure or friction.
Track is horizontal.

Find: (a) Apply continuity and momentum to show $U = U_0 M_0 / M$.
(b) Obtain a general expression for $U(t)$.

Solution: Apply continuity and momentum equations to linearly accelerating CV shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}_{xyz} \cdot d\vec{A}$

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x} \rho \, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho \, d\forall + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

($F_{S_x} = 0(1)$, $F_{B_x} = 0(2)$, $\frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho \, d\forall \approx 0(3)$)

Assumptions: (1) $F_{S_x} = 0$
(2) $F_{B_x} = 0$
(3) Neglect u within CV
(4) Uniform flow across inlet section

From continuity

$$0 = \frac{\partial}{\partial t} M_{cv} + \{-|\rho U A|\} \quad \text{or} \quad \frac{dM}{dt} = \rho U A$$

From momentum

$$-a_{rf_x} M = -\frac{dU}{dt} M = u\{-|\rho U A|\} = U \rho U A, \text{ since } u = -U$$

But from continuity, $\rho U A = \frac{dM}{dt}$, so

$$M \frac{dU}{dt} + U \frac{dM}{dt} = 0 \quad \text{or} \quad UM = \text{constant} = U_0 M_0 \; ; \; U = U_0 M_0 / M$$

U

Substituting $M = M_0 U_0 / U$ into momentum, $-\frac{dU}{dt}\frac{M_0 U_0}{U} = \rho U^2 A$, or

$$\frac{dU}{U^3} = -\frac{\rho A}{U_0 M_0} dt$$

Integrating, $\int_{U_0}^{U} \frac{dU}{U^3} = -\frac{1}{2}\frac{1}{U^2}\Big]_{U_0}^{U} = -\frac{1}{2}\left(\frac{1}{U^2} - \frac{1}{U_0^2}\right) = -\int_0^t \frac{\rho A}{U_0 M_0} dt = -\frac{\rho A}{U_0 M_0} t$

Solving for U,

$$U = \frac{U_0}{\left[1 + \frac{2\rho U_0 A}{M_0} t\right]^{\frac{1}{2}}}$$

U(t)

Problem 4.170

Given: Tank driven by jet along horizontal track. Neglect resistance. Acceleration is from rest. Initial mass is M_0. Track horizontal.

Find: (a) Apply continuity and momentum to show $M = M_0 V/(V-U)$

(b) General expression for U/V as a function of time.

Solution: Apply continuity and x component of momentum equation to linearly accelerating CV shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}_{xyz} \cdot d\vec{A}$

$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x} \rho \, d\forall = \frac{\partial}{\partial t}\int_{CV} \overset{\approx 0(3)}{u_{xyz}} \rho \, d\forall + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) $F_{S_x} = 0$

(2) $F_{B_x} = 0$

(3) Neglect u within CV

(4) Uniform flow in jet

From continuity

$$0 = \frac{\partial}{\partial t} M_{cv} + \{-|\rho(V-U)A|\} \quad \text{or} \quad \frac{dM}{dt} = \rho(V-U)A$$

From momentum

$$-a_{rf_x} M = -\frac{dU}{dt} M = u\{-|\rho(V-U)A|\} = (V-U)[-\rho(V-U)A]; \; u = V-U$$

But from continuity, $\rho(V-U)A = \frac{dM}{dt}$, and $dU = -d(V-U)$, so

$$-\frac{dU}{dt} M = \frac{d(V-U)}{dt} M = -(V-U)\frac{dM}{dt} \quad \text{or} \quad M(V-U) = \text{constant} = M_0 V$$

Thus $M = M_0 V/(V-U)$ ← M

Substituting into momentum, $-\frac{dU}{dt} M = \frac{d(V-U)}{dt} \frac{M_0 V}{(V-U)} = -\rho(V-U)^2 A$, or

$$\frac{d(V-U)}{(V-U)^3} = -\frac{\rho A}{V M_0} dt$$

Integrating, $\int_V^{V-U} \frac{d(V-U)}{(V-U)^3} = -\frac{1}{2}\left[\frac{1}{(V-U)^2} - \frac{1}{V^2}\right] = -\int_0^t \frac{\rho A}{V M_0} dt = -\frac{\rho A}{V M_0} t$

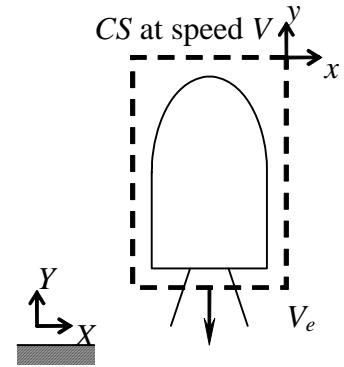
Solving,

$$\frac{U}{V} = \left\{1 - \frac{1}{\left[1 + \frac{2\rho V A}{M_0} t\right]^{1/2}}\right\}$$

← $\frac{U}{V}$

Problem 4.171

[5]

4.171 A model solid propellant rocket has a mass of 69.6 g, of which 12.5 g is fuel. The rocket produces 5.75 N of thrust for a duration of 1.7 s. For these conditions, calculate the maximum speed and height attainable in the absence of air resistance. Plot the rocket speed and the distance traveled as functions of time.

Given: Data on rocket

Find: Maximum speed and height; Plot of speed and distance versus time

Solution:

Basic equation: Momentum flux in y direction $F_{S_y} + F_{B_y} - \int_{CV} a_{rf_y}\,\rho\, dV = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\, dV + \int_{CS} v_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$

Assumptions: 1) No resistance 2) $p_e = p_{atm}$ 3) Uniform flow 4) Use relative velocities 5) Constant mass flow rate

From continuity $\frac{dM}{dt} = m_{rate} = \text{constant}$ so $M = M_0 - m_{rate}\cdot t$ (Note: Software cannot render a dot!)

Hence from momentum $-M\cdot g - a_{rfy}\cdot M = u_e\cdot(\rho_e\cdot V_e\cdot A_e) = -V_e\cdot m_{rate}$

Hence $a_{rfy} = \frac{dV}{dt} = \frac{V_e\cdot m_{rate}}{M} - g = \frac{V_e\cdot m_{rate}}{M_0 - m_{rate}\cdot t} - g$

Separating variables $dV = \left(\frac{V_e\cdot m_{rate}}{M_0 - m_{rate}\cdot t} - g\right)\cdot dt$

Integrating from V = at t = 0 to V = V at t = t

$$V = -V_e\cdot\left(\ln\left(M_0 - m_{rate}\cdot t\right) - \ln\left(M_0\right)\right) - g\cdot t = -V_e\cdot\ln\left(1 - \frac{m_{rate}\cdot t}{M_0}\right) - g\cdot t$$

$$V = -V_e\cdot\ln\left(1 - \frac{m_{rate}\cdot t}{M_0}\right) - g\cdot t \qquad \text{for} \qquad t \le t_b \quad \text{(burn time)} \qquad (1)$$

To evaluate at $t_b = 1.7$ s, we need V_e and m_{rate} $m_{rate} = \frac{m_f}{t_b}$ $m_{rate} = \frac{12.5\cdot gm}{1.7\cdot s}$ $m_{rate} = 7.35\times 10^{-3}\frac{kg}{s}$

Also note that the thrust F_t is due to momentum flux from the rocket $F_t = m_{rate}\cdot V_e$ $V_e = \frac{F_t}{m_{rate}}$ $V_e = \frac{5.75\cdot N}{7.35\times 10^{-3}\cdot\frac{kg}{s}}\times\frac{kg\cdot m}{s^2\cdot N}$ $V_e = 782\frac{m}{s}$

Hence $V_{max} = -V_e\cdot\ln\left(1 - \frac{m_{rate}\cdot t_b}{M_0}\right) - g\cdot t_b$

$$V_{max} = -782\cdot\frac{m}{s}\cdot\ln\left(1 - 7.35\times 10^{-3}\cdot\frac{kg}{s}\times\frac{1}{0.0696\cdot kg}\times 1.7\cdot s\right) - 9.81\cdot\frac{m}{s^2}\times 1.7\cdot s \qquad V_{max} = 138\frac{m}{s}$$

To obtain Y(t) we set V = dY/dt in Eq 1, and integrate to find

$$Y = \frac{V_e \cdot M_0}{m_{rate}} \cdot \left[\left(1 - \frac{m_{rate} \cdot t}{M_0}\right) \cdot \left(\ln\left(1 - \frac{m_{rate} \cdot t}{M_0}\right) - 1\right) + 1\right] - \frac{1}{2} \cdot g \cdot t^2 \qquad t \le t_b \qquad t_b = 1.7 \cdot s \qquad (2)$$

At $t = t_b$

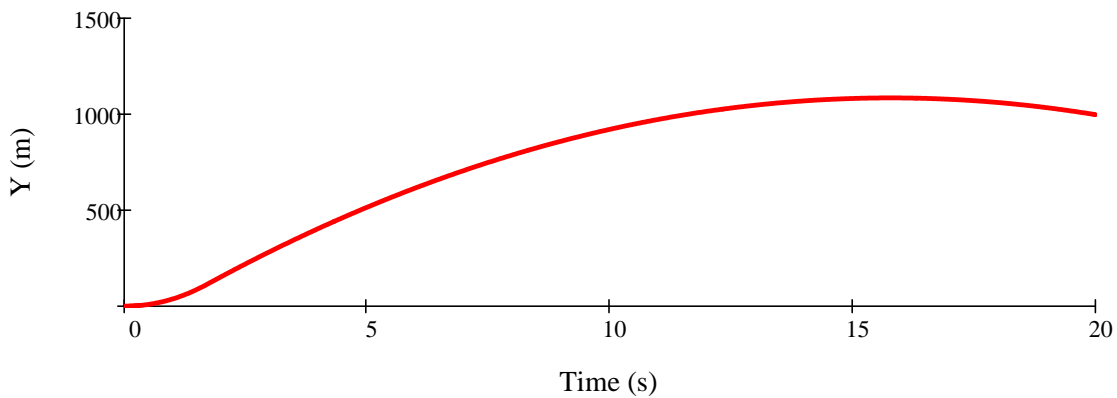
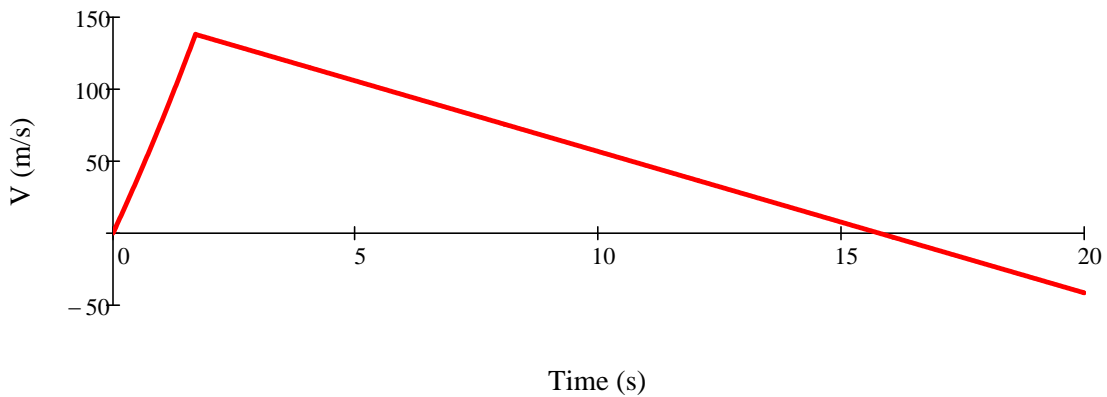
$$Y_b = 782 \cdot \frac{m}{s} \times 0.0696 \cdot kg \times \frac{s}{7.35 \times 10^{-3} \cdot kg} \cdot \left[\left(1 - \frac{0.00735 \cdot 1.7}{0.0696}\right)\left(\ln\left(1 - \frac{.00735 \cdot 1.7}{.0696}\right) - 1\right) + 1\right] \ldots$$
$$+ -\frac{1}{2} \times 9.81 \cdot \frac{m}{s^2} \times (1.7 \cdot s)^2$$

$$Y_b = 113\,m$$

After burnout the rocket is in free assent. Ignoring drag

$$V(t) = V_{max} - g \cdot (t - t_b) \qquad (3)$$

$$Y(t) = Y_b + V_{max} \cdot (t - t_b) - \frac{1}{2} \cdot g \cdot (t - t_b)^2 \qquad t > t_b \qquad (4)$$

The speed and position as functions of time are plotted below. These are obtained from Eqs 1 through 4, and can be plotted in *Excel*

Using Solver, or by differentiating y(t) and setting to zero, or by setting V(t) = 0, we find for the maximun $t = 15.8\,s$ $\qquad y_{max} = 1085\,m$

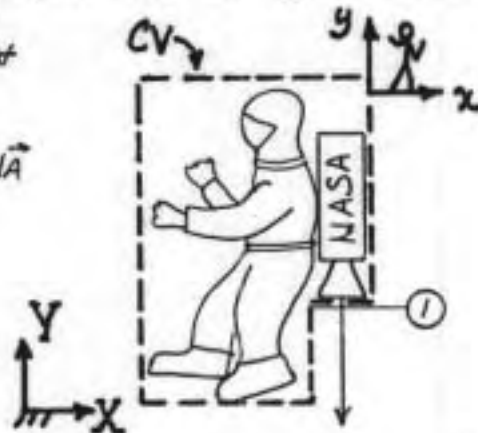
Given: Small rocket "Jet pack" used to lift astronaut above Earth. Exhaust jet speed is constant but mass flow rate varies.

Find: (a) Algebraic expression for mass flow rate needed to hover.
(b) Maximum hover time.

Solution: Apply continuity and momentum using CV & CS shown.

Basic equation: $F_{S_y} + F_{B_y} - \int_{CV} a_{rf_y} \rho \, d\forall = \frac{\partial}{\partial t}\int_{CV} v_{xyz} \rho \, d\forall + \int_{CS} v_{xyz} \rho \vec{V} \cdot d\vec{A}$

($F_{S_y} = 0$ (1), $a_{rf_y} = 0$ (2), $\frac{\partial}{\partial t} \approx 0$ (3))

Assumptions: (1) Hover; $F_{S_y} = 0$
(2) $a_{rf_y} = 0$
(3) Neglect $\partial/\partial t$ in CV
(4) Uniform flow exhaust

$V_e = 2940$ m/s
$M_0 = 130$ kg
$M_f = 40$ kg

Then

$$-Mg = v_1 \{+\dot{m}\}$$

$$v_1 = -V_e$$

$$-Mg = -V_e \dot{m}$$

so

$$\dot{m} = \frac{Mg}{V_e}$$

$\dot{m}(t)$

From conservation of mass, $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} = \frac{dM}{dt} + \dot{m}$

so $\frac{dM}{dt} = -\dot{m} = -\frac{Mg}{V_e}$ or $\frac{dM}{M} = -\frac{g}{V_e} dt$

Integrating from M_0 at $t = 0$ to $M_0 - M_f$ at t,

$$\int_{M_0}^{M_0 - M_f} \frac{dM}{M} = \ln M \Big]_{M_0}^{M_0 - M_f} = \ln\left(\frac{M_0 - M_f}{M_0}\right) = \ln\left(1 - \frac{M_f}{M_0}\right) = -\frac{gt}{V_e}$$

Solving for t,

$$t = -\frac{V_e}{g} \ln\left(1 - \frac{M_f}{M_0}\right) = -2940 \frac{m}{s} \times \frac{s^2}{9.81 \, m} \ln\left(1 - \frac{40 \, kg}{130 \, kg}\right)$$

$t = 110$ s (hover time)

t

Open-Ended Problem Statement: Several toy manufacturers sell water "rockets" that consist of plastic tanks to be partially filled with water and then pressurized with air. Upon release, the compressed air forces water out the nozzle rapidly, propelling the rocket. You are asked to help specify optimum conditions for this water-jet propulsion system. To simplify the analysis, consider horizontal motion only. Perform the analysis and design needed to define the acceleration performance of the compressed air/water-propelled rocket. Identify the fraction of tank volume that initially should be filled with compressed air to achieve optimum performance (i.e., maximum speed from the water charge). Describe the effect of varying the initial air pressure in the tank.

Discussion: The process may be modeled as a polytropic expansion of the trapped air which forces water out the jet nozzle, causing the "rocket" to accelerate. The polytropic exponent may be varied to model anything from an isothermal expansion process ($n = 1$) to an adiabatic expansion process ($n = k$), which is more likely to be an accurate model for the sudden expansion of the air.

Speed of the water jet leaving the "rocket" is proportional to the square root of the pressure difference between the tank and atmosphere.

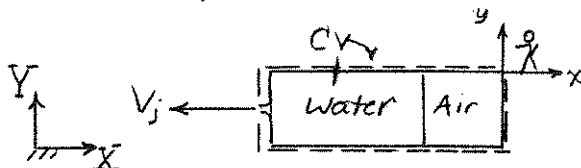
Qualitatively it is apparent that the smaller the initial volume fraction of trapped air, the larger will be the expansion ratio of the air, and the more rapid will be the pressure reduction as the air expands. This will cause the water jet speed to drop rapidly. The combination of low water jet speed and relatively large mass of water will produce sluggish acceleration.

Increasing the initial volume fraction of air will reduce the expansion ratio, so higher pressure will be maintained longer in the tank and the water jet will maintain higher speed longer. This combined with the relatively small mass of water in the tank will produce rapid acceleration.

If the initial volume fraction of air is too large, all water will be expended before the air pressure is reduced significantly. In this situation, some of the stored energy of the air will be dissipated in a relatively ineffective air jet. Consequently, for any initial pressure in the tank, there is an optimum initial air fraction.

This problem cannot be solved in closed form because of the varying air pressure, mass flow rate, and mass of water in the tank;. it can only be solved numerically. One possible integration scheme is to increment time and solve for all properties of the system at each instant. The drawback to this scheme is that the water is unlikely to be exhausted at an even increment of time. A second scheme is to increment the volume of water remaining and solve for properties using the average flow rate during the interval. This scheme is outlined below.

Model the air/water jet-propelled "rocket" using the CV and coordinates shown.

First choose dimensions and mass of "rocket" to be simulated:

Input Data:

Jet diameter:	$D_j =$	0.003	m
Tank diameter:	$D_t =$	0.035	m
Tank length:	$L =$	0.1	m
Tank mass:	$M_t =$	0.01	kg
Polytropic exponent:	$n =$	1.4	---

Next choose initial conditions for the simulation (see sample calculations below):

Problem *4.173

Initial Conditions:

Air fraction in tank:	$\alpha =$	0.5	---
Tank pressure:	$p_0 =$	200	kPa (gage)
Volume increment:	$\Delta\alpha =$	0.02	---

Compute reference parameters:

Calculated Parameters:

Jet area:	$A_j =$	7.07E-06	m^2
Tank volume:	$V_t =$	9.62E-05	m^3
Initial air volume:	$V_0 =$	4.81E-05	m^3
Initial water mass:	$M_0 =$	0.0481	kg

(These are used in the spreadsheet below.)

Then decrease the water fraction in the tank by $\Delta\alpha$:

Calculated Results:

Water Fraction, V_w/V_t (---)	Gage Pressure, p (kPa)	Water Mass, M_w (kg)	Jet Speed, V_j (m/s)	Flow Rate, dm/dt (kg/s)	Time Interval, Δt (s)	Current Time, t (s)	"Rocket" Accel., a (m/s²)	"Rocket" Speed, U (m/s)
0.50	200	0.0481	20.0	0.141	0	0	48.7	0
0.48	184	0.0461	19.2	0.135	0.0139	0.0139	47.5	0.668

The computation is made as follows:

(1) Decrease α by $\Delta\alpha$

(2) Compute p from $p = p_0\left(\frac{V_0}{V}\right)^n$

$$p = (200+101.325)\,kPa\left(\frac{0.50}{0.52}\right)^{1.4} - 101.325 = 183.9 \text{ kPa (gage)}$$

(3) Use Bernoulli to calculate jet speed

$$V_j = \sqrt{\frac{2\Delta p}{\rho}} = \left[2 \times 183.9\times10^3 \frac{N}{m^2} \times \frac{m^3}{999\,kg} \times \frac{kg\cdot m}{N\cdot s^2}\right]^{\frac{1}{2}} = 19.10 \text{ m/s}^*$$

(4) Calculate water mass using α.

(5) Use conservation of mass to compute mass flow rate

$$\dot{m} = \rho V_j A_j = 999\frac{kg}{m^3} \times 19.10\frac{m}{s} \times 7.07\times10^{-6} m^2 = 0.1349 \text{ kg/s}$$

(6) Use the average mass flow rate during the interval to approximate Δt:

$$\Delta t = \frac{\Delta m}{dm/dt} = \frac{\Delta m}{\bar{\dot{m}}} = (0.0481-0.0461)\,kg \times \frac{s}{0.138\,kg} = 0.01449 \text{ s}^*$$

(7) Use momentum to compute acceleration (note $M = M_w + M_t$):

$$a_{rf_x} = \frac{\dot{m}V_j}{M} = 0.135\frac{kg}{s} \times 19.2\frac{m}{s} \times \frac{1}{0.0461+0.0100\,kg} = 46.2 \text{ m/s}^{2*}$$

(8) Finally: use average acceleration to get speed

$$U = U_0 + \bar{a}\Delta t = 0 + 48.1\frac{m}{s^2} \times 0.0139\,s = 0.669 \text{ m/s}^*$$

---

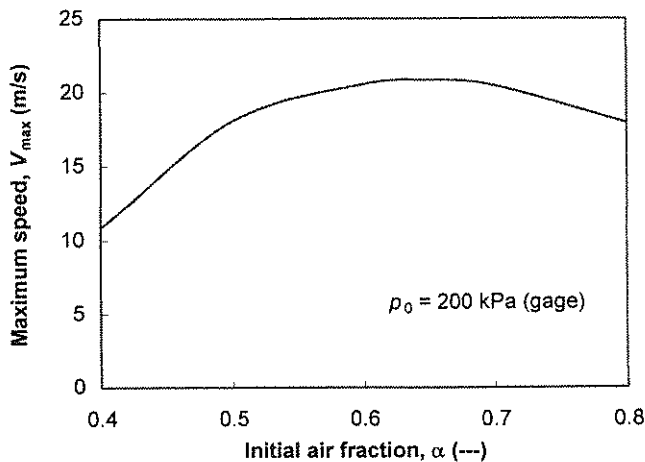
* Note effect of roundoff error.

Problem *4.173

Repeat these calculations until water is depleted or air pressure falls to zero, as shown below:

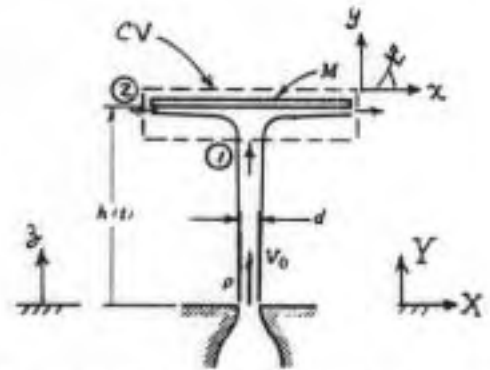
Water Fraction, V_w/V_t (---)	Gage Pressure, p (kPa)	Water Mass, M_w (kg)	Jet Speed, V_j (m/s)	Flow Rate, dm/dt (kg/s)	Time Interval, Δt (s)	Current Time, t (s)	"Rocket" Accel., a (m/s²)	"Rocket" Speed, U (m/s)
0.50	200	0.0481	20.0	0.141	0	0	48.7	0
0.48	184	0.0461	19.2	0.135	0.0139	0.0139	47.5	0.668
0.46	169	0.0442	18.4	0.130	0.0145	0.0284	45.2	1.34
0.44	156	0.0423	17.7	0.125	0.0151	0.0435	43.1	2.01
0.42	143	0.0404	16.9	0.120	0.0157	0.0592	41.2	2.67
0.40	132	0.0384	16.3	0.115	0.0164	0.0756	39.4	3.33
0.38	122	0.0365	15.6	0.110	0.0171	0.0927	37.8	3.99
0.36	112	0.0346	15.0	0.106	0.0178	0.110	36.2	4.65
0.34	103	0.0327	14.4	0.101	0.0186	0.129	34.8	5.31
0.32	94.6	0.0308	13.8	0.0972	0.0194	0.148	33.5	5.97
0.30	86.8	0.0288	13.2	0.0931	0.0202	0.169	32.2	6.63
0.28	79.5	0.0269	12.6	0.0891	0.0211	0.190	31.0	7.30
0.26	72.7	0.0250	12.1	0.0852	0.0221	0.212	29.9	7.97
0.24	66.3	0.0231	11.5	0.0814	0.0231	0.235	28.9	8.65
0.22	60.4	0.0211	11.0	0.0776	0.0242	0.259	27.9	9.34
0.20	54.7	0.0192	10.5	0.0739	0.0254	0.284	26.9	10.0
0.18	49.4	0.0173	9.95	0.0702	0.0267	0.311	26.0	10.7
0.16	44.4	0.0154	9.43	0.0666	0.0281	0.339	25.2	11.5
0.14	39.7	0.0135	8.92	0.0630	0.0297	0.369	24.3	12.2
0.12	35.2	0.0115	8.40	0.0593	0.0314	0.400	23.5	12.9
0.10	31.0	0.00961	7.88	0.0556	0.0334	0.434	22.7	13.7
0.08	27.0	0.00769	7.35	0.0519	0.0357	0.469	22.0	14.5
0.06	23.2	0.00577	6.81	0.0481	0.0384	0.508	21.2	15.3
0.04	19.6	0.00384	6.26	0.0442	0.0416	0.550	20.4	16.2
0.02	16.1	0.00192	5.68	0.0401	0.0456	0.595	19.5	17.1
0.00	12.9	0.0000	5.07	0.0358	0.0506	0.646	18.6	18.1

In this simulation, the water is depleted when $t \approx 0.65$ s; $V_{max} = 18.1$ m/s.

Varying the initial air fraction produces the following:

For this combination of parameters, a peak speed of about 20.8 m/s is attained with an initial air fraction of about 0.66.

Given: Vertical jet impinging on disk.

Disk is unconstrained vertically.

Find: (a) Differential equation for $h(t)$, if disk released from $H > h_0$, where h_0 is equilibrium height.
(b) Sketch $h(t)$ and explain.

Solution: Apply Bernoulli equation to jet, then y momentum equation to CV with linear acceleration.

Basic equations:

$$\frac{p_0}{\rho} + \frac{V_0^2}{2} + g z_0 = \frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 \qquad (p_0/\rho,\ p_1/\rho \text{ cancel};\ z_0 = 0)$$

$$F_{S_y} + F_{B_y} - \int_{CV} a_{rf_y}\,\rho\, dV = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\, dV + \int_{CS} v_{xyz}\,\rho\, \vec{V}_{xyz}\cdot d\vec{A} \qquad (F_{S_y} = 0\ (6);\ \tfrac{\partial}{\partial t} \approx 0\ (7))$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) No friction
(4) Flow along a streamline } in jet
(5) $p_1 = p_0 = p_{atm}$

(6) No pressure force on CV, so $F_{S_y} = 0$
(7) Neglect mass of liquid in CV and $v \approx 0$ in CV
(8) Uniform flow at each section
(9) Measure velocities relative to CV

From momentum

$$-(M + m_{w})g - a_{rf_y}(M + m_{w}) = v_1\{-|\rho(V_1 - U)A_1|\} + v_2\{\dot{m}_2\} \qquad (m_w \approx 0\ (7))$$

$$v_1 = V_1 - U \qquad v_2 \approx 0$$

With $a_{rf_y} = \frac{d^2h}{dt^2}$, $U = \frac{dh}{dt}$, then

$$-Mg - M\frac{d^2h}{dt^2} = -\rho\left(V_1 - \frac{dh}{dt}\right)^2 A_1$$

But from Bernoulli, $\frac{V_1^2}{2} = \frac{V_0^2}{2} - g z_1$, so $V_1 = \sqrt{V_0^2 - 2gh}$, since $z_1 = h(t)$

Also from continuity, $V_1A_1 = V_0A_0$, so $A_1 = A_0V_0/V_1$. Substituting

$$\frac{d^2h}{dt^2} = \rho\left(\sqrt{V_0^2 - 2gh} - \frac{dh}{dt}\right)^2 \frac{A_0 V_0}{M\sqrt{V_0^2 - 2gh}} - g$$

$\ddot{h}(t)$

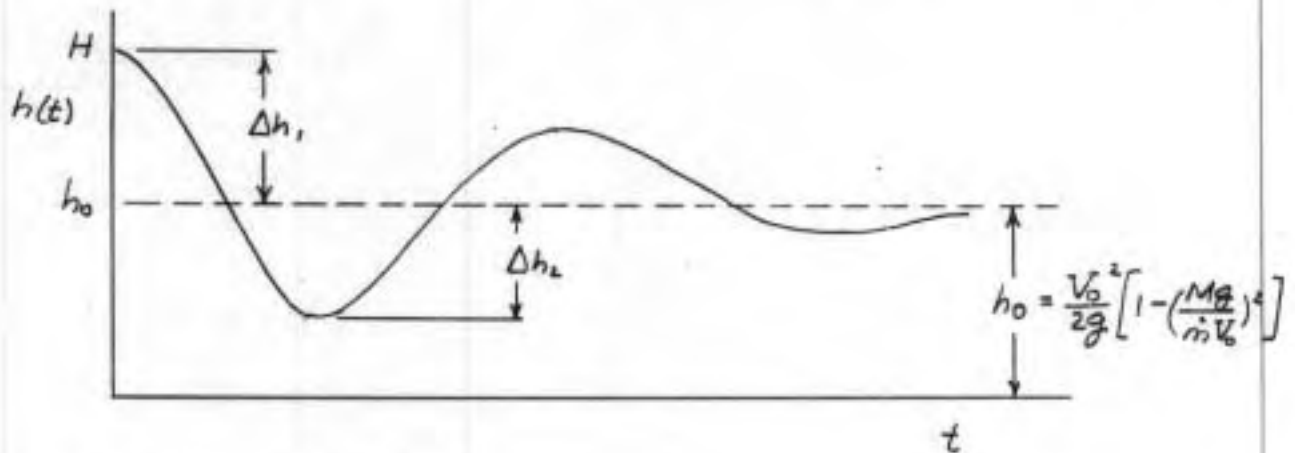
At equilibrium height, $h = h_0$, $\frac{dh}{dt} = 0$, and $\frac{d^2h}{dt^2} = 0$. Then

$$\rho\sqrt{V_0^2 - 2gh_0}\, A_0 V_0 - Mg = 0$$

Thus $\quad V_0^2 - 2gh_0 = \left(\frac{Mg}{\rho V_0 A_0}\right)^2$

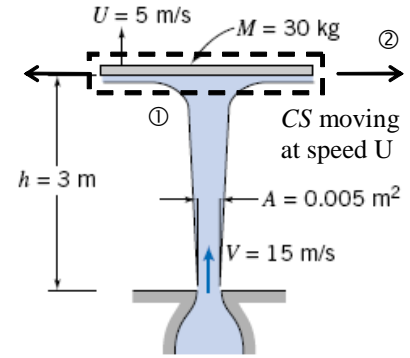
This may be solved to obtain

$$h_0 = \frac{V_0^2}{2g}\left[1 - \left(\frac{Mg}{\rho V_0^2 A_0}\right)^2\right] = \frac{V_0^2}{2g}\left[1 - \left(\frac{Mg}{\dot{m} V_0}\right)^2\right]$$

When released, $H > h_0$, and $dh/dt = 0$. Because the equation for d^2h/dt^2 is nonlinear, oscillations will occur. The expected behavior is sketched below:

Notes: (1) Expect oscillations
(2) $\Delta h_3 < \Delta h_2 < \Delta h_1$ due to nonlinear equation

Problem *4.175

[5]

*4.175 Consider the configuration of the vertical jet impinging on a horizontal disk shown in Problem 4.153. Assume the disk is released from rest at an initial height of 2 m above the jet exit plane. Solve for the subsequent motion of this disk. Identify the steady-state height of the disk.

Given: Water jet striking moving disk

Find: Motion of disk; steady state height

Solution:

Basic equations: Bernoulli; Momentum flux in z direction (treated as upwards) for linear accelerating CV

$$\frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{constant} \qquad F_{S_z} + F_{B_z} - \int_{CV} a_{rf_z}\,\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} w_{xyz}\,\rho\, d\forall + \int_{CS} w_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure 4) Uniform flow 5) velocities wrt CV (All in jet)

The Bernoulli equation becomes $\frac{V_0^2}{2} + g\cdot 0 = \frac{V_1^2}{2} + g\cdot h$ $\qquad V_1 = \sqrt{V_0^2 - 2\cdot g\cdot h}$ (1)

$$V_1 = \sqrt{\left(15\cdot\frac{m}{s}\right)^2 + 2\times 9.81\cdot\frac{m}{s^2}\cdot(0-3)\cdot m} \qquad V_1 = 12.9\frac{m}{s}$$

The momentum equation becomes

$$-M\cdot g - M\cdot a_{rfz} = w_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) + w_2\cdot\left(\rho\cdot V_2\cdot A_2\right) = \left(V_1 - U\right)\cdot\left[-\rho\cdot\left(V_1 - U\right)\cdot A_1\right] + 0$$

With $a_{rfz} = \frac{d^2h}{dt^2}$ and $U = \frac{dh}{dt}$ we get $-M\cdot g - M\cdot\frac{d^2h}{dt^2} = -\rho\cdot\left(V_1 - \frac{dh}{dt}\right)^2\cdot A_1$

Using Eq 1, and from continuity $V_1\cdot A_1 = V_0\cdot A_0$

$$\frac{d^2h}{dt^2} = \left(\sqrt{V_0^2 - 2\cdot g\cdot h} - \frac{dh}{dt}\right)^2\cdot\frac{\rho\cdot A_0\cdot V_0}{M\cdot\sqrt{V_0^2 - 2\cdot g\cdot h}} - g \qquad (2)$$

This must be solved numerically! One approach is to use Euler's method (see the *Excel* solution)

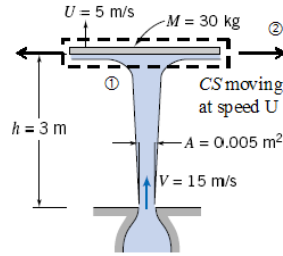
At equilibrium $h = h_0$ $\quad \frac{dh}{dt} = 0 \quad \frac{d^2h}{dt^2} = 0$ so

$$\sqrt{\left(V_0^2 - 2\cdot g\cdot h_0\right)}\cdot\rho\cdot A_0\cdot V_0 - M\cdot g = 0 \qquad \text{and} \qquad h_0 = \frac{V_0^2}{2\cdot g}\cdot\left[1 - \left(\frac{M\cdot g}{\rho\cdot V_0^2\cdot A_0}\right)^2\right]$$

Hence

$$h_0 = \frac{1}{2}\times\left(15\cdot\frac{m}{s}\right)^2\times\frac{s^2}{9.81\cdot m}\times\left[1 - \left[30\cdot kg\times 9.81\cdot\frac{m}{s^2}\times\frac{m^3}{1000\cdot kg}\times\left(\frac{s}{15\cdot m}\right)^2\times\frac{1}{.005\cdot m^2}\right]^2\right] \qquad h_0 = 10.7\,m$$

Problem *4.175 (In Excel)

*4.175 Consider the configuration of the vertical jet impinging on a horizontal disk shown in Problem 4.153. Assume the disk is released from rest at an initial height of 2 m above the jet exit plane. Solve for the subsequent motion of this disk. Identify the steady-state height of the disk.

$\Delta t =$ 0.05 s

$A_0 =$ 0.005 m^2

$g =$ 9.81 m/s^2

$V =$ 15 m/s

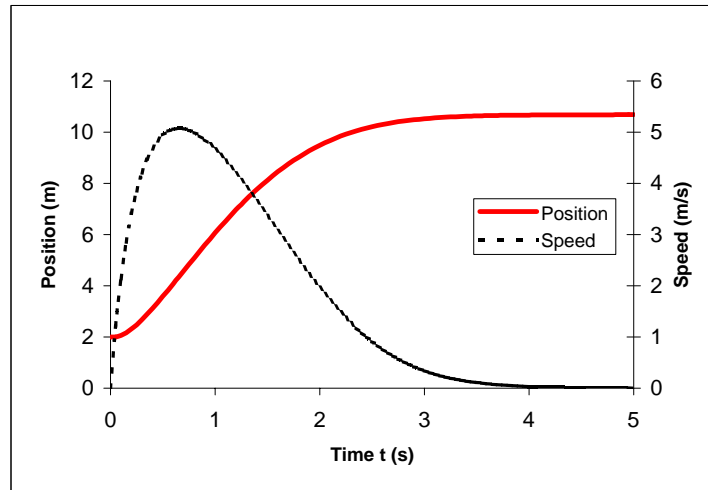
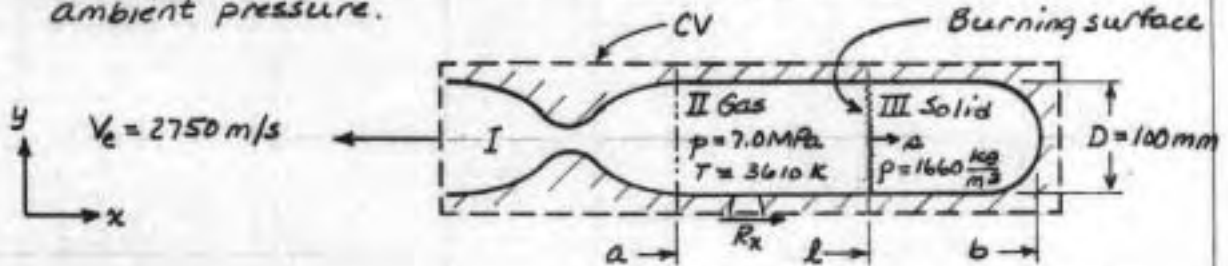
$M =$ 30 kg

$\rho =$ 1000 kg/m^3

$$h_{i+1} = h_i + \Delta t \cdot \left.\frac{dh}{dt}\right|_i$$

$$\left(\frac{dh}{dt}\right)_{i+1} = \left(\frac{dh}{dt}\right)_i + \Delta t \cdot \left.\frac{d^2h}{dt^2}\right|_i$$

$$\frac{d^2h}{dt^2} = \left(\sqrt{V_0^2 - 2\cdot g\cdot h} - \frac{dh}{dt}\right)^2 \cdot \frac{\rho\cdot A_0\cdot V_0}{M\cdot\sqrt{V_0^2 - 2\cdot g\cdot h}} - g$$

t (s)	*h* (m)	*dh/dt* (m/s)	d^2h/dt^2 (m/s²)
0.000	2.000	0.000	24.263
0.050	2.000	1.213	18.468
0.100	2.061	2.137	14.311
0.150	2.167	2.852	11.206
0.200	2.310	3.412	8.811
0.250	2.481	3.853	6.917
0.300	2.673	4.199	5.391
0.350	2.883	4.468	4.140
0.400	3.107	4.675	3.100
0.450	3.340	4.830	2.227
0.500	3.582	4.942	1.486
0.550	3.829	5.016	0.854
0.600	4.080	5.059	0.309
0.650	4.333	5.074	-0.161
0.700	4.587	5.066	-0.570
0.750	4.840	5.038	-0.926
0.800	5.092	4.991	-1.236
0.850	5.341	4.930	-1.507
0.900	5.588	4.854	-1.744
0.950	5.830	4.767	-1.951
1.000	6.069	4.669	-2.130
1.050	6.302	4.563	-2.286
1.100	6.530	4.449	-2.420
1.150	6.753	4.328	-2.535
1.200	6.969	4.201	-2.631
1.250	7.179	4.069	-2.711
1.300	7.383	3.934	-2.776
1.350	7.579	3.795	-2.826
1.400	7.769	3.654	-2.864
1.450	7.952	3.510	-2.889
1.500	8.127	3.366	-2.902
1.550	8.296	3.221	-2.904
1.600	8.457	3.076	-2.896
1.650	8.611	2.931	-2.878
1.700	8.757	2.787	-2.850
1.750	8.896	2.645	-2.814
1.800	9.029	2.504	-2.769
1.850	9.154	2.365	-2.716
1.900	9.272	2.230	-2.655
1.950	9.384	2.097	-2.588
2.000	9.488	1.967	-2.514

Given: Small solid fuel rocket motor on test stand. The fuel burns uniformly at $\dot{s} = 12.7$ mm/s. Exhaust gases leave at ambient pressure.

Treat combustion products as ideal gas with molecular mass, $M_m = 25.8$.

Find: (a) Evaluate rate of change of mass and of linear momentum within rocket motor.
(b) Express rate of change of momentum as a percentage of thrust.

__Solution:__ Apply continuity and x component of momentum equations using fixed CV shown.

Basic equations:
$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A} \quad (F_{B_x} = 0\ (2))$$

Assumptions: (1) No net pressure force; $F_{S_x} = R_x$
(2) $F_{B_x} = 0$
(3) All properties constant at each point, except at surface where combustion takes place
(4) Uniform flow at exit section

The continuity equation becomes
$$0 = \frac{\partial}{\partial t}\int_{I} \rho\, d\forall + \frac{\partial}{\partial t}\int_a^{\ell} \rho_g A\, dx + \frac{\partial}{\partial t}\int_{\ell}^{b} \rho_f A\, dx + \{|\rho_e V_e A_e|\} \quad \left(\frac{\partial}{\partial t}\int_I = 0\ (3)\right)$$

$$0 = \frac{\partial}{\partial t}[\rho_g A(\ell - a)] + \frac{\partial}{\partial t}[\rho_f A(b-\ell)] + \dot{m}_e = (\rho_g - \rho_f)A\frac{d\ell}{dt} + \dot{m}_e$$

or

$$\dot{m}_e = (\rho_f - \rho_g)A\frac{d\ell}{dt} = (\rho_f - \rho_g)A\dot{s}$$

For an ideal gas,

$$\rho_g = \frac{p_g}{RT_g} = \frac{p_g M_m}{R_u T_g} = 7.0\times10^6\,\frac{N}{m^2}\times\frac{25.8\,kg}{mol}\times\frac{mol\cdot K}{8314\,N\cdot m}\times\frac{1}{3610\,K} = 6.02\ kg/m^3$$

so

$$\dot{m}_e = (1660 - 6)\frac{kg}{m^3}\times\frac{\pi}{4}(0.1)^2 m^2\times 0.0127\frac{m}{s} = 0.165\ kg/s$$

Mass flow is out, so $\dfrac{\partial M_{cv}}{\partial t} = -0.165$ kg/s ← $\dfrac{\partial M_{cv}}{\partial t}$

From the momentum equation,

$$R_x = \frac{\partial}{\partial t}\int_I^{=0(3)} u\rho\, d\forall + \frac{\partial}{\partial t}\int_a^{\ell} u_g \rho_g A\, dx + \frac{\partial}{\partial t}\int_{\ell}^{b} u_f \rho_f A\, dx^{=0} + u_e\{|\rho_e V_e A_e|\}$$

$$= \frac{\partial}{\partial t}\left[u_g \rho_g A(\ell - a)\right] + u_e \dot{m}_e \quad ; \quad u_g = -V_g \text{ and } u_e = -V_e$$

$$R_x = -\rho_g V_g A \frac{d\ell}{dt} - V_e \dot{m}_e = -\rho_g V_g A \dot{s} - V_e \dot{m}_e$$

But from continuity, $\rho_g V_g A = \dot{m}_e$, since no mass accumulates in region I of the CV. Thus

$$R_x = -\dot{m}_e(V_e + \dot{s})$$

R_x is the force on the CV. The thrust is

$$K_x = \text{Thrust} = -R_x = \dot{m}_e(V_e + \dot{s})$$

$$K_x = 0.165 \frac{kg}{s}(2750 + 0.0127)\frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m} = 454 \ N$$

The rate of change of linear momentum within the CV is

$$\frac{\partial P_{x_{cv}}}{\partial t} = -\dot{m}_e \dot{s} = -0.165 \frac{kg}{s} \times 0.0127 \frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m} = -2.10 \ mN$$

$\frac{\partial P_{x_{cv}}}{\partial t}$

The ratio of rate of change of linear momentum to thrust is

$$\frac{\frac{\partial P_{x_{cv}}}{\partial t}}{K_x} = \frac{-\dot{m}_e \dot{s}}{\dot{m}_e(V_e + \dot{s})} = -\frac{\dot{s}}{(V_e + \dot{s})} = -\frac{0.0127 \frac{m}{s}}{(2750 + 0.0127)\frac{m}{s}} = -4.62 \times 10^{-6}$$

or

$$\frac{\frac{\partial P_{x_{cv}}}{\partial t}}{K_x} = -4.62 \times 10^{-4} \text{ percent}$$

Ratio

{Neglecting the unsteady momentum term in the analysis of this rocket motor would cause an error of approximately 1 part in 217,000. The assumption that $\partial P_{x_{cv}}/\partial t \simeq 0$ is certainly justified for engineering work.}

Problem *4.177

Open-Ended Problem Statement: The capability of the Aircraft Landing Loads and Traction Facility at NASA's Langley Research Center is to be upgraded. The facility consists of a rail-mounted carriage propelled by a water jet issuing from a pressurized tank. (The setup is identical in concept to the hydraulic catapult of Problem 4.133 The 49,000 kg carriage must accelerate to 220 knots in 122 m. (The vane turning angle is 170°.) Identify a range of water jet sizes and speeds needed to accomplish this performance. Specify the recommended operating pressure for the water jet system and determine the shape and estimated size of tankage to contain the pressurized water.

Discussion: The analysis of Example 4.11 forms the basis for the solution outlined below. Use a control volume attached to and moving with the carriage to analyze the motion. Neglect aerodynamic and rolling resistance to obtain a best-case solution. Solve the resulting differential equation of motion for carriage speed and position as functions of time, and for speed as a function of position along the rails.

Computing equations are summarized and results tabulated below. As shown in Example 4.11, analysis of the carriage motion results in the differential equation

$$\frac{dU}{dt} = \frac{\rho (V_j - U)^2 (1-\cos\theta)}{M} \tag{1}$$

Integrating with respect to time gives carriage speed versus time

$$U = V_j \frac{bt}{1+bt} \tag{2}$$

where parameter b is

$$b = \frac{\rho V_j A_j (1-\cos\theta)}{M} \tag{3}$$

Equation 2 is integrated to obtain carriage position versus time

$$x = V_j \left[t - \frac{\ln(1+bt)}{b} \right] \tag{4}$$

Substitute $dU/dt = UdU/dx$ and integrate Eq. 1 for distance traveled versus carriage speed

$$x = \frac{V_j}{b} \left[\ln(1 - U/V_j) + \frac{1}{1 - U/V_j} - 1 \right] \tag{5}$$

Relate jet speed to water tank pressure using the Bernoulli equation

$$V_j = \sqrt{2\Delta p/\rho} \tag{6}$$

The required volume of water is computed as follows:

1. Assume a range of tank pressures.
2. Compute the jet speed corresponding to each tank pressure from Eq. 6.
3. Solve for parameter b from Eq. 5 using the known maximum speed and specified distance.
4. Obtain jet area from Eq. 3.
5. Compute the time required to accelerate the carriage from Eq. 2.
6. Calculate jet diameter from jet area.
7. Compute the required volume of water from the product of mass flow rate and acceleration time.

Problem *4.177

The optimum operating pressure requires the least costly tankage. (Assume the most efficient spherical shape for pressurized tankage and constant tank pressure during acceleration.) Tankage calculations are organized as follows:

1. Obtain tank diameter from tank volume.
2. Calculate wall thickness from a force balance on the thin wall of the tank.
3. Calculate steel volume from tank surface area and wall thickness.
4. Assume steel cost is proportional to steel volume.

Sample Calculation: assume $p = 6000$ psig

$$V_j = \left[2 \times 6000 \frac{\text{lbf}}{\text{in.}^2} \times \frac{\text{ft}^3}{1.94\ \text{slug}} \times 144 \frac{\text{in.}^2}{\text{ft}^2} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2}\right]^{\frac{1}{2}} = 944\ \text{ft/s} \ ; \ \frac{U}{V_j} = \frac{371}{944} = 0.393$$

$$b = 944 \frac{\text{ft}}{\text{s}} \times \frac{1}{400\ \text{ft}}\left[\ln(1-0.393) + \frac{1}{1-0.393} - 1\right] = 0.350\ \text{s}^{-1}$$

$$A_j = \frac{bM}{\rho V_j(1-\cos\theta)} = \frac{0.350}{\text{s}} \times 3350\ \text{slug} \times \frac{\text{ft}^3}{1.94\ \text{slug}} \times \frac{\text{s}}{944\ \text{ft}} \times \frac{1}{(1-\cos 170°)} = 0.323\ \text{ft}^2$$

$$D = \sqrt{\frac{4A}{\pi}} = \left[\frac{4}{\pi} \times 0.323\ \text{ft}^2 \times 144 \frac{\text{in.}^2}{\text{ft}^2}\right]^{\frac{1}{2}} = 7.69\ \text{in.}$$

$$t = \frac{1}{b}\left(\frac{U/V_j}{1-U/V_j}\right) = \frac{\text{s}}{0.350} \times \frac{0.393}{1-0.393} = 1.85\ \text{s}$$

$$Q = V_j A = 944 \frac{\text{ft}}{\text{s}} \times 0.323\ \text{ft}^2 \times 7.48 \frac{\text{gal}}{\text{ft}^3} = 2280\ \text{gal/s}$$

$$\forall = Qt = 2280 \frac{\text{gal}}{\text{s}} \times 1.85\ \text{s} = 4220\ \text{gal}$$

$$D = (6\forall/\pi)^{1/3} = \left(\frac{6}{\pi} \times 4220\ \text{gal} \times \frac{\text{ft}^3}{7.48\ \text{gal}}\right)^{1/3} = 10.3\ \text{ft}$$

$$\Delta p \frac{\pi D^2}{4} = \pi D t \ ; \ t = \frac{pD}{4\sigma} = \frac{1}{4} \times 6000 \frac{\text{lbf}}{\text{in.}^2} \times 10.3\ \text{ft} \times \frac{\text{in.}^2}{40{,}000\ \text{lbf}} \times 12 \frac{\text{in.}}{\text{ft}} = 4.64\ \text{in.}$$

$$\forall_{steel} = \pi D^2 t = \pi \times (10.3)^2\ \text{ft}^2 \times 4.64\ \text{in.} \times \frac{\text{ft}}{12\ \text{in.}} = 129\ \text{ft}^3$$

Discussion: The results show the steel volume plummets as tank pressure is raised, with a broad minimum between 3,000 and 4000 psig.

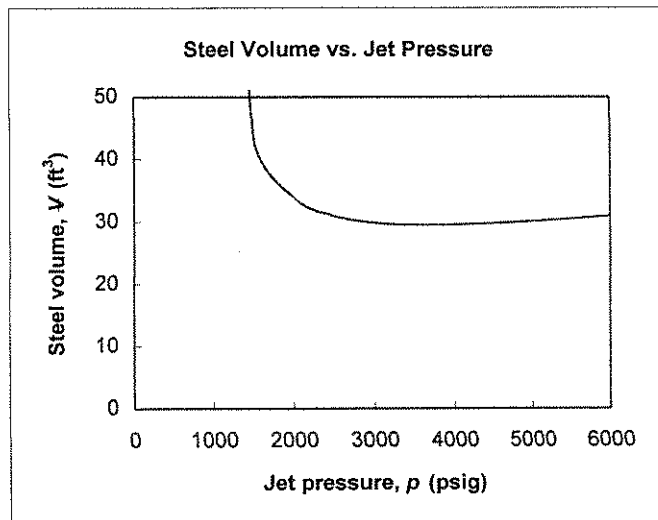
Problem *4.177

Input Data:

M =	49000	kg	3355	slug
U =	220	kt	371.3	ft/s
X =	122	m	400.3	ft
θ =	170	degrees		

Calculated Results:

Jet Pressure (psig)	Jet Speed (ft/s)	Parameter b (s^{-1})	Jet Area (ft^2)	Jet Diameter (in.)	Flow Rate (gal/s)	Flow Time (s)	Water Volume (gal)
6000	944	0.351	0.324	7.70	2285	1.85	4227
5500	904	0.380	0.367	8.20	2477	1.84	4546
5000	862	0.417	0.421	8.79	2715	1.82	4936
4500	817	0.463	0.494	9.51	3019	1.80	5426
4000	771	0.525	0.593	10.4	3419	1.77	6061
3500	721	0.610	0.737	11.6	3973	1.74	6924
3000	667	0.736	0.961	13.3	4797	1.70	8174
2500	609	0.944	1.35	15.7	6155	1.65	10172
2000	545	1.35	2.17	19.9	8830	1.58	13942
1500	472	2.53	4.67	29.3	16490	1.46	24061
1000	385	22.4	50.6	96.3	145835	1.19	173113

Jet Pressure (psig)	Water Volume (gal)	Tank Diameter (ft)	Wall Thickness (in.)	Steel Volume (ft^3)	Steel Mass (ton)
6000	4227	10.3	4.6	127.2	30.9
5500	4546	10.5	4.3	125.4	30.5
5000	4936	10.8	4.1	123.7	30.1
4500	5426	11.1	3.8	122.4	29.8
4000	6061	11.6	3.5	121.5	29.6
3500	6924	12.1	3.2	121.5	29.6
3000	8174	12.8	2.9	122.9	29.9
2500	10172	13.7	2.6	127.5	31.0
2000	13942	15.3	2.3	139.8	34.0
1500	24061	18.3	2.1	180.9	44.0
1000	173113	35.4	2.7	867.9	211.2

Open-Ended Problem Statement: A classroom demonstration of linear momentum is planned, using a water-jet propulsion system for a cart traveling on a horizontal linear air track. The track is 5 m long, and the cart mass is 155 g. The objective of the design is to obtain the best performance for the cart, using 1 L of water contained in an open cylindrical tank made from plastic sheet with density of 0.0819 g/cm^2. For stability, the maximum height of the water tank cannot exceed 0.5 m. The diameter of the smoothly rounded water jet may not exceed 10 percent of the tank diameter. Determine the best dimensions for the tank and the water jet by modeling the system performance. Plot acceleration, velocity, and distance as functions of time. Find the optimum dimensions of the water tank and jet opening from the tank. Discuss the limitations on your analysis. Discuss how the assumptions affect the predicted performance of the cart. Would the actual performance of the cart be better or worse than predicted? Why? What factors account for the difference(s)?

Discussion: This solution is an extension of Problem *4.179. The analyses for tank level, acceleration, and velocity are identical; please refer to the solution for Problem *4.179 for equations describing each of these variables as functions of time.

One new feature of this problem is computation of distance traveled. Equation 7 of Problem *4.179 could be integrated in closed form to provide an equation for distance traveled as a function of time. However, the integral would be messy, and it would provide little insight into the dependence on key parameters. Consequently, a numerical analysis has been chosen in this problem. The results are presented in the plots and spreadsheet on the next page.

We have chosen to define velocity as the output to be maximized.

A second new feature of this problem is the geometric constraints: the maximum track length is 5 m. Intuitively jet diameter should be chosen as the largest possible fraction of tank diameter for optimum performance. Using the spreadsheet to vary $\beta = d/D$ verifies that this is the case. Therefore we have used the maximum allowable ratio, $\beta = 0.1$, for all computations.

Tank height should be a factor in performance. Intuition suggests that increasing tank height should improve performance. Using the spreadsheet shows a very weak dependence on tank height. Performance is best at smaller tank heights, corresponding to the minimum tank mass.

As tank height is decreased, diameter increases because tank volume is held constant. Since diameter ratio is constant, then jet diameter increases with decreasing tank height. This effect almost overshadows the effect of tank height.

The principal limitations on the analysis are the assumptions of negligible motion resistance and no slope to the free surface of water in the tank. Actual performance of the cart would likely be less than predicted because of motion resistance.

Distance is modeled as

$$x_{i+1} = x_i + U_i \Delta t + \frac{1}{2} a_{x,i} \Delta t^2$$

The accuracy of this model for position is consistent with the accuracy of modeling the water-jet propulsion system.

Problem *4.178

Analysis of Cart Propelled by Gravity-Driven Water Jet:

Input Data:

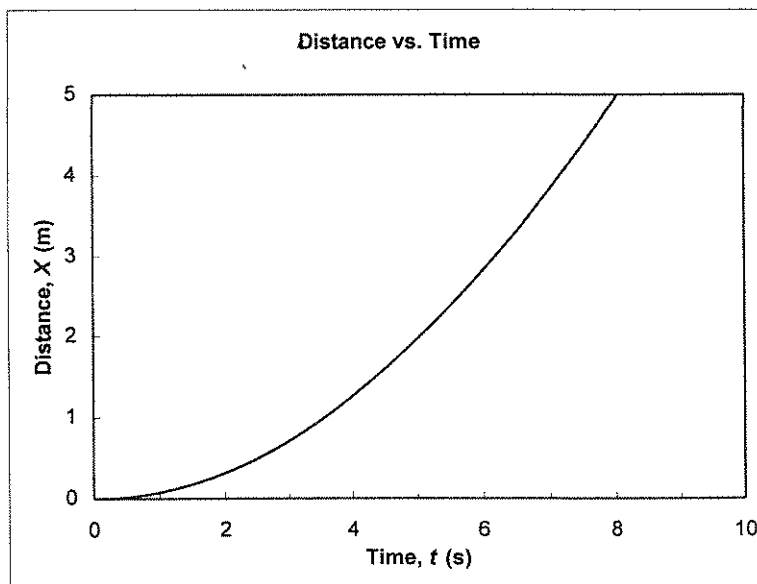
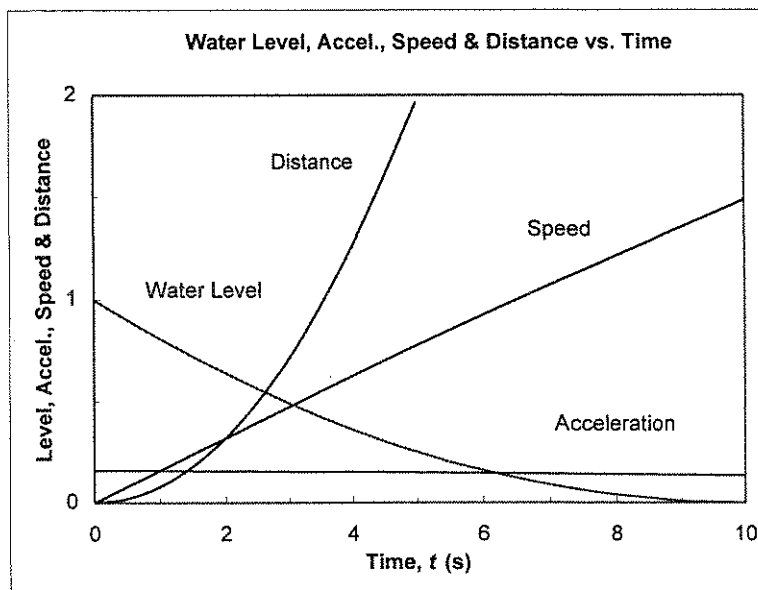
$g =$	9.81	m/s^2	Acceleration of gravity
$H =$	500	mm	Height of tank
$M_c =$	0.155	kg	Mass of cart
$V =$	1.00	L	Tank volume
$\beta =$	0.100	(---)	Ratio of jet diameter to tank diameter
$\rho =$	999	kg/m^3	Density of water
$\rho'' =$	0.819	kg/m^2	(Area) density of tank material

Calculated Parameters:

$a =$	0.471	(---)	($a^2 =$) Ratio of mass of tank to initial mass of water
$b =$	0.0313	s^{-1}	Geometric parameter of solution
$d =$	5.05	mm	Diameter of water jet
$D =$	50.5	mm	Diameter of tank
$M_0 =$	1.00	kg	Initial mass of water in tank
$M_p =$	0.0666	kg	Mass of plastic in tank
$M_t =$	0.222	kg	Mass of plastic tank plus cart

Calculated Results:

Time, t (s)	Level, y/H (---)	Accel., a_x (m/s^2)	Velocity, U (m/s)	Position, X (m)
0	1	0.161	0	0
0.5	0.903	0.160	0.080	0.0201
1.0	0.810	0.159	0.160	0.080
1.5	0.723	0.158	0.239	0.180
2.0	0.640	0.157	0.317	0.319
2.5	0.563	0.156	0.395	0.497
3.0	0.490	0.154	0.473	0.714
3.5	0.423	0.153	0.550	0.97
4.0	0.360	0.152	0.626	1.26
4.5	0.303	0.151	0.702	1.60
5.0	0.250	0.150	0.777	1.97
5.5	0.203	0.148	0.852	2.37
6.0	0.160	0.147	0.925	2.82
6.5	0.123	0.145	1.00	3.30
7.0	0.0900	0.144	1.07	3.82
7.5	0.0625	0.142	1.14	4.37
8.0	0.0400	0.141	1.21	4.96
8.03	0.0388	0.141	1.22	5.00
9.0	0.0100	0.137	1.35	
9.5	0.0025	0.135	1.42	
10.0	0.0000	0.133	1.49	

Problem *4.179

Open-Ended Problem Statement: Analyze the design and optimize the performance of a cart propelled along a horizontal track by a water jet that issues under gravity from an open cylindrical tank carried on board the cart. (A water-jet-propelled cart is shown in the diagram for Proble 4.137.) Neglect any change in slope of the liquid free surface in the tank during acceleration. Analyze the motion of the cart along a horizontal track, assuming it starts from rest and begins to accelerate when water starts to flow from the jet. Derive algebraic equations or solve numerically for the acceleration and speed of the cart as functions of time. Present results as plots of acceleration and speed versus time, neglecting the mass of the tank. Determine the dimensions of a tank of minimum mass required to accelerate the cart from rest along a horizontal track to a specified speed in a specified time interval.

Discussion: This problem solution consists of two parts. The first is to analyze the acceleration and velocity of a cart propelled by a gravity-driven water jet. The second is to optimize the dimensions of the cart and jet to accelerate to a specified speed in a specified time interval.

To analyze the problem, apply conservation of mass and the Bernoulli equation to the draining of the tank, then apply the x component of the momentum equation for a control volume to analyze the resulting linear acceleration. A representative plot of the results is presented below.

To optimize the performance of the water-jet-propelled cart, manipulate the solution dimensions until the best performance is attained.

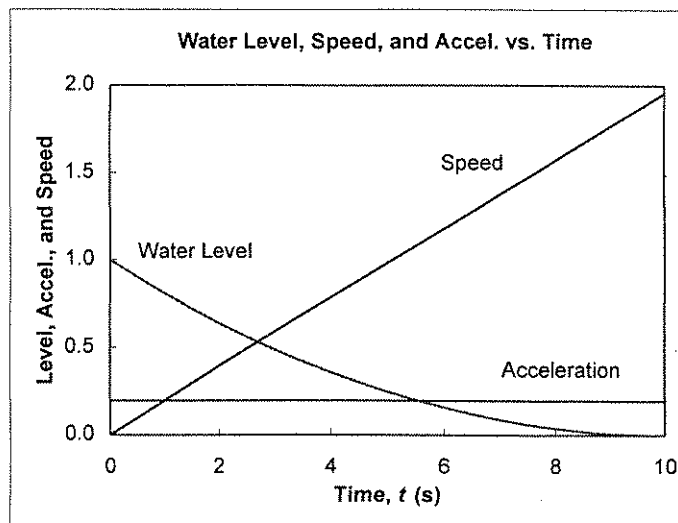
Input Data:

d =	10	mm	Diameter of water jet
D =	100	mm	Diameter of tank
g =	9.81	ft/s^2	Acceleration of gravity
H =	150	mm	Height of tank
M_t =	0.001	kg	Mass of tank
ρ =	999	kg/m^3	Density of water

Calculated Parameters:

a =	0.029	(---)	(a^2 =) Ratio of mass of tank to initial mass of water
b =	0.0572	s^{-1}	Geometric parameter of solution
M_0 =	1.18	kg	Initial mass of water in tank
β =	0.1	(---)	Ratio of jet diameter to tank diameter

Calculated Results:

Time, t (s)	Level Ratio, y/H (---)	Accel., a_x (m/s^2)	Velocity, U (m/s)
0	1	0.196	0
1	0.810	0.196	0.196
2	0.640	0.196	0.392
3	0.490	0.196	0.588
4	0.360	0.196	0.784
5	0.250	0.196	0.980
6	0.160	0.196	1.176
7	0.0900	0.196	1.37
8	0.0400	0.196	1.57
9	0.0100	0.196	1.76
10	0	0.195	1.96

Given: Cart, propelled by water jet, accelerates along horizontal track.

Find: (a) Analyze motion, derive algebraic equations for acceleration and speed of cart as functions of time
(b) Plot acceleration and speed vs. time.

Solution: Apply conservation of mass, Bernoulli, and momentum equations.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

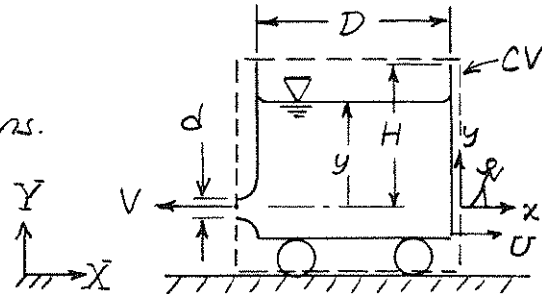
$$\frac{p_j}{\rho} + \frac{V_j^2}{2} + g y_j = \frac{p}{\rho} + \frac{V^2}{2} + g y$$

(p_j, p: $=0(8)$; y_j: $=0(9)$)

M_t = mass of tank, cart

$\beta = \dfrac{d}{D}$

$$F_{S_X} + F_{B_X} - \int_{CV} a_{rf_x} \rho \, d\forall = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$

(F_{S_X}: $=0(9)$; F_{B_X}: $=0(10)$; $\frac{\partial}{\partial t}$ term: $\approx 0(11)$)

Assumptions: (1) Uniform flow from exit jet (2) Neglect air in CV

$$0 = \frac{\partial}{\partial t}(\rho A_t y) + \{+|\rho V_j A_j|\} = \rho A_t \frac{dy}{dt} + \rho V_j A_j = -\rho A_t V + \rho V_j A_j \qquad (1)$$

Thus $V = V_j \dfrac{A_j}{A_t} = V_j \left(\dfrac{d}{D}\right)^2 = \beta^2 V_j \qquad (2)$

(3) No slope to free surface (given)
(4) Quasi-steady flow
(5) Frictionless flow
(6) Incompressible flow
(7) Flow along a streamline
(8) $p = p_j = p_{atm}$
(9) $y_j = 0$

From Bernoulli, $\dfrac{V_j^2}{2} = \dfrac{V^2}{2} + g y$ or $V_j^2 - V^2 = 2gy$

Substituting from (2), $V_j^2 - \beta^4 V_j^2 = V_j^2(1-\beta^4) = 2gy$; $V_j^2 = \dfrac{2gy}{(1-\beta^4)} \qquad (3)$

Substituting into (1), $\dfrac{dy}{dt} = -\beta^2 V_j = -\beta^2 \dfrac{\sqrt{2gy}}{(1-\beta^4)}$ or $\dfrac{dy}{y^{1/2}} = -\dfrac{\beta^2\sqrt{2g}}{1-\beta^4}\,dt$

Integrating, $2y^{1/2}\Big]_{y_0}^{y} = -\dfrac{\beta^2\sqrt{2g}}{(1-\beta^4)}\,t$ or $y^{1/2} - y_0^{1/2} = -\dfrac{\beta^2\sqrt{2g}}{2(1-\beta^4)}\,t$

Thus $\left(\dfrac{y}{y_0}\right)^{1/2} = 1 - \left[\dfrac{g\beta^4}{2y_0(1-\beta^4)}\right]^{1/2} t = 1 - bt$; $b = \left[\dfrac{g\beta^4}{2y_0(1-\beta^4)}\right]^{1/2} \qquad (4)$

From momentum (10) $F_{S_x} = 0$; no resistance
(11) $F_{B_x} = 0$; horizontal motion
(12) $u \approx 0$ in CV, so $\partial/\partial t \approx 0$

Then

$$-a_{rf_x} M(t) = u_j \{+|\rho V_j A_j|\} = -\rho V_j^2 A_j \qquad (5)$$

$$a_{rf_x} = \frac{dU}{dt} \qquad u_j = -V_j$$

But from (4), $M(t) = M_t + \rho A_t y = M_t + \rho A_t y_0 (1-bt)^2$

From (3), $V_j^2 = \frac{2gy}{1-\beta^4} = \frac{2g}{1-\beta^4} y_0 (1-bt)^2$

Substituting into (5)

$$\frac{dU}{dt}\left[M_t + \rho A_t y_0 (1-bt)^2\right] = \rho A_j \frac{2g}{1-\beta^4} y_0 (1-bt)^2 = \rho A_t y_0 \frac{2g\beta^2}{1-\beta^4}(1-bt)^2$$

Define M_0 = initial mass of water $= \rho A_t y_0$. Then

$$\frac{dU}{dt}\left[M_t + M_0 (1-bt)^2\right] = M_0 \frac{2g\beta^2}{1-\beta^4}(1-bt)^2$$

or

$$\frac{dU}{dt} = \frac{2g\beta^2}{1-\beta^4} \frac{M_0 (1-bt)^2}{M_t + M_0 (1-bt)^2} \qquad (6)$$

$\frac{dU}{dt}(t)$

To integrate, let $r = 1-bt$, $dr = -b\,dt$, and $a^2 = M_t/M_0$. Then

$$U = \int_0^U dU = \frac{2g\beta^2}{1-\beta^4}\left(-\frac{1}{b}\right)\int_0^t \frac{r^2}{a^2+r^2}\,dr = -\frac{2g\beta^2}{1-\beta^4}\frac{1}{b}\left[r - a\tan^{-1}\left(\frac{r}{a}\right)\right]_0^t$$

$$= -\frac{2g\beta^2}{1-\beta^4}\frac{1}{b}\left[(1-bt) - a\tan^{-1}\left(\frac{1-bt}{a}\right)\right]_0^t$$

$$U = -\frac{2g\beta^2}{1-\beta^4}\frac{1}{b}\left[(1-bt) - a\tan^{-1}\left(\frac{1-bt}{a}\right) - 1 + a\tan^{-1}\left(\frac{1}{a}\right)\right]$$

Simplifying, then

$$U = \frac{2g\beta^2}{1-\beta^4}\left\{t + \frac{a}{b}\left[\tan^{-1}\left(\frac{1-bt}{a}\right) - \tan^{-1}\left(\frac{1}{a}\right)\right]\right\} \qquad (7)$$

$U(t)$

$$a^2 = \frac{M_t}{M_0}\,; \quad b = \left[\frac{g\beta^4}{2y_0(1-\beta^4)}\right]^{1/2}$$

Given: Cart, propelled by water jet, accelerating on horizontal track.

$$\frac{dU}{dt} = \frac{2g\beta^2}{1-\beta^4}\frac{(1-bt)^2}{a^2+(1-bt)^2} \qquad (1)$$

$$U(t) = \frac{2g\beta^2}{1-\beta^4}\left\{t + \frac{a}{b}\left[\tan^{-1}\left(\frac{1-bt}{a}\right) - \tan^{-1}\left(\frac{1}{a}\right)\right]\right\} \qquad (2)$$

$$\beta = \frac{d}{D}, \quad a^2 = \frac{M_t}{M_0}, \quad b = \left[\frac{g\beta^4}{2y_0(1-\beta^4)}\right]^{1/2}$$

Find: (a) Shape for tank of minimum mass for given volume.
(b) Minimum water volume to reach $U = 2.5$ m/sec in $t = 25$ sec.

Solution: Mass of tank is $M = \rho_t A_s t$, where t = thickness of wall

$$A_s = A_{bottom} + A_{cylinder} = \pi\frac{D^2}{4} + \pi DH$$

Since volume is $\forall = \frac{\pi D^2}{4}H$, then $H = \frac{4\forall}{\pi D^2}$, and

$$A_s = \frac{\pi D^2}{4} + \pi D\left(\frac{4\forall}{\pi D^2}\right) = \frac{\pi D^2}{4} + \frac{4\forall}{D}$$

To minimize, set $dA_s/dD = 0$

$$\frac{dA_s}{dD} = \frac{\pi D}{2} + (-1)\frac{4\forall}{D^2} = 0 \quad \text{so} \quad D^3 = \frac{8\forall}{\pi} \quad \text{or} \quad D = \left(\frac{8\forall}{\pi}\right)^{1/3} \qquad (3) \quad D_{opt}$$

$$\text{Then} \quad \forall = \frac{\pi D^2 H}{4} = \frac{\pi D^3}{8} \quad \text{so} \quad \frac{H}{D} = \frac{1}{2} \qquad (4) \quad \left.\frac{H}{D}\right)_{opt}$$

The tank mass per volume for optimum H/D is

$$m = \frac{M}{\forall} = \frac{\rho_t\left(\frac{\pi D^2}{4} + \pi DH\right)t}{\frac{\pi D^2}{4}H} = \rho_t\left(\frac{t}{H} + \frac{4t}{D}\right) = \rho_t\frac{t}{H}\left(1 + 4\frac{H}{D}\right) = 3\rho_t\frac{t}{H}$$

Therefore mass depends on $\rho_t t$ for a given volume. The minimum mass is achieved for the smallest combination of ρ_t and t.

$$a^2 = \frac{M_t}{M_0} = \frac{M_t}{\rho\forall} = \frac{3\rho_t}{\rho}\frac{t}{H} = 3\,SG\left(\frac{t}{H}\right) \qquad (5)$$

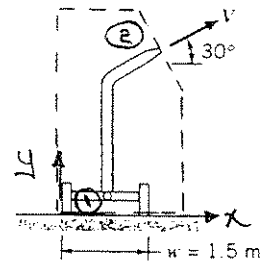
which still depends on Volume, since it contains H.

The best solution strategy seems to be: pick $\forall$, calculate H, D, β, a, and b, then plot $U(t)$.

Given: Irrigation sprinkler mounted on cart

$V = 40$ m/s $\theta = 30°$

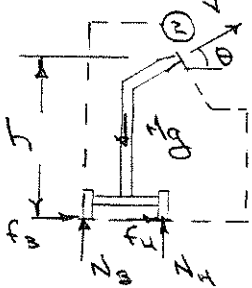
$D = 50$ mm   Flow is water

$h = 3$ m $M = 350$ kg

Find: (a) Magnitude of moment which tends to overturn the cart

(b) Value of V to cause impending motion; nature of impending motion.

(c) Effect on jet inclination on results

Plot: Jet velocity as a function of θ for the case of impending motion.

Solution:

Apply moment of momentum equation, using fixed CV shown at left. Origin of coordinates is on ground at left wheel of cart. With this coordinate system counterclockwise moments are positive (about the z axis).

Basic equation:

$$\vec{r}\times\vec{F}_s + \int_{CV}\vec{r}\times\vec{g}\,\rho\,dV + \vec{T}_s = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}\rho\,dV + \int_{CS}\vec{r}\times\vec{V}\,(\rho\vec{V}\cdot d\vec{A})$$

($\vec{T}_s$ = 0 (1); $\frac{\partial}{\partial t}$ term = 0 (2))

Assumptions:
(1) $\vec{T}_s = 0$
(2) steady flow
(3) uniform flow at nozzle outlet
(4) neglect $\vec{r}\times\vec{V}$ of inlet flow
(5) center of mass located at $x = w/2$
(6) nozzle length is short; coordinates of nozzle exit at $(x_2, y_2) = (w/2, h)$

Then

$$\vec{r}\times\vec{F}_s + \vec{r}\times M\vec{g} = \vec{r}_1\times\vec{V}_1\{-|\rho V_1 A_1|\} + \vec{r}_2\times\vec{V}_2\{|\rho_2 V_2 A_2|\}$$

(first term on right = 0 (4))

$$r_2 = \frac{w}{2}\hat{i} + h\hat{j} \qquad V_2 = V(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

and

$$wN_4\hat{k} - \frac{w}{2}Mg\hat{k} = \frac{w}{2}V\sin\theta\,\dot{m}_2\hat{k} - hV\cos\theta\,\dot{m}\hat{k}$$

$$wN_4 - \frac{w}{2}Mg = \dot{m}_2 V\left[\frac{w}{2}\sin\theta - h\cos\theta\right] \quad \text{------ (1)}$$

Rewriting Eq. 1 in the form $\Sigma M_z = 0$ {for static equilibrium}

$$wN_4 - \frac{w}{2}Mg + \dot{m}_2 V\left[h\cos\theta - \frac{w}{2}\sin\theta\right] = 0 \quad \text{--- (2)}$$

The last term in Eq 2 is the moment (due to the jet) which tends to overturn the cart.

Evaluating, $\dot{m}_2 = \rho A_2 V_2 = \rho \frac{\pi D^2}{4} V_2$

$$\dot{m}_2 = 999 \frac{kg}{m^3} \times \frac{\pi}{4} (0.05)^2 m^2 \times 40 \frac{m}{s} = 78.5 \ kg/s$$

Then with $V_2 = 40 m/s$

$$\text{Moment from jet} = 78.5 \frac{kg}{s} \times 40 \frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m} \left[3m \cos 30° - \frac{1.5m}{2} \sin 30° \right]$$

$$\text{Moment}_{jet} = 6.98 \ kN \cdot m$$ ← Moment$_{jet}$

For the case of impending tipping (about point 3) $N_4 \to 0$ and from Eq. 2

$$-\frac{W}{2} Mg + \dot{m}_2 V \left[h\cos\theta - \frac{W}{2} \sin\theta \right] = 0$$

To solve for V_2, write $\dot{m} = \rho A_2 V_2$

$$V_2^2 = \frac{W Mg}{2 \rho A_2 [h\cos\theta - \frac{W}{2}\sin\theta]} \qquad (3)$$

$$V_2^2 = \frac{1.5m}{2} \times 350kg \times 9.81 \frac{m}{s^2} \times \frac{m^3}{999 kg} \times \frac{1}{1.96 \times 10^{-3} m^2} \times \frac{1}{(3\cos 30° - 0.75 \sin 30°) m}$$

$$V_2^2 = 592 \ m^2 s^2 \qquad \therefore V_2 = 24.3 \ m/s$$ ← V_2

Thus, the maximum speed allowable without tipping is less than the value suggested.

The impending motion will be tipping since $f_3 < \mu N_3$

From the x momentum equation $f_3 = \dot{m} V_2 \cos\theta$

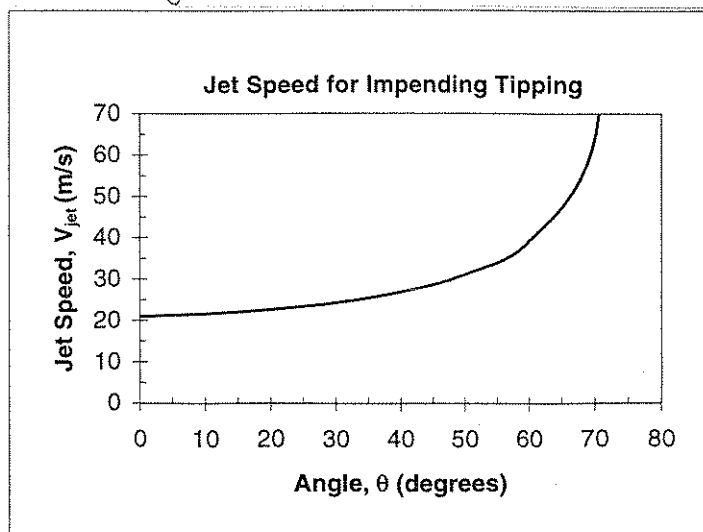
From the y momentum equation $N_3 = Mg + \dot{m} V_2 \sin\theta$

For tipping $\mu > 0.377$

From Eq. 2 we see that as θ increases the tendency to tip decreases

For impending motion from Eq. 3.

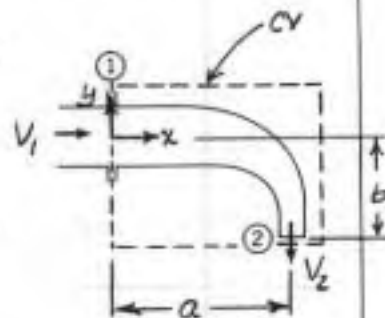
$$V = \left\{ \frac{W Mg}{2\rho A_2 \left[h\cos\theta - \frac{W}{2}\sin\theta \right]} \right\}^{1/2}$$

Problem *4.181

Given: The 90° reducing elbow of Example 4.6 discharges to atmosphere. Section ② is located 0.3 m to the right of Section ①.

Find: Estimate the moment exerted by the flange on the elbow.

Solution: Apply moment of momentum, using the CV and CS shown.

From Example Problem 4.7, $\vec{V}_2 = -16\hat{j}$ m/s, $A_1 = 0.01\ m^2$

Steady flow, $A_2 = 0.0025\ m^2$

Basic equation (fixed CV):

$$\vec{r}\times\vec{F}_s + \int_{CV}\vec{r}\times\vec{g}\,\rho\,d\forall + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}\rho\,d\forall + \int_{CS}\vec{r}\times\vec{V}\,\rho\vec{V}\cdot d\vec{A}$$

(terms marked $\approx 0(1)$, $=0(2)$, $=0(3)$)

Assumptions: (1) Neglect body forces
(2) No shafts, so $\vec{T}_{shaft} = 0$
(3) Steady flow (given)
(4) Uniform flow at each cross section
(5) Incompressible flow

Then

$$\vec{M}_{flange} = \vec{r}\times\vec{F}_s\Big)_{flange} = \vec{r}_1\times\vec{V}_1\left\{-\rho V_1 A_1\right\} + \vec{r}_2\times\vec{V}_2\left\{+\rho V_2 A_2\right\} \qquad (1)$$

$$\vec{r}_1 = 0 \qquad \left.\begin{aligned}\vec{r}_2 &= a\hat{i} - b\hat{j}\\ \vec{V}_2 &= -V_2\hat{j}\end{aligned}\right\}\ \vec{r}_2\times\vec{V}_2 = -aV_2\hat{k} + 0$$

Substituting into Eq. 1,

$$\vec{M}_{flange} = -aV_2\hat{k}\left\{+\rho V_2 A_2\right\} = -a\rho V_2^2 A_2\hat{k}$$

$$= 0.3\ m \times 999\frac{kg}{m^3}\times(16)^2\frac{m^2}{s^2}\times 0.0025\ m^2 \times \frac{N\cdot s^2}{kg\cdot m}(-\hat{k})$$

$$\vec{M}_{flange} = -192\ \hat{k}\ N\cdot m \qquad \vec{M}_{flange}$$

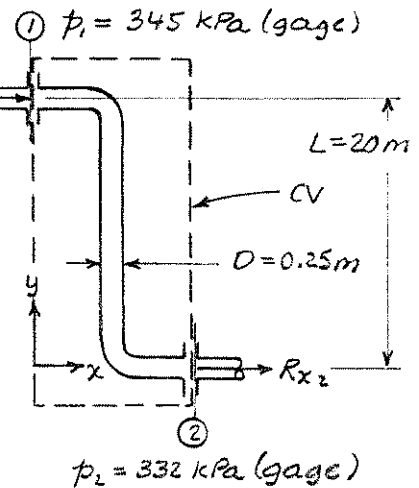
This is the torque that must be exerted on the CV by the flange.

{ Since $\vec{M}_{flange}$ is in the $-\hat{k}$ direction, it must act cw in the xy-plane. }

Problem *4.182

Given: Crude oil (SG = 0.95) flow through a pipe assembly in the horizontal configuration shown.

Find: Force and torque exerted by assembly on its supports.

Solution: No momentum components exist in the y direction. Apply x component of linear momentum and the moment of momentum equations using the CV shown. Location of coordinates is arbitrary; for simplicity, choose as shown.

Basic equations:

$$F_{S_x} + \underbrace{F_{B_x}}_{=0(1)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} u\rho\, dV}_{=0(2)} + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

$$\vec{r}\times\vec{F}_s + \underbrace{\int_{CV}\vec{r}\times\vec{g}\,\rho\, dV}_{=0(4)} + \underbrace{\vec{T}_{shaft}}_{=0(5)} = \underbrace{\frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}\rho\, dV}_{=0(2)} + \int_{CS}\vec{r}\times\vec{V}\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions:
(1) $F_{B_x} = 0$; $\vec{g}$ acts in z direction
(2) Steady flow
(3) Uniform flow at each section
(4) No z component of $\vec{r}\times\vec{g}$
(5) $\vec{T}_{shaft} = 0$

$$A = \frac{\pi D^2}{4} = \frac{\pi}{4}(0.25)^2 m^2 = 0.049\ m^2$$

From momentum equation,

$$R_{x_1} + R_{x_2} + p_1A - p_2A = u_1\{-\dot{m}\} + u_2\{\dot{m}\} = 0\ ;\quad R_{x_1} + R_{x_2} = (p_2 - p_1)A$$

From moment of momentum,

$$\vec{r}_1\times(R_{x_1}+p_1A)\hat{i} + \underbrace{\vec{r}_2}_{=0}\times(R_{x_2}-p_2A)\hat{i} = \vec{r}_1\times V_1\hat{i}\{-\dot{m}\} + \underbrace{\vec{r}_2}_{=0}\times V_2\hat{i}\{\dot{m}\}\ ;\ \vec{r}_1 = L\hat{j},\ \vec{r}_1\times\hat{i} = -L\hat{k}$$

$$-L(R_{x_1}+p_1A)\hat{k} = -LV_1(-\dot{m})\hat{k} = LV_1\dot{m}\,\hat{k} = L\frac{Q}{A}(\rho Q)\hat{k} = L\frac{\rho Q^2}{A}\hat{k}$$

$$R_{x_1} = -\frac{\rho Q^2}{A} - p_1A = -0.95\times 999\frac{kg}{m^3}\times(0.58)^2\frac{m^6}{s^2}\times\frac{1}{0.049\,m^2}\times\frac{N\cdot s^2}{kg\cdot m} - 3.45\times10^5\frac{N}{m^2}\times 0.049\,m^2 = -23.4\ kN$$

$$R_{x_2} = (p_2 - p_1)A - R_{x_1} = p_2A - p_1A + \frac{\rho Q^2}{A} + p_1A = p_2A + \frac{\rho Q^2}{A}$$

$$= 3.32\times10^5\frac{N}{m^2}\times 0.049\,m^2 + 0.95\times 999\frac{kg}{m^3}\times(0.58)^2\frac{m^6}{s^2}\times\frac{1}{0.049\,m^2}\times\frac{N\cdot s^2}{kg\cdot m} = 22.8\ kN$$

$$\vec{r}\times\vec{F}_s = \vec{r}_1\times R_{x_1}\hat{i} = L\hat{j}\times R_{x_1}\hat{i} = -LR_{x_1}\hat{k} = -20\,m\times(-46.0)kN\,\hat{k} = 468\,\hat{k}\ kN\cdot m$$

These are forces and torque on CV. The corresponding reactions are:

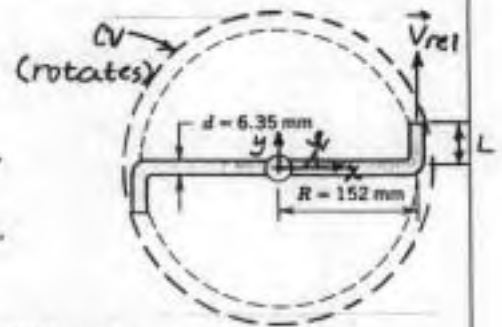
$$K_{x_1} = -R_{x_1} = 23.4\ kN\ ,\quad K_{x_2} = -R_{x_2} = -22.8\ kN$$ ← Force

$$\vec{M} = -\vec{r}\times\vec{F}_s = -468\,\hat{k}\ kN\cdot m\quad (i.e.\ clockwise)$$ ← Torque

Problem *4.183

Given: Simplified lawn sprinkler rotating in horizontal plane, $Q = 4.5$ gal/min.

Water discharges horizontally from jets.

Neglect pivot friction, inertia of sprinkler.

Find: (a) Torque needed to hold at $\omega = 0$.

(b) Angular acceleration when torque is removed.

Solution: Choose rotating CV. Apply angular momentum principle, Eq. 4.53.

Basic equation:

$$\vec{r} \times \vec{F}_s \overset{=0(1)}{} + \int_{CV} \vec{r} \times \vec{g} \rho \, d\forall \overset{=0(2)}{} + \vec{T}_{shaft}$$

$$- \int_{CV} \vec{r} \times [2\vec{\omega} \times \vec{V}_{xyz} \overset{=0(3)}{} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \overset{=0(3)}{} + \dot{\vec{\omega}} \times \vec{r}] \rho \, d\forall$$

$$= \frac{\partial}{\partial t} \int_{CV} \vec{r} \times \vec{V}_{xyz} \rho \, d\forall \overset{=0(4)}{} + \int_{CS} \vec{r} \times \vec{V}_{xyz} \, \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions:
(1) No surface forces
(2) Body torques cancel
(3) Sprinkler stationary, $\vec{\omega} = 0$
(4) Steady flow
(5) Uniform flow at each section
(6) $L \ll R$

Analyze right arm of sprinkler. From geometry $\vec{r} = r\hat{i}$ in CV, $\vec{r} = R\hat{i}$ at jet. Then

$$T\hat{k} - \int_{CV} \vec{r} \times (\dot{\vec{\omega}} \times \vec{r}) \rho \, d\forall = R\hat{i} \times V\hat{j} \, \rho \frac{Q}{2} = \rho \frac{QRV}{2} \hat{k} = \frac{\dot{m}RV}{2} \hat{k}$$

$$r\hat{i} \times (\dot{\omega}\hat{k} \times r\hat{i}) = r\hat{i} \times \dot{\omega} r \hat{j} = \dot{\omega} r^2 \hat{k}; \quad \int_{CV} = \dot{\omega} \frac{R^3}{3} \rho A \, \hat{k}$$

Dropping $\hat{k}$, $T - \frac{\dot{\omega} \rho A R^3}{3} = \frac{\dot{m} R V}{2}$. When arm is stationary, $\dot{\omega} = 0$, and

$$T = \frac{\dot{m}RV}{2} \qquad \dot{m} = \rho Q = 999 \frac{kg}{m^3} \times 4.5 \frac{gal}{min} \times 231 \frac{in.^3}{gal} \times (0.0254)^3 \frac{m^3}{in.^3} \times \frac{min}{60\,s} = 0.284 \frac{kg}{s}$$

$$V = \frac{Q}{2A} = \frac{2Q}{\pi d^2} = \frac{2}{\pi} \times 2.84 \times 10^{-4} \frac{m^3}{s} \times \frac{1}{(0.00635)^2 m^2} = 4.48 \text{ m/s}$$

$$T = \frac{1}{2} \times 0.284 \frac{kg}{sec} \times 0.152 \text{ m} \times 4.48 \frac{m}{sec} = 0.0967 \text{ N·m (per arm)}$$

For two arms, $T_2 = 2T = 2 \times 0.0967 \text{ N·m} = 0.193 \text{ N·m}$ $\longleftarrow T_2$

When torque is removed, angular acceleration would be the same for each arm. Thus

$$\dot{\omega} = \frac{\dot{m}RV}{2} \times \frac{3}{\rho A R^3} = \frac{3 \dot{m} V}{2 \rho A R^2}$$

$$\dot{\omega} = \frac{3}{2} \times 0.284 \frac{kg}{s} \times 4.48 \frac{m}{s} \times \frac{m^3}{999 \, kg} \times \frac{4}{\pi (0.00635)^2 m^2} \times \frac{1}{(0.152)^2 m^2} = 2610 \text{ rad/s}^2$$ $\longleftarrow \dot{\omega}$

Problem *4.184

Given: Simplified lawn sprinkler rotating in horizontal plane, $Q = 4.5$ gal/min.

Water discharges horizontally from jets.

Neglect pivot friction, inertia of sprinkler.

Find: (a) Derive a differential equation for angular speed as a function of time.
(b) Evaluate steady-state angular speed.

Solution: Choose rotating CV. Apply angular momentum principle, Eq. 4.53.

Basic equation:

$$\vec{r}\times\vec{F}_s + \int_{CV}\vec{r}\times\vec{g}\,\rho\,d\forall + \vec{T}_{shaft} - \int_{CV}\vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \dot{\vec{\omega}}\times\vec{r}\right]\rho\,d\forall$$

$$= \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}_{xyz}\,\rho\,d\forall + \int_{CS}\vec{r}\times\vec{V}_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

(terms marked $=0(1)$, $=0(2)$, $=0(3)$, $=0(4)$, $=0(5)$)

Assumptions: (1) $\vec{F}_s = 0$, (2) Body torques cancel, (3) $\vec{T}_{shaft} = 0$, (4) No $\hat{k}$ component of centripetal acceleration, (5) Steady flow, (6) $L \ll R$

Analyze right arm of sprinkler. From geometry, $\vec{r} = r\hat{i}$ in CV, $\vec{r} = R\hat{i}$ at jet. Then

$$-\int_{CV} r\hat{i}\times\left[2\omega\hat{k}\times V\hat{i} + \dot{\omega}\hat{k}\times r\hat{i}\right]\rho A\,dr = R\hat{i}\times V\hat{j}\,\rho\frac{Q}{2} = \rho\frac{QRV}{2}\hat{k}$$

$$r\hat{i}\times\left[2\omega V(+\hat{j}) + \dot{\omega}r(+\hat{j})\right] = (2\omega rV + \dot{\omega}r^2)(+\hat{k});\ -\int_{CV} = -\left(\omega R^2 V + \dot{\omega}\frac{R^3}{3}\right)\rho A$$

Dropping $\hat{k}$,

$$-\omega\rho VAR^2 - \frac{\dot{\omega}\rho AR^3}{3} = \frac{\rho QRV}{2} \quad\text{or}\quad \dot{\omega} = \frac{3}{\rho AR^3}\left[-\omega\rho VAR^2 - \frac{\rho QRV}{2}\right] \qquad \text{O.D.E.}$$

Thus $\frac{d\omega}{dt} = -a - b\omega$, where $a = \frac{3}{\rho AR^3}\frac{\rho QRV}{2} = \frac{3}{2}\frac{QV}{AR^2} = \frac{3V^2}{R^2}$, $b = \frac{3\rho VAR^2}{\rho AR^3} = \frac{3V}{R}$

$\frac{d\omega}{dt} = 0$ when $-a - b\omega_{max} = 0$, i.e., when $\omega_{max} = -a/b$. ↑ Note $V = \frac{Q}{2A}$ (one arm)

$$Q = 4.5\,\frac{gal}{min}\times 231\,\frac{in^3}{gal}\times(0.0254)^3\,\frac{m^3}{in^3}\times\frac{min}{60\,s} = 2.84\times10^{-4}\ m^3/s$$

$$\omega_{max} = -\frac{a}{b} = -\frac{3V^2}{R^2}\times\frac{R}{3V} = -\frac{V}{R} = -4.48\,\frac{m}{s}\times\frac{1}{0.152\,m} = -29.5\ rad/s \quad (-281\ rpm) \qquad \omega_{max}$$

{Note it is not necessary to solve the differential equation to find ω_{max}.}

Given: Simplified lawn sprinkler rotating in horizontal plane, $Q = 4.5$ gal/min.

Water discharges horizontally from jets.

Neglect inertia of sprinkler; $T_f = 0.045$ ft·lbf

Find: (a) Derive a differential equation for angular speed as a function of time.
(b) Evaluate steady-state angular speed.

Solution: Choose rotating CV. Apply angular momentum principle, Eq. 4.53.

Basic equation:
$$\vec{r}\times\vec{F}_s + \int_{CV}\vec{r}\times\vec{g}\,\rho\,dV + \vec{T}_{shaft} - \int_{CV}\vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \dot{\vec{\omega}}\times\vec{r}\right]\rho\,dV = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}_{xyz}\,\rho\,dV + \int_{CS}\vec{r}\times\vec{V}_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$
($\vec{r}\times\vec{F}_s = 0(1)$; $\int_{CV}\vec{r}\times\vec{g}\rho dV = 0(2)$; $\vec{\omega}\times(\vec{\omega}\times\vec{r}) = 0(4)$; $\frac{\partial}{\partial t} = 0(5)$)

Assumptions: (1) $\vec{F}_s = 0$, (2) Body torques cancel, (3) $\vec{T}_{shaft} = 0.045$ ft·lbf, (4) No $\hat{k}$ component of centripetal acceleration, (5) Steady flow, (6) $L \ll R$.

Analyze right arm of sprinkler. From geometry, $\vec{r} = r\hat{i}$ in CV, $\vec{r} = R\hat{i}$ at jet. Then
$$-\int_{CV} r\hat{i}\times[2\omega\hat{k}\times V\hat{i} + \dot{\omega}\hat{k}\times r\hat{i}]\rho A\,dr = R\hat{i}\times V\hat{j}\,\rho\frac{Q}{2} = \rho\frac{QRV}{2}\hat{k}$$
$$r\hat{i}\times[2\omega V\hat{j} + \dot{\omega}r\hat{j}] = (2\omega Vr + \dot{\omega}r^2)\hat{k};\quad -\int_{CV} = -\left(\omega VR^2 + \frac{\dot{\omega}R^3}{3}\right)\rho A\,\hat{k}$$

For both arms, dropping $\hat{k}$, $\{T = 0.045 \text{ ft·lbf} = 0.0610 \text{ N·m}\}$
$$T - 2\omega\rho VAR^2 - \frac{2\dot{\omega}\rho AR^3}{3} = \rho QRV \quad\text{or}\quad \dot{\omega} = \frac{3}{2\rho AR^3}\left[T - \rho QRV - 2\omega\rho VAR^2\right]$$ O.D.E.

Thus $\frac{d\omega}{dt} = a - b\omega$, where $a = \frac{3}{2\rho AR^3}(T - \rho QRV)$; $b = \frac{3}{2\rho AR^3}\times 2\rho VAR^2 = 3\frac{V}{R}$

The steady-state speed occurs when $\frac{d\omega}{dt} = 0$, i.e. when $\omega_{max} = \frac{a}{b}$

$$Q = 4.5\frac{\text{gal}}{\text{min}}\times 231\frac{\text{in}^3}{\text{gal}}\times(0.0254)^3\frac{\text{m}^3}{\text{in}^3}\times\frac{\text{min}}{60\text{ s}} = 2.84\times10^{-4}\text{ m}^3/\text{s};\quad A = \frac{\pi d^2}{4} = 3.17\times10^{-5}\text{ m}^2$$

From the O.D.E., $\omega_{max} = \frac{T - \rho QRV}{2\rho VAR^2}$

$$\omega_{max} = \frac{1}{2}\left[0.0610\text{ N·m}\times\frac{\text{kg·m}}{\text{N·s}^2} - 999\frac{\text{kg}}{\text{m}^3}\times 2.84\times10^{-4}\frac{\text{m}^3}{\text{s}}\times 0.152\text{ m}\times 4.48\frac{\text{m}}{\text{s}}\right]\frac{\text{m}^3}{999\text{ kg}}\times\frac{\text{s}}{4.48\text{ m}}\times\frac{1}{3.17\times10^{-5}\text{ m}^2}\times\frac{1}{(0.152)^2\text{ m}^2}$$

$\omega_{max} = -20.2$ rad/s $(-193$ rpm$)$ ω_{max}

Problem *4.186

*4.186 Water flows in a uniform flow out of the 5 mm slots of the rotating spray system as shown. The flow rate is 15 kg/s. Find the torque required to hold the system stationary, and the steady-state speed of rotation after it is released.

Given: Data on rotating spray system

Find: Torque required to hold stationary; steady-state speed

Solution:

The given data is $\rho = 999\cdot\frac{kg}{m^3}$ $\quad m_{flow} = 15\cdot\frac{kg}{s}$ $\quad D = 0.015\cdot m$ $\quad r_o = 0.25\cdot m$ $\quad r_i = 0.05\cdot m$ $\quad \delta = 0.005\cdot m$

Governing equation: Rotating CV

$$\vec{r}\times\vec{F}_s + \int_{CV}\vec{r}\times\vec{g}\,\rho\,dV + \vec{T}_{shaft} - \int_{CV}\vec{r}\times[2\vec{\omega}\times\vec{V}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \dot{\vec{\omega}}\times\vec{r}]\,\rho\,dV = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}_{xyz}\,\rho\,dV + \int_{CS}\vec{r}\times\vec{V}_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A} \quad (4.52)$$

For no rotation (ω = 0) this equation reduces to a single scalar equation

$$T_{shaft} = \int \vec{r}\times\vec{V}_{xyz}\cdot\rho\cdot\vec{V}_{xyz}\,d\vec{A}$$

or

$$T_{shaft} = 2\cdot\delta\cdot\int_{r_i}^{r_o} r\cdot V\cdot\rho\cdot V\,dr = 2\cdot\rho\cdot V^2\cdot\delta\cdot\int_{r_i}^{r_o} r\,dr = \rho\cdot V^2\cdot\delta\cdot\left(r_o^{\,2} - r_i^{\,2}\right)$$

where V is the exit velocity with respect to the CV

$$V = \frac{\frac{m_{flow}}{\rho}}{2\cdot\delta\cdot\left(r_o - r_i\right)}$$

Hence

$$T_{shaft} = \rho\cdot\left[\frac{\frac{m_{flow}}{\rho}}{2\cdot\delta\cdot\left(r_o - r_i\right)}\right]^2\cdot\delta\cdot\left(r_o^{\,2} - r_i^{\,2}\right) \qquad T_{shaft} = \frac{m_{flow}^{\,2}}{4\cdot\rho\cdot\delta}\cdot\frac{\left(r_o + r_i\right)}{\left(r_o - r_i\right)}$$

$$T_{shaft} = \frac{1}{4}\times\left(15\cdot\frac{kg}{s}\right)^2\times\frac{m^3}{999\cdot kg}\times\frac{1}{0.005\cdot m}\times\frac{(0.25 + 0.05)}{(0.25 - 0.05)} \qquad T_{shaft} = 16.9\,N\cdot m$$

For the steady rotation speed the equation becomes

$$-\int \vec{r}\times\left(2\cdot\vec{\omega}\times\vec{V}_{xyz}\right)\cdot\rho\,dV = \int \vec{r}\times\vec{V}_{xyz}\cdot\rho\cdot\vec{V}_{xyz}\,d\vec{A}$$

The volume integral term $-\int \vec{r}\times\left(2\cdot\vec{\omega}\times\vec{V}_{xyz}\right)\cdot\rho\,dV$ must be evaluated for the CV. The velocity in the CV varies with r. This variation can be found from mass conservation

For an infinitesimal CV of length dr and cross-section A at radial position r, if the flow in is Q, the flow out is $Q + dQ$, and the loss through the slot is $V\delta dr$. Hence mass conservation leads to

$$(Q + dQ) + V\cdot\delta\cdot d\,dQ = -V\cdot\delta\cdot dr \qquad Q(r) = -V\cdot\delta\cdot r + \text{const}$$

At the inlet ($r = r_i$) $\quad Q = Q_i = \dfrac{m_{flow}}{2\cdot\rho}$

Hence $\quad Q = Q_i + V\cdot\delta\cdot(r_i - r) = \dfrac{m_{flow}}{2\cdot\rho} + \dfrac{m_{flow}}{2\cdot\rho\cdot\delta\cdot(r_o - r_i)}\cdot\delta\cdot(r_i - r) \qquad Q = \dfrac{m_{flow}}{2\cdot\rho}\cdot\left(1 + \dfrac{r_i - r}{r_o - r_i}\right) = \dfrac{m_{flow}}{2\cdot\rho}\cdot\left(\dfrac{r_o - r}{r_o - r_i}\right)$

and along each rotor the water speed is $\quad v(r) = \dfrac{Q}{A} = \dfrac{m_{flow}}{2\cdot\rho\cdot A}\cdot\left(\dfrac{r_o - r}{r_o - r_i}\right)$

Hence the term $-\int \vec{r}\times\left(2\cdot\vec{\omega}\times\overrightarrow{V_{xyz}}\right)\cdot\rho\, dV$ becomes

$$-\int \vec{r}\times\left(2\cdot\vec{\omega}\times\overrightarrow{V_{xyz}}\right)\cdot\rho\, dV = 4\cdot\rho\cdot A\cdot\omega\cdot\int_{r_i}^{r_o} r\cdot v(r)\, dr = 4\cdot\rho\cdot\omega\cdot\int_{r_i}^{r_o} r\cdot\frac{m_{flow}}{2\cdot\rho}\cdot\left(\frac{r_o - r}{r_o - r_i}\right) dr$$

or

$$-\int \vec{r}\times\left(2\cdot\vec{\omega}\times\overrightarrow{V_{xyz}}\right)\cdot\rho\, dV = 2\cdot m_{flow}\cdot\omega\cdot\int_{r_i}^{r_o} r\cdot\left(\frac{r_o - r}{r_o - r_i}\right) dr = m_{flow}\cdot\omega\cdot\frac{r_o^{\,3} + r_i^{\,2}\cdot(2\cdot r_i - 3\cdot r_o)}{3\cdot(r_o - r_i)}$$

Recall that

$$\int \vec{r}\times\overrightarrow{V_{xyz}}\cdot\rho\cdot\overrightarrow{V_{xyz}}\, d\vec{A} = \rho\cdot V^2\cdot\delta\cdot\left(r_o^{\,2} - r_i^{\,2}\right)$$

Hence equation

$$-\int \vec{r}\times\left(2\cdot\vec{\omega}\times\overrightarrow{V_{xyz}}\right)\cdot\rho\, dV = \int \vec{r}\times\overrightarrow{V_{xyz}}\cdot\rho\cdot\overrightarrow{V_{xyz}}\, d\vec{A}$$

becomes

$$m_{flow}\cdot\omega\cdot\frac{r_o^{\,3} + r_i^{\,2}\cdot(2\cdot r_i - 3\cdot r_o)}{3\cdot(r_o - r_i)} = \rho\cdot V^2\cdot\delta\cdot\left(r_o^{\,2} - r_i^{\,2}\right)$$

Solving for ω

$$\omega = \frac{3\cdot(r_o - r_i)\cdot\rho\cdot V^2\cdot\delta\cdot\left(r_o^{\,2} - r_i^{\,2}\right)}{m_{flow}\cdot\left[r_o^{\,3} + r_i^{\,2}\cdot(2\cdot r_i - 3\cdot r_o)\right]} \qquad \omega = 461\,\text{rpm}$$

Problem *4.187

[3]

4.187 If the same flow rate in the rotating spray system of Problem 4.186 is not uniform but instead varies linearly from a maximum at the outer radius to zero at a point 50 mm from the axis, find the torque required to hold it stationary, and the steady-state speed of rotation.

Given: Data on rotating spray system

Find: Torque required to hold stationary; steady-state speed

Solution:

The given data is $\rho = 999 \cdot \frac{kg}{m^3}$ $\quad m_{flow} = 15 \cdot \frac{kg}{s}$ $\quad D = 0.015 \cdot m$ $\quad r_o = 0.25 \cdot m$ $\quad r_i = 0.05 \cdot m$ $\quad \delta = 0.005 \cdot m$

Governing equation: Rotating CV

$$\vec{r} \times \vec{F}_s + \int_{CV} \vec{r} \times \vec{g}\,\rho\, dV + \vec{T}_{shaft} - \int_{CV} \vec{r} \times [2\vec{\omega} \times \vec{V}_{xyz} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r}]\,\rho\, dV = \frac{\partial}{\partial t}\int_{CV} \vec{r} \times \vec{V}_{xyz}\,\rho\, dV + \int_{CS} \vec{r} \times \vec{V}_{xyz}\,\rho \vec{V}_{xyz} \cdot d\vec{A} \qquad (4.52)$$

For no rotation ($\omega = 0$) this equation reduces to a single scalar equation

$$T_{shaft} = \int \vec{r} \times \vec{V_{xyz}} \rho \cdot \vec{V_{xyz}} d\vec{A} \qquad \text{or} \qquad T_{shaft} = 2 \cdot \delta \cdot \int_{r_i}^{r_o} r \cdot V \cdot \rho \cdot V\, dr$$

where *V* is the exit velocity with respect to the CV. We need to find *V*(*r*). To do this we use mass conservation, and the fact that the distribution is linear

$$V(r) = V_{max} \cdot \frac{(r - r_i)}{(r_o - r_i)} \qquad \text{and} \qquad 2 \cdot \frac{1}{2} \cdot V_{max} \cdot (r_o - r_i) \cdot \delta = \frac{m_{flow}}{\rho}$$

so

$$V(r) = \frac{m_{flow}}{\rho \cdot \delta} \cdot \frac{(r - r_i)}{(r_o - r_i)^2}$$

Hence

$$T_{shaft} = 2 \cdot \rho \cdot \delta \cdot \int_{r_i}^{r_o} r \cdot V^2\, dr = 2 \cdot \frac{m_{flow}^2}{\rho \cdot \delta} \cdot \int_{r_i}^{r_o} r \cdot \left[\frac{(r - r_i)}{(r_o - r_i)^2}\right]^2 dr \qquad T_{shaft} = \frac{m_{flow}^2 \cdot (r_i + 3 \cdot r_o)}{6 \cdot \rho \cdot \delta \cdot (r_o - r_i)}$$

$$T_{shaft} = \frac{1}{6} \times \left(15 \cdot \frac{kg}{s}\right)^2 \times \frac{m^3}{999 \cdot kg} \times \frac{1}{0.005 \cdot m} \times \frac{(0.05 + 3 \cdot 0.25)}{(0.25 - 0.05)} \qquad T_{shaft} = 30 \cdot N \cdot m$$

For the steady rotation speed the equation becomes

$$-\int \vec{r} \times (2 \cdot \vec{\omega} \times \vec{V_{xyz}}) \cdot \rho\, dV = \int \vec{r} \times \vec{V_{xyz}} \cdot \rho \cdot \vec{V_{xyz}}\, d\vec{A}$$

The volume integral term $-\int \vec{r} \times \left(2\cdot\vec{\omega}\times\overrightarrow{V_{xyz}}\right)\cdot\rho\, dV$ must be evaluated for the CV. The velocity in the CV varies with *r*. This variation can be found from mass conservation

For an infinitesmal CV of length *dr* and cross-section *A* at radial position *r*, if the flow in is *Q*, the flow out is *Q* + *dQ*, and the loss through the slot is *V*δdr Hence mass conservation leads to

$$(Q + dQ) + V\cdot\delta\cdot dr - Q = 0 \qquad dQ = -V\cdot\delta\cdot dr \qquad Q(r) = Q_i - \delta\cdot\int_{r_i}^{r} \frac{m_{flow}}{\rho\cdot\delta}\cdot\frac{(r - r_i)}{(r_o - r_i)^2}\, dr = Q_i - \int_{r_i}^{r} \frac{m_{flow}}{\rho}\cdot\frac{(r - r_i)}{(r_o - r_i)^2}\, dr$$

At the inlet ($r = r_i$) $\qquad Q = Q_i = \dfrac{m_{flow}}{2\cdot\rho}$

Hence $\qquad Q(r) = \dfrac{m_{flow}}{2\cdot\rho}\cdot\left[1 - \dfrac{(r - r_i)^2}{(r_o - r_i)^2}\right]$

and along each rotor the water speed is $\qquad v(r) = \dfrac{Q}{A} = \dfrac{m_{flow}}{2\cdot\rho\cdot A}\cdot\left[1 - \dfrac{(r - r_i)^2}{(r_o - r_i)^2}\right]$

Hence the term $-\int \vec{r} \times \left(2\cdot\vec{\omega}\times\overrightarrow{V_{xyz}}\right)\cdot\rho\, dV$ becomes

$$4\cdot\rho\cdot A\cdot\omega\cdot\left(\int_{r_i}^{r_o} r\cdot v(r)\, dr\right) = 4\cdot\rho\cdot\omega\cdot\int_{r_i}^{r_o} \frac{m_{flow}}{2\cdot\rho}\cdot r\cdot\left[1 - \frac{(r - r_i)^2}{(r_o - r_i)^2}\right] dr$$

or

$$2\cdot m_{flow}\cdot\omega\cdot\int_{r_i}^{r_o} r\cdot\left[1 - \frac{(r_o - r)^2}{(r_o - r_i)^2}\right] dr = m_{flow}\cdot\omega\cdot\left(\frac{1}{6}\cdot r_o^2 + \frac{1}{3}\cdot r_i\cdot r_o - \frac{1}{2}\cdot r_i^2\right)$$

Recall that $\qquad \int \vec{r}\times\overrightarrow{V_{xyz}}\cdot\rho\cdot\overrightarrow{V_{xyz}}\, d\vec{A} = \dfrac{m_{flow}^2\cdot(r_i + 3\cdot r_o)}{6\cdot(r_o - r_i)\cdot\rho\cdot\delta}$

Hence equation $\qquad -\int \vec{r} \times \left(2\cdot\vec{\omega}\times\overrightarrow{V_{xyz}}\right)\cdot\rho\, dV = \int \vec{r}\times\overrightarrow{V_{xyz}}\cdot\rho\cdot\overrightarrow{V_{xyz}}\, d\vec{A}$

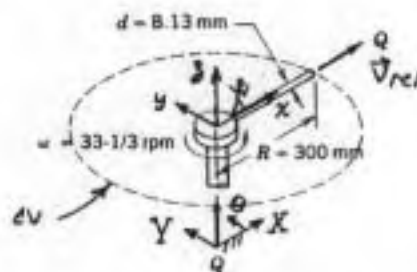
becomes $\qquad m_{flow}\cdot\omega\cdot\left(\dfrac{1}{6}\cdot r_o^2 + \dfrac{1}{3}\cdot r_i\cdot r_o - \dfrac{1}{2}\cdot r_i^2\right) = \dfrac{m_{flow}^2\cdot(r_i + 3\cdot r_o)}{6\cdot(r_o - r_i)\cdot\rho\cdot\delta}$

Solving for ω $\qquad \omega = \dfrac{m_{flow}\cdot(r_i + 3\cdot r_o)}{\left(r_o^2 + 2\cdot r_i\cdot r_o - 3\cdot r_i^2\right)\cdot(r_o - r_i)\cdot\rho\cdot\delta} \qquad \omega = 1434\cdot rpm$

Problem *4.188

Given: Single rotating tube with water.

$Q = 13.8$ L/min

Find: Torque that must be applied to maintain steady rotation using:
(a) Rotating control volume.
(b) Fixed control volume.

Solution: Apply angular momentum principle. $\left\{\omega = 33\tfrac{1}{3}\,\frac{rev}{min} = 3.49\ rad/s\right\}$

(a) Rotating CV: use relative velocities, Eq. 4.53:

Basic equation:
$$\vec{r}\times\vec{F}_s + \int_{CV}\vec{r}\times\vec{g}\,\rho\,d\forall + \vec{T}_{shaft} - \int_{CV}\vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \dot{\vec{\omega}}\times\vec{r}\right]\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}_{xyz}\,\rho\,d\forall + \int_{CS}\vec{r}\times\vec{V}_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$
(terms marked $=0(1)$, $=0(2)$, $=0(3)$, $=0(4)$, $=0(5)$, $=0(6)$)

Assumptions: (1) $\vec{F}_s = 0$, (2) Body torques cancel, (3) No $\hat{k}$ in centripetal accel, (4) $\dot{\vec{\omega}} = 0$, (5) Steady flow, (6) $\vec{r}\times\vec{V} = 0$

Then
$$T_{shaft}\hat{k} = \int_{CV}\vec{r}\times(2\vec{\omega}\times\vec{V})\rho\,d\forall = \int_0^R r\hat{i}\times(2\omega\hat{k}\times V\hat{i})\rho A\,dr = \omega\rho V A R^2\hat{k} = \omega\rho Q R^2\hat{k}$$

$$T_{shaft} = 3.49\,\frac{rad}{s}\times 999\,\frac{kg}{m^3}\times 13.8\times10^{-3}\,\frac{m^3}{min}\times(0.3)^2 m^2\times\frac{min}{60\,s}\times\frac{N\cdot s^2}{kg\cdot m} = 0.0722\ N\cdot m$$ T

(b) Fixed control volume: use absolute velocities, Eq. 4.47:

Basic equation:
$$\vec{r}\times\vec{F}_s + \int_{CV}\vec{r}\times\vec{g}\,\rho\,d\forall + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}\rho\,d\forall + \int_{CS}\vec{r}\times\vec{V}\rho\vec{V}_{xyz}\cdot d\vec{A}$$
(terms marked $=0(1)$, $=0(2)$)

Relative to fixed coordinates XY, $\vec{r} = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

$$\vec{V} = V(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) + r\omega(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$

$$\vec{r}\times\vec{V} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ r\cos\theta & r\sin\theta & 0\\ V\cos\theta - r\omega\sin\theta & V\sin\theta + r\omega\cos\theta & 0\end{vmatrix} = \hat{k}\left(rV\sin\theta\cos\theta + \omega^2 r^2\cos^2\theta - rV\sin\theta\cos\theta + \omega^2 r^2\sin^2\theta\right) = \omega r^2\hat{k}$$

Thus $\partial/\partial t = 0$ and $\int_{CS}\vec{r}\times\vec{V}\rho\vec{V}_{xyz}\cdot d\vec{A} = \omega R^2\hat{k}\{+\rho Q\} = \omega\rho Q R^2\hat{k}$ and

$T_{shaft}\hat{k} = \omega\rho Q R^2\hat{k}$ (as before); $T = 0.0722\ N\cdot m$ T

{Note that when applied correctly, either choice of CV produces the same result.}

Problem *4.189

Given: Lawn sprinkler rotating in horizontal plane.

Neglect friction. $Q = 68\ L/min$

Find: steady-state angular speed for $\theta = 30°$.

Plot: steady-state angular speed for $0 \le \theta \le 90°$.

Solution: Choose rotating CV. Apply angular momentum principle, Eq. 4.53.

Basic equation:

$$\vec{r}\times\vec{F}_s + \int_{CV}\vec{r}\times\vec{g}\,\rho\,d\forall + \vec{T}_{shaft} - \int_{CV}\vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \dot{\vec{\omega}}\times\vec{r}\right]\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}_{xyz}\,\rho\,d\forall + \int_{CS}\vec{r}\times\vec{V}_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

(with $\vec{r}\times\vec{F}_s = 0(1)$, $\int_{CV}\vec{r}\times\vec{g}\rho d\forall = 0(2)$, $\vec{T}_{shaft} = 0(3)$, $\vec{\omega}\times(\vec{\omega}\times\vec{r}) = 0(5)$, $\dot{\vec{\omega}}\times\vec{r} = 0(6)$, $\frac{\partial}{\partial t} = 0(6)$)

Assumptions: (1) $\vec{F}_s = 0$, (2) Body torques cancel, (3) $\vec{T}_{shaft} = 0$, (4) Neglect aerodynamic drag, (5) No $\hat{k}$ component of centripetal acceleration, (6) Steady flow, (7) $L \ll R$

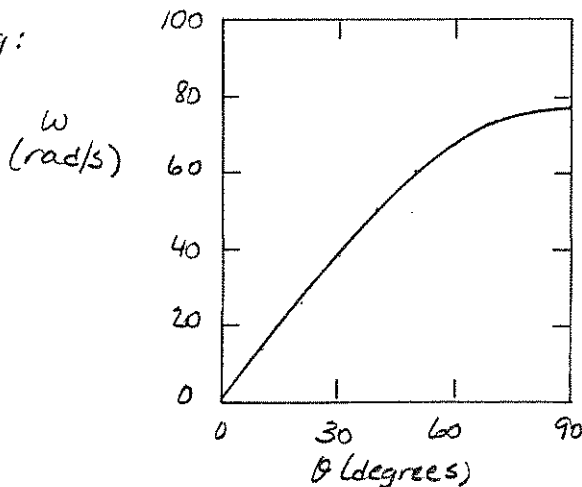
Analyze one arm of sprinkler. From geometry, $\vec{r} = r\hat{i}$ in CV, $\vec{r} = R\hat{i}$ at jet. Then

$$-\int_{CV}\vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz}\right]\rho\,d\forall = R\hat{i}\times(-V\sin\theta\,\hat{j})\,\rho\frac{Q}{3} = -\rho\frac{QRV}{3}\sin\theta\,\hat{k}$$

$$r\hat{i}\times(2\omega\hat{k}\times V\hat{i}) = 2\omega V r\,\hat{k}\ ;\ -\int_{CV} = -\omega V R^2\rho A\,\hat{k}$$

Dropping $\hat{k}$, $\quad -\omega V R^2\rho A = -\rho\frac{QRV}{3}\sin\theta$, so with $VA = Q/3$,

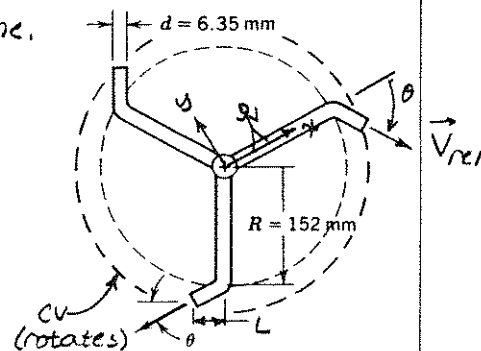
$$\omega = \frac{V}{R}\sin\theta\ ;\ V = \frac{Q}{3A} = \frac{4Q}{3\pi d^2} = \frac{4}{3\pi}\times 68\times10^{-3}\frac{m^3}{min}\times\frac{1}{(0.00635)^2 m^2}\times\frac{min}{60\ s} = 11.9\ m/s$$

$$\omega = 11.9\frac{m}{s}\times\frac{1}{0.152\ m}\times\sin\theta = 78.3\sin\theta\ rad/s$$

ω

Plotting:

For $\theta = 30°$,

$\omega = 78.3\sin 30°$

$\omega = 39.1\ rad/s$

$\omega\ (\theta = 30°)$

Plot

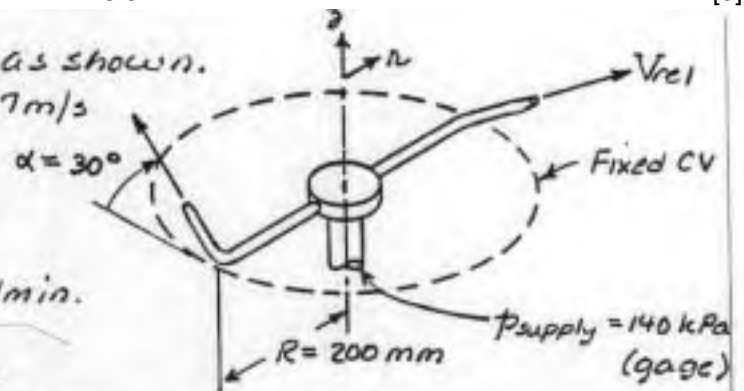
Problem *4.190

Given: Small lawn sprinkler as shown.

$V_{rel} = 17 \text{ m/s}$

Friction torque at pivot is $T_f = 0.18 \text{ N}\cdot\text{m}$.

Flowrate is $Q = 4.0 \text{ liter/min}$.

$\alpha = 30°$, $R = 200 \text{ mm}$, $p_{supply} = 140 \text{ kPa (gage)}$, Fixed CV

Find: Torque to hold stationary.

Solution: Apply moment of momentum using fixed CV enclosing sprinkler arms.

Basic equation:

$$\vec{r} \times \vec{F}_s \overset{=0(1)}{} + \int_{CV} \vec{r} \times \vec{g}\, \rho\, d\forall \overset{=0(2)}{} + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV} \vec{r} \times \vec{V}\, \rho\, d\forall \overset{=0(3)}{} + \int_{CS} \vec{r} \times \vec{V} \rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) Neglect torque due to surface forces
(2) Torques due to body forces cancel by symmetry
(3) Steady flow
(4) Uniform flow leaving each jet

Then

$$-T_f \hat{k} = (\vec{r} \times \vec{V})_{in} \{-\rho Q\} + 2(\vec{r} \times \vec{V})_{jet} \{\tfrac{1}{2}\rho Q\}$$

$(\vec{r} \times \vec{V})_{in} \approx 0$ $\qquad \vec{r} = R\hat{e}_r$

$\vec{V} = (R\omega - V_{rel}\cos\alpha)\hat{e}_\theta + V_{rel}\sin\alpha\, \hat{e}_z$

The absolute velocity of the jet leaving sprinkler is $\vec{V} = V_{rel}[\cos\alpha(-\hat{e}_\theta) + \sin\alpha(\hat{e}_z)]$

Then $(\vec{r} \times \vec{V})_z = \{R\hat{e}_r \times V_{rel}[\cos\alpha(-\hat{e}_\theta) + \sin\alpha(\hat{e}_z)]\}_z = \{RV_{rel}\cos\alpha(-\hat{e}_z) + RV_{rel}\sin\alpha(-\hat{e}_\theta)\}_z$

$(\vec{r} \times \vec{V})_z = -RV_{rel}\cos\alpha$

Substituting, $T_{shaft} = T_{ext} - T_f = 2(-RV_{rel}\cos\alpha)(\tfrac{1}{2}\rho Q)$

Thus $T_{ext} = T_f - \rho Q R V_{rel}\cos\alpha$

$$= 0.18 \text{ N}\cdot\text{m} - 999\frac{\text{kg}}{\text{m}^3} \times 4\frac{\text{L}}{\text{min}} \times 0.2\text{ m} \times 17\frac{\text{m}}{\text{s}} \times 0.866 \times \frac{\text{m}^3}{1000\text{ L}} \times \frac{\text{min}}{60\text{ s}} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$T_{ext} = -0.0161 \text{ N}\cdot\text{m}$ (to hold sprinkler stationary) $\leftarrow T_{ext}$

{ Since $T_{ext} < 0$, it must be applied in the minus z direction to oppose motion. }

Given: Small lawn sprinkler as shown.

$V_{rel} = 17\ m/s$

Friction torque at pivot is zero. $I = 0.1\ kg \cdot m^2$

Flowrate is $Q = 4.0\ liter/min$.

$\alpha = 30°$

$R = 200\ mm$

$p_{supply} = 140\ kPa$ (gage)

Find: Initial angular acceleration from rest.

Solution: Apply moment of momentum using fixed CV enclosing sprinkler arms.

Basic equation:

$$\vec{r} \times \vec{F}_s + \int_{CV} \vec{r} \times \vec{g}\, \rho\, dV + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV} \vec{r} \times \vec{V} \rho\, dV + \int_{CS} \vec{r} \times \vec{V} \rho \vec{V} \cdot d\vec{A}$$

(terms marked $=0(1)$, $=0(2)$, $=0(3)$)

Assumptions: (1) Neglect torque due to surface forces
(2) Torques due to body forces cancel by symmetry
(3) Steady flow
(4) Uniform flow leaving each jet

Then

$$-T_f \hat{k} = (\vec{r} \times \vec{V})_{in}\{-\rho Q\} + 2(\vec{r} \times \vec{V})_{jet}\{\tfrac{1}{2}\rho Q\}$$

$$(\vec{r} \times \vec{V})_{in} \approx 0 \qquad \vec{r} = R\hat{e}_r$$

$$\vec{V} = (R\omega - V_{rel}\cos\alpha)\hat{e}_\theta + V_{rel}\sin\alpha\, \hat{e}_z$$

The jet leaves the sprinkler at $\vec{V}(abs) = V_{rel}[\cos\alpha(-\hat{e}_\theta) + \sin\alpha(\hat{e}_z)]$

Then $\vec{r} \times \vec{V} = R\hat{e}_r \times V_{rel}[\cos\alpha(-\hat{e}_\theta) + \sin\alpha(\hat{e}_z)] = RV_{rel}[\cos\alpha(-\hat{e}_z) + \sin\alpha(-\hat{e}_\theta)]$

Summing moments on the rotor, $\Sigma \vec{M} = I\dot{\vec{\omega}}$. Thus

$$\dot{\omega} = \frac{\Sigma T}{I} = \frac{\rho Q R V_{rel}\cos\alpha - T_f}{I}$$

$$= \left[\frac{999\,kg}{m^3} \times \frac{4\,L}{min} \times 0.2\,m \times \frac{17\,m}{s} \times 0.866 \times \frac{m^3}{1000\,L} \times \frac{min}{60\,s} - 0.18\,N \cdot m \times \frac{kg \cdot m}{N \cdot s^2}\right]\frac{1}{0.1\,kg \cdot m^2}$$

$$\dot{\omega} = 0.161\ rad/s^2 \qquad \dot{\omega}$$

{It is not necessary to use a rotating CV, because at the instant considered, $\vec{\omega} = 0$ and I is known.}

Problem *4.192

Given: Small lawn sprinkler as shown.

$V_{rel} = 17\ m/s$

Friction torque at pivot is $T_f = 0.18\ N \cdot m$.

Flowrate is $Q = 4.0\ liter/min$.

$\alpha = 30°$, $R = 200\ mm$, $p_{supply} = 140\ kPa$ (gage)

Find: (a) Steady speed of rotation.
(b) Area covered by spray.

Solution: Apply moment of momentum using fixed CV enclosing sprinkler arms.

Basic equation:

$$\vec{r} \times \vec{F}_s + \int_{CV} \vec{r} \times \vec{g}\,\rho\, dV + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV} \vec{r} \times \vec{V}\,\rho\, dV + \int_{CS} \vec{r} \times \vec{V}\,\rho\,\vec{V} \cdot d\vec{A}$$

(first term $= 0(1)$, second term $= 0(2)$, time derivative term $= 0(3)$)

Assumptions: (1) Neglect torque due to surface forces
(2) Torques due to body forces cancel by symmetry
(3) Steady flow
(4) Uniform flow leaving each jet

Then

$$-T_f \hat{k} = (\vec{r} \times \vec{V})_{in}\{-\rho Q\} + 2(\vec{r} \times \vec{V})_{jet}\{\tfrac{1}{2}\rho Q\}$$

$(\vec{r} \times \vec{V})_{in} \approx 0$ $\qquad \vec{r} = R\hat{e}_r$

$\vec{V} = (R\omega - V_{rel}\cos\alpha)\hat{e}_\theta + V_{rel}\sin\alpha\,\hat{e}_z$

$(\vec{r} \times \vec{V})_z = R(R\omega - V_{rel}\cos\alpha)$

or

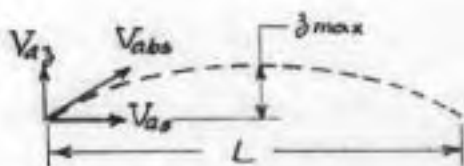
$$-T_f = R(R\omega - V_{rel}\cos\alpha)\rho Q$$

Thus

$$\omega = \frac{V_{rel}\cos\alpha}{R} - \frac{T_f}{\rho Q R^2}$$

$$= 17\frac{m}{s} \times \frac{\cos 30°}{0.2\,m} - 0.18\,N\cdot m \times \frac{m^3}{999\,kg} \times \frac{min}{4.0\,L} \times \frac{1}{(0.2)^2 m^2} \times \frac{60\,s}{min} \times \frac{10^3\,L}{m^3} \times \frac{kg \cdot m}{N \cdot s^2}$$

$$\omega = 6.04\,\frac{rad}{s} \quad \text{or} \quad 57.7\ rpm \qquad \omega$$

Treat the spray outside each nozzle as moving without air resistance:

For each particle, $\frac{dV_{az}}{dt} = -g$, so $V_z = V_{z0} - gt$

At z_{max}, $V_z = 0$, so $t = \frac{V_{z0}}{g}$; flight time is $2t$.

$$L = 2t\,V_{a\theta} = \frac{2V_{az}V_{a\theta}}{g} = \frac{2V_{rel}\sin\alpha\,(V_{rel}\cos\alpha - R\omega)}{g}$$

$$L = 2 \times 17\frac{m}{s} \times \sin 30°\left(17\frac{m}{s} \times \cos 30° - 0.2\,m \times 6.04\frac{rad}{s}\right)\frac{s^2}{9.81\,m} = 23.4\ m$$

$R_{spray} = \sqrt{R^2 + L^2} = 23.4\,m$; $A_{spray} = \pi R_{spray}^2 = \pi \times (23.4)^2\,m^2 = 1720\ m^2$ $\qquad A_{spray}$

Problem *4.193

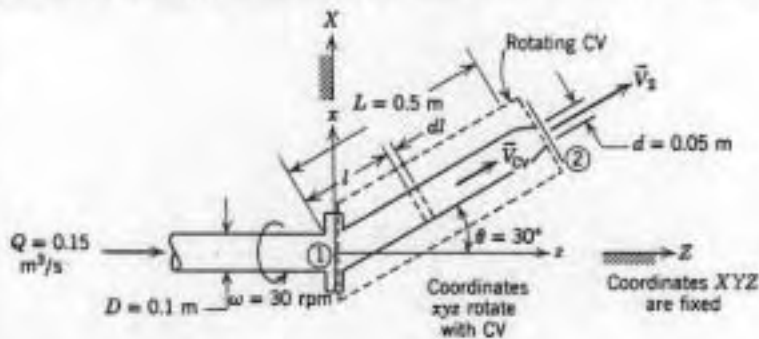
Open-Ended Problem Statement: When a garden hose is used to fill a bucket, water in the bucket may develop a swirling motion. Why does this happen? How could the amount of swirl be calculated approximately?

Discussion: Frequently when filling a bucket the hose is held so that the water stream entering the bucket is not vertical. If, in addition, the water stream is off-center in the bucket, then flow entering the bucket has a tangential component of velocity, a swirl component.

The tangential component of the water velocity entering the bucket has a moment-of-momentum (swirl) with respect to a control volume drawn around the stationary bucket. This entering swirl can only be reduced by a torque acting to oppose it. Viscous forces among fluid layers will tend to transfer swirl to other layers so that eventually all of the water in the bucket has a swirling motion.

Swirl in the bucket may be influenced by viscosity. The swirl may tend to nearly a rigid-body motion to minimize viscous forces between annular layers of water in the bucket. The rigid-body motion assumption may be a reasonable model to calculate the total angular momentum (moment-of-momentum) of the water in the bucket.

Problem *4.194

Given: Nozzle assembly rotating steadily, as shown in the sketch.

Find: (a) Torque required to drive the nozzle assembly
(b) Reaction torques at the flange.

Solution: Apply the moment of momentum equation to the rotating CV shown.

Basic equation:

$$\vec{r}\times\vec{F}_s + \int_{CV}\vec{r}\times\vec{g}\,\rho\,d\forall + \vec{T}_{shaft} \quad (\approx 0(2))$$

$$-\int_{CV}\vec{r}\times[2\vec{\omega}\times\vec{V}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \dot{\vec{\omega}}\times\vec{r}]\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}_{xyz}\,\rho\,d\forall + \int_{CS}\vec{r}\times\vec{V}_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

($\dot{\vec{\omega}}\times\vec{r} = 0(3)$; $\frac{\partial}{\partial t}\int = 0(5)$; $\int_{CS} = 0(7)$)

Assumptions: (1) Let $\vec{T}_{cv}$ represent all torques acting on the CV
(2) Neglect torque due to body force
(3) Constant angular speed
(4) Neglect mass of arm compared to water inside
(5) Steady flow in CV
(6) Neglect nozzle length compared to L
(7) $\vec{r}$ colinear with $\vec{V}$, so $\vec{r}\times\vec{V}_{xyz} = 0$

Then

$$\vec{T}_{cv} = \int_{CV}\vec{r}\times[2\vec{\omega}\times\vec{V}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\vec{r})]\rho\,d\forall$$

Since $\vec{\omega} = \omega\hat{k}$ and $\vec{r} = l(\sin\theta\hat{i} + \cos\theta\hat{k})$, then

$$\vec{\omega}\times\vec{r} = \omega l\sin\theta\hat{j}$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \omega\hat{k}\times\omega l\sin\theta\hat{j} = \omega^2 l\sin\theta(-\hat{i})$$

and $\vec{r}\times[\vec{\omega}\times(\vec{\omega}\times\vec{r})] = l(\sin\theta\hat{i} + \cos\theta\hat{k})\times\omega^2 l\sin\theta(-\hat{i}) = \omega^2 l^2\sin\theta\cos\theta(-\hat{j})$

Since $\vec{V}_{xyz} = V_{cv}(\sin\theta\hat{i} + \cos\theta\hat{k})$, then

$$2\vec{\omega}\times\vec{V}_{xyz} = 2\omega\hat{k}\times V_{cv}(\sin\theta\hat{i} + \cos\theta\hat{k}) = 2\omega V_{cv}\sin\theta\hat{j}$$

and $\vec{r}\times[2\vec{\omega}\times\vec{V}_{xyz}] = l(\sin\theta\hat{i} + \cos\theta\hat{k})\times 2\omega V_{cv}\sin\theta\hat{j} = 2\omega l V_{cv}\sin^2\theta\hat{k}$

$$+ 2\omega l V_{cv}\sin\theta\cos\theta(-\hat{i})$$

Substituting and introducing $d\forall = A\,dl$,

$$\vec{T}_{cv} = \int_0^L (-2\omega l V_{cv} \sin\theta\cos\theta\,\hat{i} - \omega^2 l^2 \sin\theta\cos\theta\,\hat{j} + 2\omega l V_{cv}\sin^2\theta\,\hat{k})\,\rho A\,dl$$

$$\vec{T}_{cv} = \left[-\omega L^2 V_{cv}\sin\theta\cos\theta\,\hat{i} - \frac{\omega^2 L^3}{3}\sin\theta\cos\theta\,\hat{j} + \omega L^2 V_{cv}\sin^2\theta\,\hat{k}\right]\rho A$$

The shaft torque needed to maintain steady rotation of the assembly is

$$T_{shaft} = T_{cv_z} = \omega L^2 V_{cv}\sin^2\theta\,\rho A = \omega L^2 \frac{Q}{A}\sin^2\theta\,\rho A = \rho Q \omega L^2 \sin^2\theta$$

$$= 999\,\frac{kg}{m^3} \times 0.15\,\frac{m^3}{s} \times 30\,\frac{rev}{min} \times (0.5)^2\,m^2 \times (0.5)^2 \times 2\pi\,\frac{rad}{rev} \times \frac{min}{60\,s} \times \frac{N\cdot s^2}{kg\cdot m}$$

$T_{shaft} = 29.4\ N\cdot m$ ← T_{shaft}

The reaction moments acting on the flange are

$$M_x = -T_{cv_x} = \omega L^2 V_{cv}\sin\theta\cos\theta\,\rho A = \rho Q \omega L^2 \sin\theta\cos\theta$$

$$= 999\,\frac{kg}{m^3} \times 0.15\,\frac{m^3}{s} \times 30\,\frac{rev}{min} \times (0.5)^2\,m^2 \times (0.5)(0.866)\ 2\pi\,\frac{rad}{rev} \times \frac{min}{60\,s} \times \frac{N\cdot s^2}{kg\cdot m}$$

$M_x = 51.0\ N\cdot m$ (applied to flange by CV) ← M_x

$$M_y = -T_{cv_y} = \frac{1}{3}\rho\omega^2 L^3 A\sin\theta\cos\theta$$

$$= \frac{1}{3} \times 999\,\frac{kg}{m^3}\left[30\,\frac{rev}{min} \times 2\pi\,\frac{rad}{rev} \times \frac{min}{60\,s}\right]^2 (0.5)^3\,m^3 \times \frac{\pi}{4}(0.1)^2\,m^2 \times (0.5)(0.866) \times \frac{N\cdot s^2}{kg\cdot m}$$

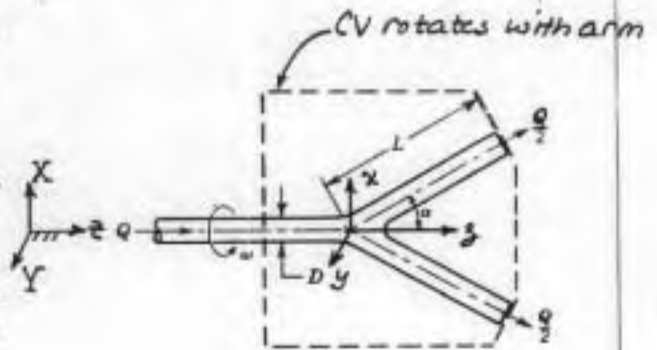
$M_y = 1.40\ N\cdot m$ (applied to flange by CV) ← M_y

{ Torques due to the masses of water, tube, and nozzle must be considered in the overall design. }

Given: Branched pipe with symmetrical legs as shown.

Angular momentum zero at inlet, relative to nonrotating frame.

Find: (a) External torque expression
(b) Additional torque to produce angular acceleration of $\dot{\omega}$.

Solution: Apply moment of momentum equation using rotating CV.

Basic equation:

$$\vec{r}\times\vec{F}_s^{\,=0(1)} + \int_{CV}\vec{r}\times\vec{g}^{\,=0(2)}\rho\, d\forall + \vec{T}_{shaft}$$

$$-\int_{CV}\vec{r}\times[2\vec{\omega}\times\vec{V}_{xyz}+\vec{\omega}\times(\vec{\omega}\times\vec{r})+\dot{\vec{\omega}}\times\vec{r}]\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}_{xyz}\rho\, d\forall^{\,=0(3)} + \int_{CS}\vec{r}\times\vec{V}_{xyz}^{\,=0(4)}\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) No surface forces
(2) Body-forces produce no torque about axis (symmetry)
(3) Flow steady in rotating frame
(4) $\vec{r}$ and $\vec{V}_{xyz}$ are colinear: $\vec{r}\times\vec{V}_{xyz}=0$

Then

$$\vec{T}_{shaft} = \int_{CV}\vec{r}\times[2\vec{\omega}\times\vec{V}_{xyz}+\vec{\omega}\times(\vec{\omega}\times\vec{r})+\dot{\vec{\omega}}\times\vec{r}]\rho\, d\forall$$

Using the coordinates above, $\vec{\omega}=\omega\hat{k}$ $\qquad \dot{\vec{\omega}}=\dot{\omega}\hat{k}$

$\vec{r} = r(\cos\alpha\,\hat{k}+\sin\alpha\,\hat{\imath})$ (upper tube)

$\vec{V}_{xyz} = \frac{Q}{2A}(\cos\alpha\,\hat{k}+\sin\alpha\,\hat{\imath})$ (upper tube); $A=\frac{\pi D^2}{4}$

and

$$\dot{\vec{\omega}}\times\vec{r} = \dot{\omega}r\sin\alpha\,\hat{\jmath}$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \omega\hat{k}\times\omega r\sin\alpha\,\hat{\jmath} = -\omega^2 r\sin\alpha\,\hat{\imath}$$

$$2\vec{\omega}\times\vec{V}_{xyz} = 2\omega\frac{Q}{2A}\sin\alpha\,\hat{\jmath} = \frac{\omega Q}{A}\sin\alpha\,\hat{\jmath}$$

Thus for the upper tube,

$$\vec{T}_{shaft} = \int_0^L\left\{r(\cos\alpha\,\hat{k}+\sin\alpha\,\hat{\imath})\times\left[\left(\frac{\omega Q}{A}+\dot{\omega}r\right)\sin\alpha\,\hat{\jmath}-\omega^2 r\sin\alpha\,\hat{\imath}\right]\right\}\rho A\, dr$$

$$=\int_0^L\left[\left(\frac{r\omega Q}{A}+\dot{\omega}r^2\right)(\sin\alpha\cos\alpha)\hat{\imath}+\left(\frac{r\omega Q}{A}+\dot{\omega}r^2\right)\sin^2\alpha\,\hat{k}+\omega^2r^2\sin\alpha\cos\alpha(-\hat{\jmath})\right]\rho A\, dr$$

$$\vec{T}_{shaft}(upper) = \left[\left(\frac{L^2\omega Q}{2A}+\frac{\dot{\omega}L^3}{3}\right)\sin\alpha\cos\alpha\,\hat{\imath}+\left(\frac{L^2\omega Q}{2A}+\frac{\dot{\omega}L^3}{3}\right)\sin^2\alpha\,\hat{k}+\frac{\omega^2L^3}{3}\sin\alpha\cos\alpha(-\hat{\jmath})\right]\rho A$$

For the lower tube, $\vec{\omega} = \omega\hat{k}$ $\qquad \dot{\vec{\omega}} = \dot{\omega}\hat{k}$

$$\vec{r} = r(\cos\alpha\,\hat{k} - \sin\alpha\,\hat{i}) \quad \text{(lower tube)}$$

$$\vec{V}_{xyz} = \frac{Q}{2A}(\cos\alpha\,\hat{k} - \sin\alpha\,\hat{j}) \quad \text{(lower tube)}$$

and

$$\dot{\vec{\omega}} \times \vec{r} = -r\dot{\omega}\sin\alpha\,\hat{j}$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \omega\hat{k} \times (-r\omega\sin\alpha\,\hat{j}) = r\omega^2\sin\alpha\,\hat{i}$$

$$2\vec{\omega} \times \vec{V}_{xyz} = 2\omega\frac{Q}{2A}(-\sin\alpha)(\hat{j}) = -\frac{\omega Q}{A}\sin\alpha\,\hat{j}$$

Thus for the lower tube,

$$\vec{T}_{shaft} = \int_0^L \left\{ r(\cos\alpha\,\hat{k} - \sin\alpha\,\hat{i}) \times \left[\left(\frac{\omega Q}{A} + r\dot{\omega}\right)\sin\alpha(-\hat{j}) + r\omega^2\sin\alpha \right] \right\} \rho A\,dr$$

$$= \int_0^L \left[\left(\frac{r\omega Q}{A} + r^2\dot{\omega}\right)\sin\alpha\cos\alpha(-\hat{i}) + \left(\frac{r\omega Q}{A} + r^2\dot{\omega}\right)\sin^2\alpha\,\hat{k} + r^2\omega^2\sin\alpha\cos\alpha\,\hat{j} \right] \rho A\,dr$$

$$\vec{T}_{shaft}(\text{lower}) = \left[\left(\frac{L^2\omega Q}{2A} + \frac{L^3\dot{\omega}}{3}\right)\sin\alpha\cos\alpha\,\hat{i} + \left(\frac{L^2\omega Q}{2A} + \frac{L^3\dot{\omega}}{3}\right)\sin^2\alpha\,\hat{k} + \frac{L^3\omega^2}{3}\sin\alpha\cos\alpha\,\hat{j} \right] \rho A$$

Summing these expressions gives

$$\vec{T}_{shaft}(\text{total}) = \left(\frac{L^2\omega Q}{A} + \frac{2L^3\dot{\omega}}{3}\right)\sin^2\alpha\,\rho A\,\hat{k}$$

Thus the steady-state portion of the torque is

$$\vec{T}_{shaft}(\text{steady state}) = \left(\frac{L^2\omega Q}{A}\right)\sin^2\alpha\,\rho A\,\hat{k} = L^2\rho\omega Q\sin^2\alpha\,\hat{k}$$

Stea

The additional torque needed to provide angular acceleration, $\dot{\omega}$, is

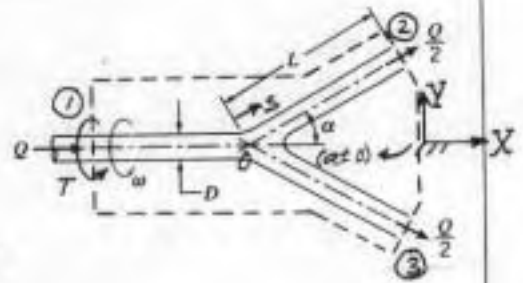
$$\vec{T}_{shaft}(\text{acceleration}) = \frac{2L^3\rho\dot{\omega}A}{3}\sin^2\alpha\,\hat{k}$$

Accel

{ Torques of individual tubes about the x and y axes are reacted internally; they must be considered in design of the tube. }

(b) Using fixed CV:

Basic equation: $\vec{r}\times\vec{F}_s + \int_{CV} \vec{r}\times\vec{g}\rho\, dV + \vec{T}_{shaft}$ (terms $=0(1)$, $=0(2)$)

$$= \frac{\partial}{\partial t}\int_{CV} \vec{r}\times\vec{V}\rho\, dV + \int_{CS} \vec{r}\times\vec{V}\rho\vec{V}\cdot d\vec{A}$$ (first term $=0(3)$)

Assumptions: (1) No surface forces
(2) Body forces symmetric (no moment about X axis)
(3) No change in angular momentum within CV w.r.t. time
(4) Symmetry in two branches
(5) Uniform flow at each cross-section

Then $\vec{T}_s = T\hat{I} = \vec{r}_1\times\vec{V}_1\{-\rho Q\} + \vec{r}_2\times\vec{V}_2\{+\rho\frac{Q}{2}\} + \vec{r}_3\times\vec{V}_3\{+\rho\frac{Q}{2}\} = 2\vec{r}_2\times\vec{V}_2\{\rho\frac{Q}{2}\}$

$\vec{r}_1 = 0$ $\quad \vec{r}_2 = L\sin\alpha\,\hat{J}$; $\vec{V}_2 = \omega r_2\,\hat{K}$; $\vec{r}_2\times\vec{V}_2 = \omega L^2\sin^2\alpha\,\hat{I}$

or

$T_{ss} = \rho\omega Q L^2 \sin^2\alpha$ (steady-state torque) $\quad T_{ss}$

The torque required for acceleration is $T_{acc} = I\dot{\omega}$, where $I = \int r^2 dm$

For one leg of the branch, $I = \int r^2 dm = \int_0^L (s\sin\alpha)^2 \rho A\, ds = \frac{\rho A L^3}{3}\sin^2\alpha$

(6) Neglect mass of pipe

For both sides, $I = \frac{2\rho A L^3}{3}\sin^2\alpha$.

Thus

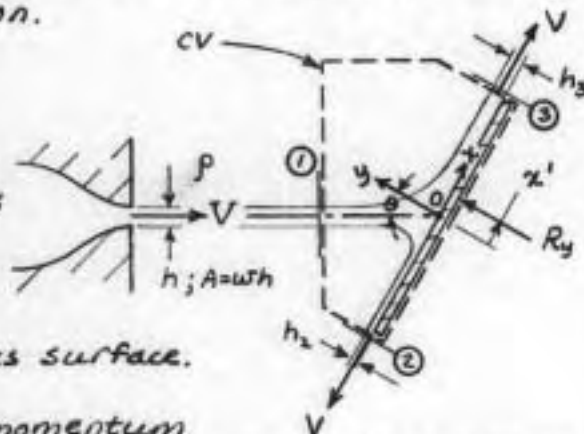
$T_{acc} = \frac{2\rho\dot{\omega}AL^3}{3}\sin^2\alpha$ (torque required for angular acceleration) $\quad T_{acc}$

The total torque that must be applied is

$T = T_{ss} + T_{acc} = \rho\omega Q L^2\sin^2\alpha + \frac{2\rho\dot{\omega}A}{3}L^3\sin^2\alpha$ $\quad T_{total}$

Given: Thin sheet of liquid, of width, w, and thickness, h, striking inclined flat plate, as shown.

Neglect any viscous effects.

Find: (a) Magnitude and line of action of resultant force as functions of θ.
(b) Equilibrium angle of plate if force is applied at point O, where jet centerline intersects surface.

Solution: Apply continuity, linear momentum and moment of momentum using CV and coordinates shown.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u \rho\, d\forall + \int_{CS} u \rho \vec{V}\cdot d\vec{A}$$

$$F_{Sy} + F_{By} = \frac{\partial}{\partial t}\int_{CV} v \rho\, d\forall + \int_{CS} v \rho \vec{V}\cdot d\vec{A}$$

$$\vec{r}\times\vec{F}_S + \int_{CV} \vec{r}\times\vec{g}\,\rho\, d\forall + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV} \vec{r}\times\vec{V}\rho\, d\forall + \int_{CS} \vec{r}\times\vec{V}\rho\vec{V}\cdot d\vec{A}$$

(Cancellations marked: $=0(1)$ on unsteady terms, $=0(4)$ on F_{Sx}, $=0(5)$ on body forces, $=0(6)$ on $\vec{T}_{shaft}$.)

Assumptions:
(1) Steady flow
(2) Uniform flow at each section
(3) No net pressure forces; $F_{Sx} = R_x$, $F_{Sy} = R_y$
(4) No viscous effects; $R_x = 0$ and $V_1 = V_2 = V_3 = V$
(5) Neglect body forces and torques
(6) $\vec{T}_{shaft} = 0$
(7) Incompressible flow, ρ = constant

Then from continuity,

$$0 = \{-|\rho V w h_1|\} + \{|\rho V w h_2|\} + \{|\rho V w h_3|\} \quad \text{or} \quad h_1 = h_2 + h_3 = h \qquad (1)$$

From x momentum

$$0 = u_1\{-|\rho V w h_1|\} + u_2\{|\rho V w h_2|\} + u_3\{|\rho V w h_3|\}$$

$$u_1 = V\sin\theta \qquad u_2 = -V \qquad u_3 = V$$

$$0 = \rho V^2 w\,(-h_1\sin\theta - h_2 + h_3) \quad \text{or} \quad h_3 - h_2 = h_1\sin\theta = h\sin\theta \qquad (2)$$

Combining Eqs. 1 and 2,

$$h_2 = h\left(\frac{1-\sin\theta}{2}\right) \qquad (3)$$

$$h_3 = h\left(\frac{1+\sin\theta}{2}\right) \qquad (4)$$

From y momentum, $R_y = v_1\{-|\rho V w h_1|\} + v_2\{|\rho V w h_2|\} + v_3\{|\rho V w h_3|\}$

$$v_1 = -V\cos\theta \qquad v_2 = 0 \qquad v_3 = 0$$

$$R_y = \rho V^2 w h \cos\theta \qquad (5) \quad R_y$$

From moment of momentum,

$$\vec{r}' \times \vec{F}_s = \vec{r}_1 \times \vec{V}_1\{-|\rho V w h_1|\} + \vec{r}_2 \times \vec{V}_2\{|\rho V w h_2|\} + \vec{r}_3 \times \vec{V}_3\{|\rho V w h_3|\}$$

$\vec{r}' = x'\hat{i}$ $\quad \vec{r}_1 \times \vec{V}_1 = 0$ $\quad \vec{r}_2 = \frac{h_2}{2}\hat{j}$ $\quad \vec{r}_3 = \frac{h_3}{2}\hat{j}$

$\vec{F}_s = R_y\hat{j}$ $\quad \vec{V}_2 = -V\hat{i}$ $\quad \vec{V}_3 = V\hat{i}$

$\vec{r}' \times \vec{F}_s = x' R_y \hat{k}$ $\quad \vec{r}_2 \times \vec{V}_2 = \frac{h_2 V}{2}\hat{k}$ $\quad \vec{r}_3 \times \vec{V}_3 = -\frac{h_3 V}{2}\hat{k}$

Combining and dropping $\hat{k}$,

$$x' R_y = \frac{1}{2}\rho V^2 w h_2^2 - \frac{1}{2}\rho V^2 w h_3^2 = \frac{1}{2}\rho V^2 w (h_2^2 - h_3^2)$$

or

$$x' = \frac{\rho V^2 w (h_2^2 - h_3^2)}{2R_y} = \frac{\rho V^2 w (h_2 + h_3)(h_2 - h_3)}{2R_y}$$

Substituting from Eqs. 3, 4 and 5,

$$x' = \frac{\rho V^2 w h^2 \left(\frac{1-\sin\theta}{2} + \frac{1+\sin\theta}{2}\right)\left(\frac{1-\sin\theta}{2} - \frac{1+\sin\theta}{2}\right)}{2\rho V^2 w h \cos\theta} = \frac{h(-\sin\theta)}{2\cos\theta}$$

or

$$x' = -\frac{h}{2}\tan\theta \qquad (6) \quad x'$$

Note that $x' < 0$. This means that R_y must be applied below point O.

If R_y is applied at point O, then $x' = 0$. For equilibrium, from Eq. 6, $\theta = 0$. Thus if force is applied at point O, plate will be in equilibrium when perpendicular to jet.

Problem *4.197

Given: The rotating lawn sprinkler of Example Problem 4.14.

Find: (a) Jet angle α for maximum speed of rotation.
(b) What jet angle will provide the maximum area of coverage by the spray?
(c) Draw a velocity diagram to show the absolute velocity of the water jet leaving the nozzle.
(d) What governs the steady rotational speed of the sprinkler?
(e) Does the rotational speed of the sprinkler affect the area covered by the spray?
(f) How would you estimate the area of coverage?
(g) For fixed α, what might be done to increase or reduce the area covered by the spray?

Solution: The results of Example Problem 4.14 were computed assuming steady flow of water and constant frictional retarding torque at the sprinkler pivot.

$$T_f = R(V_{rel}\cos\alpha - \omega R)\rho Q$$

From these results,

$$\omega = \frac{V_{rel}\cos\alpha}{R} - \frac{T_f}{\rho Q R^2}$$

Thus rotational speed of the sprinkler increases as cos α increases, i.e., as α decreases. The maximum rotational speed occurs when $\alpha = 0$. Then cos $\alpha = 1$ and the rotational speed is

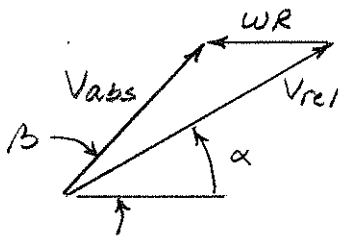
$$\omega = \frac{V_{rel}}{R} - \frac{T_f}{\rho Q R^2}$$

For the conditions of Example Problem 4.14 the maximum rotational speed is

$$\omega = 4.97\,\frac{m}{s} \times \frac{1}{0.150\,m} - 0.0718\,N\cdot m \times \frac{m^3}{999\,kg} \times \frac{min}{7.5\,L} \times \frac{1}{(0.150)^2\,m^2} \times \frac{1000\,L}{m^3} \times \frac{60\,s}{min} = 7.58\,rad/s$$

ω_{max}

The steady rotation speed ω of the sprinkler is governed by torque T_f and angle α.

Maximum coverage by the spray occurs when the "carry" of each jet stream is the longest. When aerodynamic drag on the stream is neglected, maximum carry occurs when the absolute velocity of the stream leaves the sprinkler at $\beta = 45°$, as shown in the velocity diagram below.

Note $\vec{V}_{abs} = \vec{V}_{rel} - \omega R \hat{e}_\theta$

Both the magnitude and direction of $\vec{V}_{abs}$ vary with ω!

For $\omega = 0$, the relative velocity angle α and absolute velocity angle β are equal. Therefore maximum carry occurs when $\alpha = 45°$ (see graph on next page).

Any rotation rate ω reduces the magnitude V_{abs} and increases the angle β of the absolute velocity leaving the sprinkler jet. When $\omega > 0$, then $\beta > \alpha$, so for maximum carry α must be less than 45°. Consequently rotation reduces the carry of the stream and the area of coverage; at specified α the area of coverage decreases with increasing ω.

For the conditions of Example Problem 4.14 ($\omega = 30$ rpm), optimum carry occurs at $\alpha \approx 42°$, and the coverage area is reduced from approximately 20 m^2 with a fixed sprinkler to 15 m^2 with 30 rpm rotation. If the rotation speed is increased (by decreasing pivot friction or decreasing nozzle angle α), coverage area may be reduced still further, to 9 m^2 or less.

$$A \approx \pi (x_{max})^2$$

Problem *4.197

Analysis of Ground Area Covered by Rotating Lawn Sprinkler:

Variables:

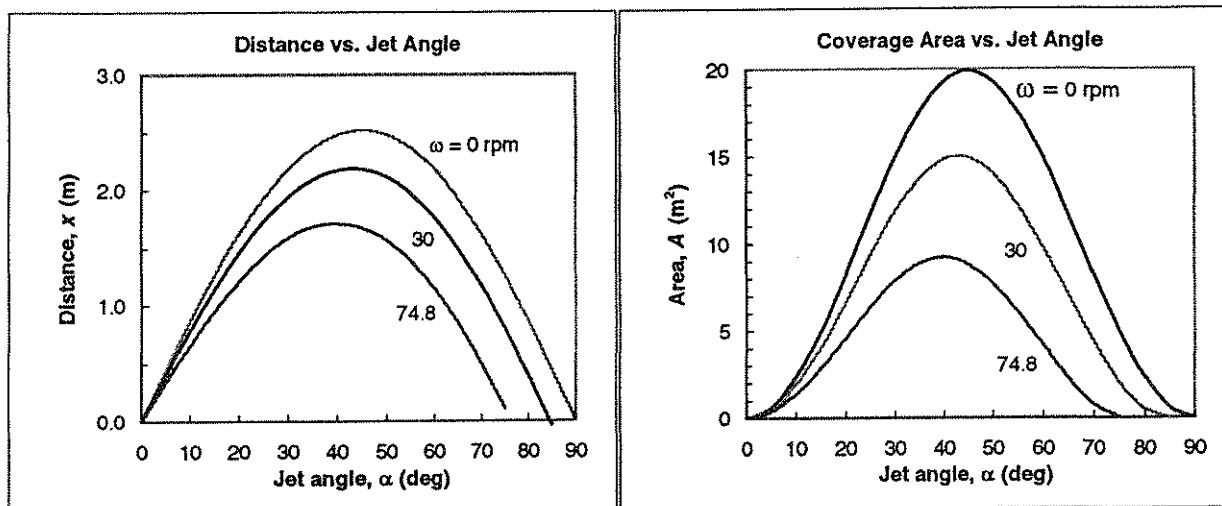
A = ground area covered by spray stream
x = ground distance reached by spray stream
α = angle of jet above ground plane
β = angle of absolute velocity above ground plane

Input Data:

R = 0.150 m
V_{rel} = 4.97 m/s (Q = 7.5 L/min)

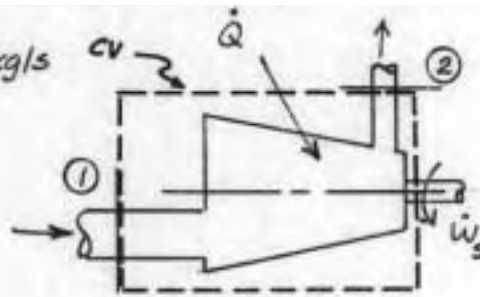
Results:

	ω (rpm) =	0		30		74.8
	ωR (m/s) =	0		0.471		1.17
α (deg)	x_{max} (m)	A (m^2)	x_{max} (m)	A (m^2)	x_{max} (m)	A (m^2)
0	0.00	0.00	0.00	0.00	0.00	0.00
5	0.437	0.601	0.396	0.492	0.333	0.349
10	0.861	2.33	0.778	1.90	0.654	1.35
15	1.26	4.98	1.14	4.05	0.951	2.84
20	1.62	8.23	1.46	6.65	1.21	4.61
25	1.93	11.7	1.73	9.37	1.43	6.39
30	2.18	14.9	1.94	11.8	1.59	7.90
35	2.37	17.6	2.09	13.8	1.68	8.90
40	2.48	19.3	2.17	14.8	1.71	9.23
45	2.52	19.9	2.18	14.9	1.68	8.83
50	2.48	19.3	2.11	14.0	1.57	7.72
55	2.37	17.6	1.97	12.3	1.39	6.08
60	2.18	14.9	1.77	9.81	1.15	4.15
65	1.93	11.7	1.50	7.03	0.850	2.269
70	1.62	8.23	1.17	4.30	0.500	0.785
75	1.26	4.98	0.798	2.00	0.109	0.037
78	1.02	3.30	0.557	0.975		
80	0.861	2.33	0.391	0.480		
85	0.437	0.601	-0.04	0.00		
90	0.00	0.00				

Problem 4.198

Given: Compressor, $\dot{m} = 1.0$ kg/s

$p_1 = 101$ kPa (abs)

$T_1 = 288$ K

$V_1 = 75$ m/s

$p_2 = 200$ kPa (abs)

$T_2 = 345$ K

$V_2 = 125$ m/s

$\frac{dQ}{dm} = -18$ kJ/kg

Find: Power required.

Solution: Apply first law of thermodynamics, using CV shown.

B.E. $$\dot{Q} - \dot{W}_s - \dot{W}_{shear} = \frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall + \int_{CS} (e + pv)\rho \vec{V} \cdot d\vec{A}$$

($\dot{W}_{shear} = 0(1)$, $\frac{\partial}{\partial t} = 0(2)$)

Assume: (1) $\dot{W}_{shear} = 0$
(2) Steady flow
(3) Uniform flow at each section
(4) Neglect Δz
(5) Ideal gas, $p = \rho RT$, $\Delta h = c_p \Delta T$; $c_p = 1.00$ kJ/kg·K
(6) From continuity, $\dot{m}_1 = \dot{m}_2 = \dot{m}$

Then
$$\dot{Q} - \dot{W}_s = \left(u_2 + \frac{V_2^2}{2} + gz_2 + p_2 v_2\right)\{|\dot{m}|\} + \left(u_1 + \frac{V_1^2}{2} + gz_1 + p_1 v_1\right)\{-|\dot{m}|\}$$

($gz_2 \approx 0(4)$, $gz_1 \approx 0(4)$)

Note that $h = u + pv$, and $\dot{Q} = \dot{m}\frac{dQ}{dm}$, so

$$\dot{W}_{in} = -\dot{W}_s = \dot{m}\left(\frac{V_2^2 - V_1^2}{2} + h_2 - h_1 - \frac{dQ}{dm}\right) = \dot{m}\left[\frac{V_2^2 - V_1^2}{2} + c_p(T_2 - T_1) - \frac{dQ}{dm}\right]$$

or

$$\dot{W}_{in} = 1.0\frac{\text{kg}}{\text{s}}\left\{\frac{1}{2}\left[(125)^2 - (75)^2\right]\frac{\text{m}^2}{\text{s}^2} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} \times \frac{\text{kJ}}{1000\,\text{N}\cdot\text{m}}\right.$$

$$\left. + 1.00\frac{\text{kJ}}{\text{kg}\cdot\text{K}}(345 - 288)\,\text{K} - \left(-18\frac{\text{kJ}}{\text{kg}}\right)\right\}\frac{\text{kW}\cdot\text{s}}{\text{kJ}}$$

$$\dot{W}_{in} = 80.0\ \text{kW}$$

$\dot{W}_{in}$

Problem 4.199

[3]

4.199 Compressed air is stored in a pressure bottle with a volume of 0.5 m^3, at 20 MPa and 60°C. At a certain instant a valve is opened and mass flows from the bottle at $\dot{m} = 0.05$ kg/s. Find the rate of change of temperature in the bottle at this instant.

Given: Compressed air bottle

Find: Rate of temperature change

Solution:

Basic equations: Continuity; First Law of Thermodynamics for a CV

$$\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0 \qquad \dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\,\rho\, d\forall + \int_{CS}\left(u + pv + \frac{V^2}{2} + gz\right)\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Adiabatic 2) No work 3) Neglect KE 4) Uniform properties at exit 5) Ideal gas

From continuity $\frac{\partial}{\partial t}M_{CV} + m_{exit} = 0$ where m_{exit} is the mass flow rate at the exit (Note: Software does not allow a dot!)

$$\frac{\partial}{\partial t}M_{CV} = -m_{exit}$$

From the 1st law $0 = \frac{\partial}{\partial t}\int u\, dM + \left(u + \frac{p}{\rho}\right)\cdot m_{exit} = u\cdot\left(\frac{\partial}{\partial t}M\right) + M\cdot\left(\frac{\partial}{\partial t}u\right) + \left(u + \frac{p}{\rho}\right)\cdot m_{exit}$

Hence $u\cdot\left(-m_{exit}\right) + M\cdot c_V\cdot\frac{dT}{dt} + u\cdot m_{exit} + \frac{p}{\rho}\cdot m_{exit} = 0 \qquad \frac{dT}{dt} = -\frac{m_{exit}\cdot p}{M\cdot c_V\cdot\rho}$

But $M = \rho\cdot Vol$ so $\frac{dT}{dt} = -\frac{m_{exit}\cdot p}{Vol\cdot c_V\cdot\rho^2}$

For air $\rho = \frac{p}{R\cdot T} \qquad \rho = 20\times 10^6\cdot\frac{N}{m^2}\times\frac{kg\cdot K}{286.9\cdot N\cdot m}\times\frac{1}{(60+273)\cdot K} \qquad \rho = 209\frac{kg}{m^3}$

Hence $\frac{dT}{dt} = -0.05\cdot\frac{kg}{s}\times 20\times 10^6\cdot\frac{N}{m^2}\times\frac{1}{0.5\cdot m^3}\times\frac{kg\cdot K}{717.4\cdot N\cdot m}\times\left(\frac{m^3}{209\cdot kg}\right)^2 = -0.064\cdot\frac{K}{s}$

Problem 4.200

[3]

4.200 A centrifugal water pump with a 0.1-m diameter inlet and a 0.1-m diameter discharge pipe has a flow rate of 0.02 m^3/s. The inlet pressure is 0.2 m Hg vacuum and the exit pressure is 240 kPa. The inlet and outlet sections are located at the same elevation. The measured power input is 6.75 kW. Determine the pump efficiency.

Given: Data on centrifugal water pump

Find: Pump efficiency

Solution:

Basic equations:

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\,\rho\, d\Psi + \int_{CS}\left(u + pv + \frac{V^2}{2} + gz\right)\rho \vec{V}\cdot d\vec{A} \qquad (4.56)$$

$$\Delta p = SG_{Hg}\cdot\rho\cdot g\cdot\Delta h \qquad \eta = \frac{W_s}{P_{in}}$$

Available data:

$D_1 = 0.1\cdot m$ $\quad D_2 = 0.1\cdot m$ $\quad Q = 0.02\cdot\frac{m^3}{s}$ $\quad P_{in} = 6.75\cdot kW$

$\rho = 1000\frac{kg}{m^3}$ $\quad SG_{Hg} = 13.6$ $\quad h_1 = -0.2\cdot m$ $\quad p_2 = 240\cdot kPa$

Assumptions: 1) Adiabatic 2) Only shaft work 3) Steady 4) Neglect Δu 5) Δz = 0 6) Incompressible 7) Uniform flow

Then

$$-W_s = \left(p_1\cdot v_1 + \frac{V_1^{\,2}}{2}\right)\cdot\left(-m_{rate}\right) + \left(p_2\cdot v_2 + \frac{V_2^{\,2}}{2}\right)\cdot\left(m_{rate}\right)$$

Since $m_{rate} = \rho\cdot Q$ and $V_1 = V_2$ (from continuity)

$$-W_s = \rho\cdot Q\cdot\left(p_2\cdot v_2 - p_1\cdot v_1\right) = Q\cdot\left(p_2 - p_1\right)$$

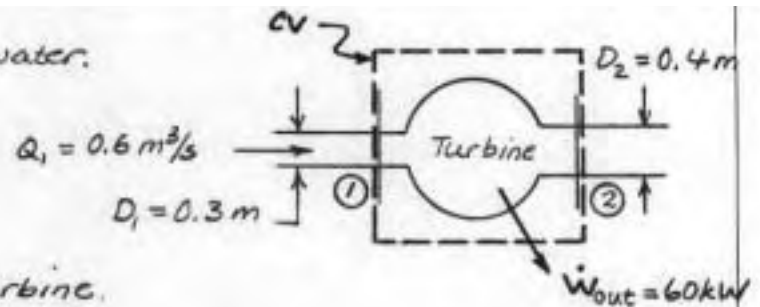
$p_1 = \rho_{Hg}\cdot g\cdot h$ or $p_1 = SG_{Hg}\cdot\rho\cdot g\cdot h_1$ $\quad p_1 = -26.7\,kPa$

$W_s = Q\cdot\left(p_1 - p_2\right)$ $\quad W_s = -5.33\,kW$ The negative sign indicates work in

$$\eta = \frac{|W_s|}{P_{in}} \qquad \eta = 79.0\,\%$$

Given: Turbine operating on water.

$Q_1 = 0.6\ m^3/s$, $D_1 = 0.3\ m$, $D_2 = 0.4\ m$, $\dot{W}_{out} = 60\ kW$

Turbine, CV, ①, ②

Find: Pressure drop across turbine.

Solution: Apply continuity, energy equations, using CV shown.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} \quad (=0(1))$$

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall + \int_{CS}\left(u + \frac{V^2}{2} + gz + pv\right)\rho\vec{V}\cdot d\vec{A}$$

$(\dot{Q} = 0(4),\ \dot{W}_{shear} = 0(5),\ \dot{W}_{other} = 0(5),\ \frac{\partial}{\partial t} = 0(1),\ u \approx 0(6),\ gz \approx 0(7))$

Assumptions:
(1) Steady flow
(2) Uniform flow at each section
(3) Incompressible flow
(4) $\dot{Q} = 0$
(5) $\dot{W}_{shear} = 0$ by choice of CV ; $\dot{W}_{other} = 0$
(6) Neglect Δu
(7) Neglect Δz

Then

$$0 = \{-|\rho V_1 A_1|\} + \{|\rho V_2 A_2|\} \quad \text{or} \quad V_2 = V_1 \frac{A_1}{A_2} = V_1\left(\frac{D_1}{D_2}\right)^2$$

and

$$-\dot{W}_s = \left(\frac{V_1^2}{2} + p_1 v\right)\{-|\rho V_1 A_1|\} + \left(\frac{V_2^2}{2} + p_2 v\right)\{|\rho V_2 A_2|\}$$

$$-\dot{W}_s = -\left[\frac{V_1^2 - V_2^2}{2} + (p_1 - p_2)v\right]\rho Q = -\left\{\frac{V_1^2}{2}\left[1 - \left(\frac{D_1}{D_2}\right)^4\right] + (p_1 - p_2)v\right\}\rho Q$$

or

$$p_1 - p_2 = \frac{1}{v}\left\{\frac{\dot{W}_s}{\rho Q} - \frac{V_1^2}{2}\left[1 - \left(\frac{D_1}{D_2}\right)^4\right]\right\} = \frac{\dot{W}_s}{Q} - \frac{\rho V_1^2}{2}\left[1 - \left(\frac{D_1}{D_2}\right)^4\right]$$

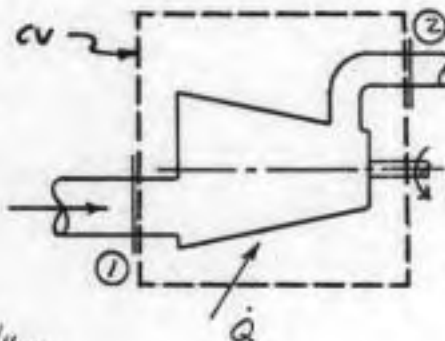
But $V_1 = \frac{Q}{A_1} = 0.6\frac{m^3}{s} \times \frac{4}{\pi}\frac{1}{(0.3)^2 m^2} = 8.49\ m/s$, and $\dot{W}_s = \dot{W}_{out} = 60\ kW$, so

$$p_1 - p_2 = (60\ kW)\, 10^3 \frac{N\cdot m}{kW\cdot s} \times \frac{s}{0.6\ m^3} - \frac{1}{2} \times 999\frac{kg}{m^3} \times (8.49)^2\frac{m^2}{s^2}\left[1 - \left(\frac{0.3}{0.4}\right)^4\right]\frac{N\cdot s^2}{kg\cdot m}$$

$$p_1 - p_2 = 75.4\ kPa$$

$p_1 - p_2$

Problem 4.202

Given: Compressor operating at conditions shown

Find: Heat transfer, in Btu/lbm.

Solution: Apply energy equation to CV shown.

Basic equations: $p = \rho RT$, $\Delta h = c_p \Delta T$

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\rho\, dV + \int_{CS}\left(u + pv + \frac{V^2}{2} + gz\right)\rho \vec{V}\cdot d\vec{A}$$

($\dot{W}_{shear} = 0(2)$, $\dot{W}_{other} = 0(2)$, $\frac{\partial}{\partial t} = 0(3)$, $gz = 0(5)$)

Assumptions: (1) Ideal gas, constant specific heat
(2) $\dot{W}_{shear} = 0$ by choice of CV; $\dot{W}_{other} = 0$
(3) Steady flow
(4) Uniform flow at each section
(5) Neglect Δz
(6) $V_1 \approx 0$

By definition $h \equiv u + pv$, so

$$\dot{Q} - \dot{W}_s = \left(h_1 + \frac{V_1^2}{2}\right)\{-|\dot{m}|\} + \left(h_2 + \frac{V_2^2}{2}\right)\{|\dot{m}|\} = \dot{m}\left[\frac{V_2^2}{2} + c_p(T_2 - T_1)\right]$$

($\frac{V_1^2}{2} \approx 0(6)$)

or

$$\frac{\delta Q}{dm} = \frac{\dot{Q}}{\dot{m}} = \frac{\dot{W}_s}{\dot{m}} + \frac{V_2^2}{2} + c_p(T_2 - T_1)$$

Noting $\dot{W}_s = -3200$ hp, so

$$\frac{\delta Q}{dm} = -3200\ hp \times \frac{2545\ Btu}{hp \cdot hr} \times \frac{s}{20\ lbm} \times \frac{hr}{3600\ s} + 0.240 \frac{Btu}{lbm \cdot °F} \times (500 - 80)°F$$

$$+ \frac{(500)^2}{2}\frac{ft^2}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \frac{slug}{32.2\ lbm} \times \frac{Btu}{778\ ft \cdot lbf}$$

$$\frac{\delta Q}{dm} = -7.32\ Btu/lbm$$

$\frac{\delta Q}{dm}$

Therefore heat transfer is <u>out of</u> CV, since $\delta Q/dm < 0$. The rate of heat transfer is

$$\dot{Q} = -7.32 \frac{Btu}{lbm} \times 20 \frac{lbm}{s} = -146\ Btu/s$$

$\dot{Q}$

Problem 4.203

Given: Flow through turbomachine shown. Fluid is air.

$\dot{m} = 0.8$ kg/s

$T_1 = 288$ K

$p_1 = 101$ kPa (abs)

CV

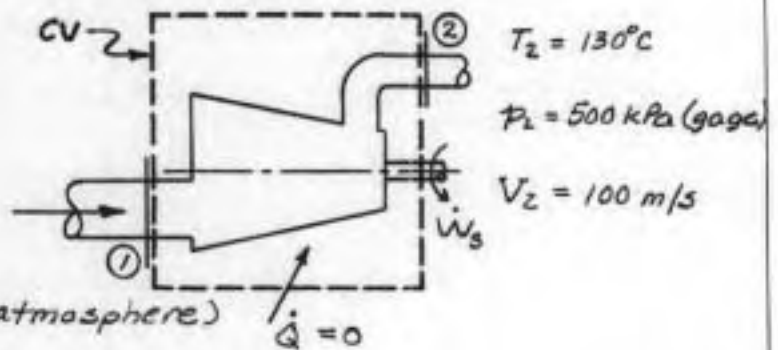
$T_2 = 130°$C

$p_2 = 500$ kPa (gage)

$V_2 = 100$ m/s

$\dot{W}_s$

$V_1 \approx 0$ (from atmosphere)

$\dot{Q} = 0$

Find: Shaft work interaction with surroundings.

Solution: Apply energy equation, using CV shown.

Basic equations: $p = \rho R T$, $\Delta h = c_p \Delta T$

$$\overset{=0(7)}{\dot{Q}} - \dot{W}_s - \overset{=0(2)}{\dot{W}_{shear}} - \overset{=0(2)}{\dot{W}_{other}} = \overset{=0(3)}{\frac{\partial}{\partial t}} \int_{CV} e \rho \, d\forall + \int_{CS} \left(u + pv + \frac{V^2}{2} + \overset{\approx 0(5)}{gz}\right) \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) Ideal gas, constant specific heat
(2) $\dot{W}_{shear} = 0$ by choice of CV; $\dot{W}_{other} = 0$
(3) Steady flow
(4) Uniform flow at each section
(5) Neglect Δz
(6) $V_1 \approx 0$
(7) $\dot{Q} = 0$

By definition, $h \equiv u + pv$, so

$$-\dot{W}_s = \left(h_1 + \overset{\approx 0(6)}{\frac{V_1^2}{2}}\right)\{-|\dot{m}|\} + \left(h_2 + \frac{V_2^2}{2}\right)\{|\dot{m}|\} = \dot{m}\left(h_2 - h_1 + \frac{V_2^2}{2}\right)$$

or

$$-\dot{W}_s = \dot{m}\left(h_2 - h_1 + \frac{V_2^2}{2}\right) = \dot{m}\left[c_p(T_2 - T_1) + \frac{V_2^2}{2}\right]$$

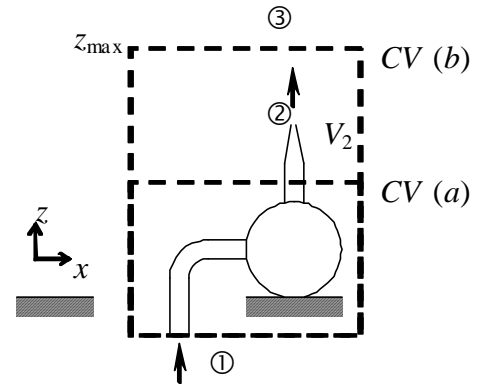
$$= 0.8 \frac{\text{kg}}{\text{s}} \left[1.00 \frac{\text{kJ}}{\text{kg} \cdot \text{K}} (403 - 288)\,\text{K} + \frac{(100)^2}{2}\frac{\text{m}^2}{\text{s}^2} \times \frac{\text{N} \cdot \text{s}^2}{\text{kg} \cdot \text{m}} \times \frac{\text{kJ}}{10^3\,\text{N} \cdot \text{m}}\right] \frac{\text{kW} \cdot \text{s}}{\text{kJ}}$$

$-\dot{W}_s = 96.0$ kW or $\dot{W}_s = -96.0$ kW

$\dot{W}_s$

{Power is into CV because $\dot{W}_s < 0$.}

Problem 4.204

4.204 All major harbors are equipped with fire boats for extinguishing ship fires. A 3-in. diameter hose is attached to the discharge of a 15-hp pump on such a boat. The nozzle attached to the end of the hose has a diameter of 1 in. If the nozzle discharge is held 10 ft above the surface of the water, determine the volume flow rate through the nozzle, the maximum height to which the water will rise, and the force on the boat if the water jet is directed horizontally over the stern.

Given: Data on fire boat hose system

Find: Volume flow rate of nozzle; Maximum water height; Force on boat

Solution:

Basic equation: First Law of Thermodynamics for a CV

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\,\rho\,dV + \int_{CS}\left(u + pv + \frac{V^2}{2} + gz\right)\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Neglect losses 2) No work 3) Neglect KE at 1 4) Uniform properties at exit 5) Incompressible 6) p_{atm} at 1 and 2

Hence for CV (a) $\quad -W_s = \left(\frac{V_2^2}{2} + g\cdot z_2\right)\cdot m_{exit} \quad m_{exit} = \rho\cdot V_2\cdot A_2$ where m_{exit} is mass flow rate (Note: Software cannot render a dot!)

Hence, for V_2 (to get the flow rate) we need to solve $\quad \left(\frac{1}{2}\cdot V_2^2 + g\cdot z_2\right)\cdot\rho\cdot V_2\cdot A_2 = -W_s \quad$ which is a cubic for V_2!

To solve this we could ignore the gravity term, solve for velocity, and then check that the gravity term is in fact minor. Alternatively we could manually iterate, or use a calculator or Excel, to solve. The answer is $\quad V_2 = 114\frac{ft}{s}$

Hence the flow rate is $\quad Q = V_2\cdot A_2 = V_2\cdot\frac{\pi\cdot D_2^2}{4} \quad Q = 114\cdot\frac{ft}{s}\times\frac{\pi}{4}\times\left(\frac{1}{12}\cdot ft\right)^2 \quad Q = 0.622\frac{ft^3}{s} \quad Q = 279\,gpm$

To find z_{max}, use the first law again to (to CV (b)) to get $\quad -W_s = g\cdot z_{max}\cdot m_{exit}$

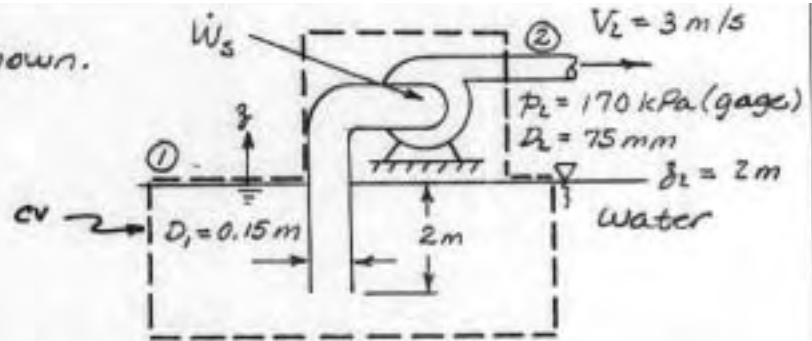
$$z_{max} = -\frac{W_s}{g\cdot m_{exit}} = -\frac{W_s}{g\cdot\rho\cdot Q} \quad z_{max} = 15\cdot hp\times\frac{\frac{550\cdot ft\cdot lbf}{s}}{1\cdot hp}\times\frac{s^2}{32.2\cdot ft}\times\frac{ft^3}{1.94\cdot slug}\times\frac{s}{0.622\cdot ft^3}\times\frac{slug\cdot ft}{s^2\cdot lbf} \quad z_{max} = 212\,ft$$

For the force in the x direction when jet is horizontal we need x momentum

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

Then $\quad R_x = u_1\cdot(-\rho\cdot V_1\cdot A_1) + u_2\cdot(\rho\cdot V_2\cdot A_2) = 0 + V_2\cdot\rho\cdot Q \quad R_x = \rho\cdot Q\cdot V_2$

$$R_x = 1.94\cdot\frac{slug}{ft^3}\times 0.622\cdot\frac{ft^3}{s}\times 114\cdot\frac{ft}{s}\times\frac{lbf\cdot s^2}{slug\cdot ft} \quad R_x = 138\,lbf$$

Problem 4.205

Find: Power required.

<u>Solution</u>: Apply first law to CV shown, noting that flow enters with negligible velocity at section ①.

Basic equation:

$$\dot{Q} - \dot{W}_{shaft} - \overset{=0(1)}{\dot{W}_{shear}} - \overset{=0(1)}{\dot{W}_{other}} = \overset{=0(2)}{\frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall} + \int_{CS}\left(e + \frac{p}{\rho}\right)\rho\vec{V}\cdot d\vec{A}$$

$$e = u + \frac{V^2}{2} + gz$$

Assumptions: (1) $\dot{W}_{shear} = \dot{W}_{other} = 0$

(2) Steady flow

(3) $V_1 \approx 0$

(4) $z_1 = 0$

(5) $p_1 = 0$ (gage)

(6) Uniform flow at each section

(7) Incompressible flow; $V_1 A_1 = V_2 A_2$

Then

$$\dot{Q} - \dot{W}_s = \Big(u_1 + \overset{\approx 0(3)}{\frac{V_1^2}{2}} + \overset{=0(4)}{gz_1} + \overset{=0(5)}{\frac{p_1}{\rho}}\Big)\{-\dot{m}\} + \left(u_2 + \frac{V_2^2}{2} + gz_2 + \frac{p_2}{\rho}\right)\{\dot{m}\}$$

or

$$-\dot{W}_s = \dot{m}\left[\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \left(u_2 - u_1 - \frac{\delta Q}{dm}\right)\right]$$

Obtain the ideal or minimum power input by neglecting thermal effects. Thus

$$-\dot{W}_{s,ideal} = \dot{m}\left[\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2\right]$$

For the system,

$$\dot{m} = \rho V_2 A_2 = 999\,\frac{kg}{m^3} \times 3\,\frac{m}{s} \times \frac{\pi}{4}(0.075)^2\,m^2 = 13.2\ kg/s$$

and

$$-\dot{W}_{s,ideal} = 13.2\,\frac{kg}{s}\left[1.70\times10^5\,\frac{N}{m^2} \times \frac{m^3}{999\,kg} + \frac{1}{2}(3)^2\frac{m^2}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} + 9.81\,\frac{m}{s^2} \times 2\,m \times \frac{N\cdot s^2}{kg\cdot m}\right]$$

$$\dot{W}_{s,ideal} = -2560\,\frac{N\cdot m}{s} \times \frac{kW\cdot s}{10^3\,N\cdot m} = -2.56\ kW$$

Finally

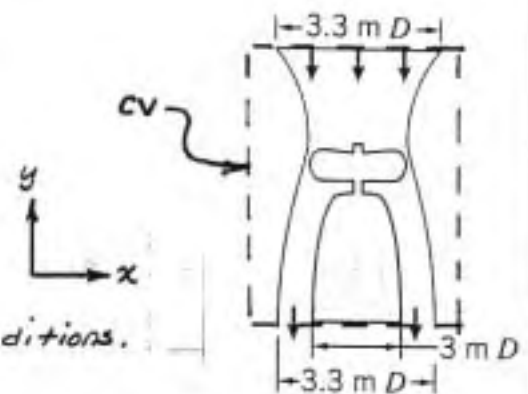
$$\dot{W}_{s,actual} = \frac{\dot{W}_{s,ideal}}{\eta} = \frac{-2.56\ kW}{0.75} = -3.41\ kW$$

$\dot{W}_{s,actual}$

Given: Helicopter-type craft hovering

Mass, $M = 1500\ kg$

Assume atmospheric pressure at outlet, and treat as steady, uniform, incompressible flow.

Assume air is at standard conditions.

Find: (a) Speed of air leaving craft.
(b) Minimum power required.

Solution: Use inertial CV and coordinates shown. Apply continuity and momentum to determine V_2, then apply energy to find power.

Basic equations: $p = \rho RT$; $\Delta h = c_p \Delta T$; $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} \quad (=0(2))$$

$$F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{CV} w \rho\, d\forall + \int_{CS} w \rho \vec{V} \cdot d\vec{A} \quad (=0(2))$$

Assumptions: (1) Air is an ideal gas, $c_p = \text{constant}$
(2) Steady flow
(3) Incompressible flow
(4) Uniform flow at each section
(5) Uniform pressure at inlet; $F_{S_z} = (p_{atm} - p_1) A_1 = -p_{1g} A_1$

Then

$$\rho = \frac{p}{RT} = 1.01 \times 10^5 \frac{N}{m^2} \times \frac{kg \cdot K}{287\ N \cdot m} \times \frac{1}{288\ K} = 1.22\ kg/m^3$$

and from continuity

$$0 = \{-|\rho V_1 A_1|\} + \{|\rho V_2 A_2|\} = \rho (V_2 A_2 - V_1 A_1) \quad \text{or} \quad V_1 = V_2\left(\frac{A_2}{A_1}\right)$$

Now $A_1 = \frac{\pi}{4} D_o^2 = \frac{\pi}{4}(3.3)^2 m^2 = 8.55\ m^2$

$$A_2 = \frac{\pi}{4}(D_o^2 - D_i^2) = \frac{\pi}{4}\left[(3.3)^2 - (3.0)^2\right] m^2 = 1.48\ m^2$$

From momentum

$$-p_{1g} A_1 - Mg = w_1 \{-|\rho V_1 A_1|\} + w_2 \{|\rho V_2 A_2|\}$$

$$w_1 = -V_1 \qquad w_2 = -V_2 \qquad \text{and} \quad \rho V_1 A_1 = \rho V_2 A_2$$

$$-p_{1g} A_1 - Mg = V_1 \rho V_2 A_2 - V_2 \rho V_2 A_2 = -\rho V_2 A_2 (V_2 - V_1)$$

For steady, incompressible flow without friction, along a streamline from atmosphere to ①, Bernoulli gives, neglecting Δz,

$$p_{atm} + \frac{1}{2}\rho V_0^2 + g z_0 = p_1 + \frac{1}{2}\rho V_1^2 + g z_1 \quad \text{so} \quad p_{1g} = -\frac{1}{2}\rho V_1^2$$

($V_0 \approx 0$)

Using continuity, $p_{1g}A_1 = -\frac{1}{2}\rho V_1^2 A_1 = -\frac{1}{2}\rho V_2 A_2 V_1 = -\frac{1}{2}\rho V_2^2 A_2 \frac{A_2}{A_1}$

Substituting into the momentum equation and using continuity,

$$\frac{1}{2}\rho V_2^2 A_2 \frac{A_2}{A_1} - Mg = -\rho V_2^2 A_2\left(1 - \frac{V_1}{V_2}\right) = -\rho V_2^2 A_2\left(1 - \frac{A_2}{A_1}\right) \quad \text{or} \quad Mg = \rho V_2^2 A_2\left(1 - \frac{1}{2}\frac{A_2}{A_1}\right)$$

Thus

$$V_2 = \sqrt{\frac{Mg}{\rho A_2\left(1 - \frac{1}{2}\frac{A_2}{A_1}\right)}} = \left[1500\,kg \times 9.81\frac{m}{s^2} \times \frac{m^3}{1.22\,kg} \times \frac{1}{1.48\,m^2} \times \frac{1}{\left(1 - \frac{1}{2}\frac{1.48}{8.55}\right)}\right]^{\frac{1}{2}} = 94.5\ m/s$$

V_2

Basic equation:

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall + \int_{CS}\left(u + pv + \frac{V^2}{2} + gz\right)\rho \vec{V}\cdot d\vec{A}$$

($\dot{W}_{shear} = 0(6)$, $\dot{W}_{other} = 0(6)$, $\frac{\partial}{\partial t} = 0(2)$, pv: $=0(7)$, $gz \approx 0(8)$)

Additional assumptions: (6) $\dot{W}_{shear} = \dot{W}_{other} = 0$
(7) pv = constant
(8) Neglect Δz

Then

$$-\dot{W}_s = \left(u_1 + \frac{V_1^2}{2}\right)\{-|\dot{m}|\} + \left(u_2 + \frac{V_2^2}{2}\right)\{|\dot{m}|\} - \dot{Q}$$

$$-\dot{W}_s = \dot{m}\left(\frac{V_2^2 - V_1^2}{2}\right) + \dot{m}\left(u_2 - u_1 - \frac{dQ}{dm}\right)$$

The term $\left(u_2 - u_1 - \frac{dQ}{dm}\right)$ represents nonmechanical energy. The minimum possible work would be attained when the nonmechanical energy is zero. Thus

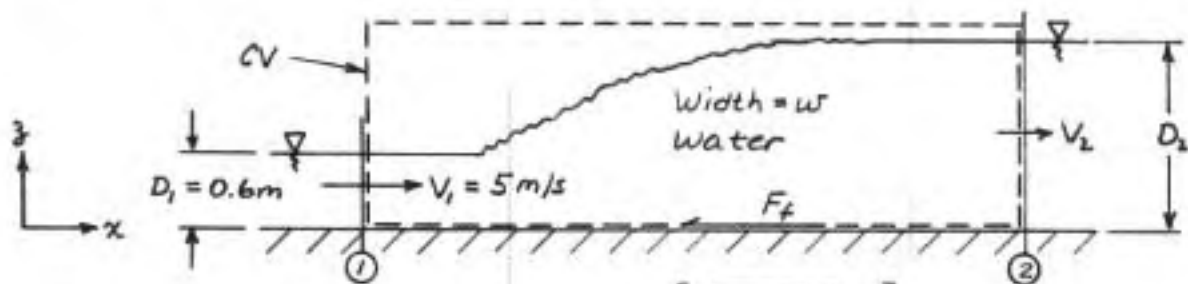
$$-\dot{W}_s\Big)_{min} = \dot{m}\left(\frac{V_2^2 - V_1^2}{2}\right) = \dot{m}\frac{V_2^2}{2}\left[1 - \left(\frac{V_1}{V_2}\right)^2\right] = \frac{\rho A_2 V_2^3}{2}\left[1 - \left(\frac{A_2}{A_1}\right)^2\right]$$

$$-\dot{W}_s = \frac{1}{2} \times 1.22\frac{kg}{m^3} \times 1.48\,m^2 \times (94.5)^3\frac{m^3}{s^3}\left[1 - \left(\frac{1.48}{8.55}\right)^2\right]\frac{N\cdot s^2}{kg\cdot m} \times \frac{kW\cdot s}{10^3\,N\cdot m}$$

$$\dot{W}_s\Big)_{min} = -739\ kW \quad (\text{input})$$

$\dot{W}_s$

{The power required for hovering in a real craft would be greater due to flow losses, nonuniformities, etc.}

Given: Liquid flow in a wide, horizontal open channel, as shown.

Find: (a) Show that in general, $D_2 = \frac{D_1}{2}\left[\sqrt{1 + \frac{8V_1^2}{gD_1}} - 1\right]$

(b) Change in mechanical energy across hydraulic jump.
(c) Temperature rise if no heat transfer.

Solution: Apply continuity, x component of momentum, and energy equations using CV shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$ ($\frac{\partial}{\partial t}$ term $=0(1)$)

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} V_x \rho\, d\forall + \int_{CS} V_x \rho \vec{V} \cdot d\vec{A}$$ ($F_{Bx} = 0(8)$, $\frac{\partial}{\partial t}$ term $= 0(1)$)

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall + \int_{CS} (e + pv)\rho \vec{V} \cdot d\vec{A};$$ ($\dot{Q} = 0(6)$, $\dot{W}_s = 0(7)$, $\dot{W}_{shear} = 0(7)$, $\dot{W}_{other} = 0(7)$, $\frac{\partial}{\partial t}$ term $= 0(1)$)

$$e = u + \frac{V^2}{2} + gz$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) Hydrostatic pressure distribution at sections ①, ②, so $p = \rho g (D - z)$
(5) Neglect friction force, F_f, on CV
(6) $\dot{Q} = 0$
(7) $\dot{W}_s = \dot{W}_{shear} = \dot{W}_{other} = 0$
(8) $F_{Bx} = 0$, since channel is horizontal

From continuity,

$$0 = \{-|\rho V_1 A_1|\} + \{|\rho V_2 A_2|\} = -\rho V_1 w D_1 + \rho V_2 w D_2 \,;\quad V_1 D_1 = V_2 D_2$$

From momentum,

$$F_{Sx} = \underbrace{\rho g \frac{D_1}{2} w D_1 - \rho g \frac{D_2}{2} w D_2}_{\text{hydrostatic forces}} = V_{x_1}\{-|\rho V_1 w D_1|\} + V_{x_2}\{|\rho V_2 w D_2|\}$$

$$V_{x_1} = V_1 \qquad V_{x_2} = V_2$$

or

$$\frac{g}{2}(D_1^2 - D_2^2) = V_1 D_1 (V_2 - V_1) = V_1^2 D_1 \left(\frac{V_2}{V_1} - 1\right) = V_1^2 D_1 \left(\frac{D_1}{D_2} - 1\right)$$

or

$$\frac{g}{2}(D_1 + D_2)\cancel{(D_1 - D_2)} = V_1^2 \frac{D_1}{D_2} \cancel{(D_1 - D_2)}$$

Thus $\frac{gD_1}{2}\left(1+\frac{D_2}{D_1}\right) = V_1^2\frac{D_1}{D_2}$ or $\frac{D_2}{D_1}\left(1+\frac{D_2}{D_1}\right) = \frac{2V_1^2}{gD_1}$ or $\left(\frac{D_2}{D_1}\right)^2 + \frac{D_2}{D_1} - \frac{2V_1^2}{gD_1} = 0$

Using the quadratic equation,

$$\frac{D_2}{D_1} = \frac{1}{2}\left[-1 \pm \sqrt{1+\frac{8V_1^2}{gD_1}}\right] \quad \text{or} \quad D_2 = \frac{D_1}{2}\left[\sqrt{1+\frac{8V_1^2}{gD_1}} - 1\right]$$

D_2

Solving for D_2

$$D_2 = \frac{1}{2} \times 0.6\,m\left[\sqrt{1 + 8 \times (5)^2\frac{m^2}{s^2} \times \frac{s^2}{9.81\,m} \times \frac{1}{0.6\,m}} - 1\right] = 1.47\,m$$

$$V_2 = \frac{D_1}{D_2}V_1 = \frac{0.6}{1.47} \times 5\frac{m}{s} = 2.04\,m/s$$

From the energy equation, with $e_{mech} = \frac{V^2}{2} + gz + \frac{p}{\rho}$, and $dA = w\,dz$, the mechanical energy fluxes are

$$mef_1 = \int_0^{D_1}\left[\frac{V_1^2}{2} + gz + \frac{1}{\rho}\rho g(D-z)\right]\rho V_1 w\,dz = \left(\frac{V_1^2}{2} + gD_1\right)\rho V_1 w D_1$$

$$mef_2 = \int_0^{D_2}\left[\frac{V_2^2}{2} + gz + \frac{1}{\rho}\rho g(D-z)\right]\rho V_2 w\,dz = \left(\frac{V_2^2}{2} + gD_2\right)\rho V_2 w D_2$$

and

$$\Delta mef = mef_2 - mef_1 = \left[\frac{V_2^2 - V_1^2}{2} + g(D_2 - D_1)\right]\rho V_1 w D_1\text{, since } V_1 D_1 = V_2 D_2$$

Thus $\frac{\Delta mef}{\dot{m}} = \frac{1}{2}\left[V_2^2 - V_1^2 + 2g(D_2 - D_1)\right]$

$$\frac{\Delta mef}{\dot{m}} = \frac{1}{2}\left[(2.04)^2\frac{m^2}{s^2} - (5)^2\frac{m^2}{s^2} + 2 \times 9.81\frac{m}{s^2}(1.47-0.6)m\right]\frac{N \cdot s^2}{kg \cdot m} = -1.88\,N\cdot m/kg$$

$\frac{\Delta mef}{\dot{m}}$

From the energy equation,

$$0 = \left[u_1 + \frac{V_1^2}{2} + gz + \frac{1}{\rho}\rho g(D-z)\right]\left\{-|\rho V_1 w D_1|\right\} + \left[u_2 + \frac{V_2^2}{2} + gz + \frac{1}{\rho}\rho g(D-z)\right]\left\{|\rho V_2 w D_2|\right\}$$

or

$$0 = (u_2 - u_1)\dot{m} + \Delta mef$$

Thus

$$u_2 - u_1 = c_v(T_2 - T_1) = -\frac{\Delta mef}{\dot{m}}$$

$$\Delta T = T_2 - T_1 = -\frac{\Delta mef}{\dot{m}\,c_v} = -\left(-1.88\frac{N\cdot m}{kg}\right)\frac{kg\cdot K}{1\,kcal} \times \frac{kcal}{4187\,J} = 4.49\times10^{-4}\,K$$

ΔT

{This small temperature change would be almost impossible to measure.}

Problem 5.1

Given: Velocity fields listed below

Find: Which are possible two-dimensional, incompressible flow cases?

Solution: Apply the continuity equation in differential form.

Basic equation: $\frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \frac{\partial}{\partial z}\rho w + \frac{\partial \rho}{\partial t} = 0$ (the $\frac{\partial}{\partial z}\rho w$ term is $= 0$ (1); the $\frac{\partial \rho}{\partial t}$ term is $= 0$ (2))

Assumptions: (1) Two-dimensional flow, $\vec{V} = \vec{V}(x,y)$, so $\frac{\partial}{\partial z} = 0$

(2) Incompressible flow

ρ = constant, so $\frac{\partial \rho}{\partial t} = 0$, $\frac{\partial \rho}{\partial(\text{distance})} = 0$

Then, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ is criterion.

(a) $u = 2x^2 + y^2 - x^2 y$, $v = x^3 + x(y^2 - 2y)$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = (4x - 2xy) + x(2y - 2)$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 4x - 2xy + 2xy - 2x \neq 0$

so $\rho \neq$ constant

(b) $u = 2xy - x^2 + y$, $v = 2xy - y^2 + x^2$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = (2y - 2x) + (2x - 2y) = 0$

so possible

(c) $u = xt + 2y$, $v = xt^2 - yt$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = t - t = 0$, so possible

(d) $u = (x + 2y)xt$, $v = -(2x + y)yt$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = (2xt + 2yt) + (-2xt - 2yt) = 0$

so possible

Problem 5.2

5.2 Which of the following sets of equations represent possible three-dimensional incompressible flow cases?

a. $u = y^2 + 2xz;\ v = -2yz + x^2yz;\ w = \frac{1}{2}x^2z^2 + x^3y^4$

b. $u = xyzt;\ v = -xyzt^2;\ w = (z^2/2)(xt^2 - yt)$

c. $u = x^2 + y + z^2;\ v = x - y + z;\ w = -2xz + y^2 + z$

Given: Velocity fields

Find: Which are 3D incompressible

Solution:

Basic equation: $$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0$$

Assumption: Incompressible flow

a) $u(x,y,z,t) = y^2 + 2\cdot x\cdot z$ $\quad v(x,y,z,t) = -2\cdot y\cdot z + x^2\cdot y\cdot z$ $\quad w(x,y,z,t) = \frac{1}{2}\cdot x^2\cdot z^2 + x^3\cdot y^4$

$\frac{\partial}{\partial x}u(x,y,z,t) \rightarrow 2\cdot z$ $\quad \frac{\partial}{\partial y}v(x,y,z,t) \rightarrow x^2\cdot z - 2\cdot z$ $\quad \frac{\partial}{\partial z}w(x,y,z,t) \rightarrow x^2\cdot z$

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0$ INCOMPRESSIBLE

b) $u(x,y,z,t) = x\cdot y\cdot z\cdot t$ $\quad v(x,y,z,t) = -x\cdot y\cdot z\cdot t^2$ $\quad w(x,y,z,t) = \frac{z^2}{2}\cdot\left(x\cdot t^2 - y\cdot t\right)$

$\frac{\partial}{\partial x}u(x,y,z,t) \rightarrow t\cdot y\cdot z$ $\quad \frac{\partial}{\partial y}v(x,y,z,t) \rightarrow -t^2\cdot x\cdot z$ $\quad \frac{\partial}{\partial z}w(x,y,z,t) \rightarrow z\cdot\left(t^2\cdot x - t\cdot y\right)$

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0$ INCOMPRESSIBLE

c) $u(x,y,z,t) = x^2 + y + z^2$ $\quad v(x,y,z,t) = x - y + z$ $\quad w(x,y,z,t) = -2\cdot x\cdot z + y^2 + z$

$\frac{\partial}{\partial x}u(x,y,z,t) \rightarrow 2\cdot x$ $\quad \frac{\partial}{\partial y}v(x,y,z,t) \rightarrow -1$ $\quad \frac{\partial}{\partial z}w(x,y,z,t) \rightarrow 1 - 2\cdot x$

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0$ INCOMPRESSIBLE

Problem 5.3

[1]

Given: Velocity field $u = Ax + By + Cz$

$v = Dx + Ey + Fz$

$w = Gx + Hy + Jz$

Find: The relationship among coefficients A thru J for this to be an incompressible flow field.

Solution: Flow must satisfy differential form of continuity.

Basic equation: $\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0$ (with $\frac{\partial \rho}{\partial t} = 0$)

Assumption: Incompressible flow, so $\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial x} = 0$

Then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

For the given flow field, $\frac{\partial u}{\partial x} = A$, $\frac{\partial v}{\partial y} = E$, $\frac{\partial w}{\partial z} = J$. Thus

$A + E + J = 0$, and

B, C, D, F, G, H are arbitrary

Problem 5.4

[2]

5.4 For a flow in the xy plane, the x component of velocity is given by $u = Ax(y - B)$, where $A = 1\ \text{ft}^{-1} \cdot \text{s}^{-1}$, $B = 6$ ft, and x and y are measured in feet. Find a possible y component for steady, incompressible flow. Is it also valid for unsteady, incompressible flow? Why? How many y components are possible?

Given: x component of velocity

Find: y component for incompressible flow; Valid for unsteady?; How many y components?

Solution:

Basic equation: $$\frac{\partial}{\partial x}(\rho \cdot u) + \frac{\partial}{\partial y}(\rho \cdot v) + \frac{\partial}{\partial z}(\rho \cdot w) + \frac{\partial}{\partial t}\rho = 0$$

Assumption: Incompressible flow; flow in x-y plane

Hence $$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0 \qquad \text{or} \qquad \frac{\partial}{\partial y}v = -\frac{\partial}{\partial x}u = -\frac{\partial}{\partial x}[A \cdot x \cdot (y - B)] = -A \cdot (y - B)$$

Integrating $$v(x,y) = -\int A \cdot (y - B)\, dy = -A \cdot \left(\frac{y^2}{2} - B \cdot y\right) + f(x)$$

This basic equation is valid for steady and unsteady flow (t is not explicit)

There are an infinite number of solutions, since f(x) can be any function of x. The simplest is f(x) = 0

$$v(x,y) = -A \cdot \left(\frac{y^2}{2} - B \cdot y\right) \qquad v(x,y) = 6 \cdot y - \frac{y^2}{2}$$

Problem 5.5

5.5 For a flow in the xy plane, the x component of velocity is given by $u = x^3 - 3xy^2$. Determine a possible y component for steady, incompressible flow. Is it also valid for unsteady, incompressible flow? Why? How many possible y components are there?

Given: x component of velocity

Find: y component for incompressible flow; Valid for unsteady? How many y components?

Solution:

Basic equation: $$\frac{\partial}{\partial x}(\rho \cdot u) + \frac{\partial}{\partial y}(\rho \cdot v) + \frac{\partial}{\partial z}(\rho \cdot w) + \frac{\partial}{\partial t}\rho = 0$$

Assumption: Incompressible flow; flow in x-y plane

Hence $$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0 \quad \text{or} \quad \frac{\partial}{\partial y}v = -\frac{\partial}{\partial x}u = -\frac{\partial}{\partial x}\left(x^3 - 3 \cdot x \cdot y^2\right) = -\left(3 \cdot x^2 - 3 \cdot y^2\right)$$

Integrating $$v(x,y) = -\int \left(3 \cdot x^2 - 3 \cdot y^2\right) dy = -3 \cdot x^2 \cdot y + y^3 + f(x)$$

This basic equation is valid for steady and unsteady flow (t is not explicit)

There are an infinite number of solutions, since f(x) can be any function of x. The simplest is f(x) = 0 $v(x,y) = y^3 - 3 \cdot x^2 \cdot y$

Problem 5.6

Given: Steady, incompressible flow field in the xy plane has an x component of velocity given by
$$u = \frac{A}{x}, \text{ where } A = 2\ m^2/s \text{ and } x \text{ is in meters.}$$

Find: the simplest y component of velocity for this flow field.

Solution:

Apply the continuity equation for the conditions given

Basic equation: $\nabla \cdot \rho \vec{V} + \frac{\partial \rho}{\partial t} = 0$

For steady flow $\frac{\partial \rho}{\partial t} = 0$ and for two-dimensional flow in the xy plane, $\frac{\partial}{\partial z}() = 0$. Thus the basic equation reduces to
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Then
$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{A}{x}\right) = \frac{A}{x^2}$$

and
$$v = \int \frac{\partial v}{\partial y}\, dy + f(x) = \int \frac{A}{x^2}\, dy + f(x) = \frac{Ay}{x^2} + f(x)$$

The simplest y component of velocity is obtained with $f(x) = 0$

$$\therefore\ v = \frac{Ay}{x^2} \longleftarrow \quad v$$

Problem 5.7

[2]

5.7 The y component of velocity in a steady, incompressible flow field in the xy plane is $v = Axy(y^2 - x^2)$, where $A = 2\ m^{-3} \cdot s^{-1}$ and x and y are measured in meters. Find the simplest x component of velocity for this flow field.

Given: y component of velocity

Find: x component for incompressible flow; Simplest x components?

Solution:

Basic equation: $$\frac{\partial}{\partial x}(\rho \cdot u) + \frac{\partial}{\partial y}(\rho \cdot v) + \frac{\partial}{\partial z}(\rho \cdot w) + \frac{\partial}{\partial t}\rho = 0$$

Assumption: Incompressible flow; flow in x-y plane

Hence $$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$$ or $$\frac{\partial}{\partial x}u = -\frac{\partial}{\partial y}v = -\frac{\partial}{\partial y}\left[A \cdot x \cdot y \cdot \left(y^2 - x^2\right)\right] = -\left[A \cdot x \cdot \left(y^2 - x^2\right) + A \cdot x \cdot y \cdot 2 \cdot y\right]$$

Integrating $$u(x,y) = -\int A \cdot \left(3 \cdot x \cdot y^2 - x^3\right) dx = -\frac{3}{2} \cdot A \cdot x^2 \cdot y^2 + \frac{1}{4} \cdot A \cdot x^4 + f(y)$$

This basic equation is valid for steady and unsteady flow (t is not explicit)

There are an infinite number of solutions, since f(y) can be any function of y. The simplest is f(y) = 0

$$u(x,y) = \frac{1}{4} \cdot A \cdot x^4 - \frac{3}{2} \cdot A \cdot x^2 \cdot y^2 \qquad u(x,y) = \frac{1}{2} \cdot x^4 - 3 \cdot x^2 y^2$$

Problem 5.8

5.8 The x component of velocity in a steady incompressible flow field in the xy plane is $u = Ae^{x/b}\cos(y/b)$, where $A = 10$ m/s, $b = 5$ m, and x and y are measured in meters. Find the simplest y component of velocity for this flow field.

Given:
x component of velocity

Find:
y component for incompressible flow; Valid for unsteady? How many y components?

Solution:

Basic equation:
$$\frac{\partial}{\partial x}(\rho\cdot u) + \frac{\partial}{\partial y}(\rho\cdot v) + \frac{\partial}{\partial z}(\rho\cdot w) + \frac{\partial}{\partial t}\rho = 0$$

Assumption: Incompressible flow; flow in x-y plane

Hence
$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0 \qquad \text{or} \qquad \frac{\partial}{\partial y}v = -\frac{\partial}{\partial x}u = -\frac{\partial}{\partial x}\left(A\cdot e^{\frac{x}{b}}\cdot\cos\left(\frac{y}{b}\right)\right) = -\left(\frac{A}{b}\cdot e^{\frac{x}{b}}\cdot\cos\left(\frac{y}{b}\right)\right)$$

Integrating
$$v(x,y) = -\int \frac{A}{b}\cdot e^{\frac{x}{b}}\cdot\cos\left(\frac{y}{b}\right)dy = -A\cdot e^{\frac{x}{b}}\cdot\sin\left(\frac{y}{b}\right) + f(x)$$

This basic equation is valid for steady and unsteady flow (t is not explicit)

There are an infinite number of solutions, since f(x) can be any function of x. The simplest is f(x) = 0

$$v(x,y) = -A\cdot e^{\frac{x}{b}}\cdot\sin\left(\frac{y}{b}\right) \qquad v(x,y) = -10\cdot e^{\frac{x}{5}}\cdot\sin\left(\frac{y}{5}\right)$$

Problem 5.9

5.9 The y component of velocity in a steady incompressible flow field in the xy plane is

$$v = \frac{2xy}{(x^2 + y^2)^2}$$

Show that the simplest expression for the x component of velocity is

$$u = \frac{1}{(x^2 + y^2)} - \frac{2y^2}{(x^2 + y^2)^2}$$

Given: y component of velocity

Find: x component for incompressible flow; Simplest x component

Solution:

Basic equation: $$\frac{\partial}{\partial x}(\rho \cdot u) + \frac{\partial}{\partial y}(\rho \cdot v) + \frac{\partial}{\partial z}(\rho \cdot w) + \frac{\partial}{\partial t}\rho = 0$$

Assumption: Incompressible flow; flow in x-y plane

Hence $$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0 \qquad \text{or} \qquad \frac{\partial}{\partial x}u = -\frac{\partial}{\partial y}v = -\frac{\partial}{\partial y}\left[\frac{2 \cdot x \cdot y}{\left(x^2 + y^2\right)^2}\right] = -\left[\frac{2 \cdot x \cdot \left(x^2 - 3 \cdot y^2\right)}{\left(x^2 + y^2\right)^3}\right]$$

Integrating $$u(x,y) = -\int \left[\frac{2 \cdot x \cdot \left(x^2 - 3 \cdot y^2\right)}{\left(x^2 + y^2\right)^3}\right] dx = \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2} + f(y) = \frac{x^2 + y^2 - 2 \cdot y^2}{\left(x^2 + y^2\right)^2} + f(y)$$

$$u(x,y) = \frac{1}{x^2 + y^2} - \frac{2 \cdot y^2}{\left(x^2 + y^2\right)^2} + f(y)$$

The simplest form is $$u(x,y) = \frac{1}{x^2 + y^2} - \frac{2 \cdot y^2}{\left(x^2 + y^2\right)^2}$$

Note: Instead of this approach we could have verified that u and v satisfy continuity

$$\frac{\partial}{\partial x}\left[\frac{1}{x^2 + y^2} - \frac{2 \cdot y^2}{\left(x^2 + y^2\right)^2}\right] + \frac{\partial}{\partial y}\left[\frac{2 \cdot x \cdot y}{\left(x^2 + y^2\right)^2}\right] \rightarrow 0$$

However, this does not verify the solution is the simplest

Problem 5.10

Given: Approximate profile for laminar boundary layer

$$u = c\,U\frac{y}{x^{1/2}}$$

Find: (a) Show simplest v is $v = \frac{U}{4}\frac{y}{x}$

(b) Evaluate maximum value of v/U where $\delta = 5\,mm$, $x = 0.5\,m$.

Solution: Apply continuity for incompressible flow

Basic equation: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ (2-D flow: $\frac{\partial w}{\partial z}$ crossed out)

Thus

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\left(-\frac{1}{2}\right)c\,U\frac{y}{x^{3/2}}$$

$$v = \int\frac{\partial v}{\partial y}dy + f(x) = \int\frac{1}{2}c\,U\frac{y}{x^{3/2}}dy + f(x) = \frac{1}{4}c\,U\frac{y^2}{x^{3/2}} + f(x)$$

or

$$v = \frac{U}{4}\frac{y}{x} \quad [f(x) = 0 \text{ since } v = 0 \text{ along } y = 0]$$

v

From

$$\frac{v}{U} = \frac{1}{4}\frac{y}{x}$$

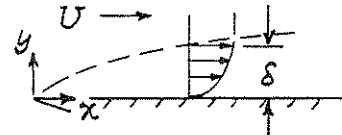
maximum value occurs at $y = \delta$. At the location given,

$$\left.\frac{v}{U}\right)_{max} = \frac{1}{4}\frac{\delta}{x} = \frac{1}{4}\frac{0.005\,m}{0.5\,m} = 0.0025$$

$\left.\frac{v}{U}\right)_{m}$

Problem 5.11

Given: Laminar boundary layer, parabolic approximate profile.

$$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \qquad \delta = cx^{1/2}$$

Find: Show $\frac{v}{U} = \frac{\delta}{x}\left[\frac{1}{2}\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right]$ for incompressible flow.

Plot: $\frac{v}{U}$ vs. $\frac{y}{\delta}$, evaluate max. at $x = 0.5$ m, if $\delta = 5$ mm.

Solution: Apply conservation of mass for incompressible flow.

Basic equation: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ (third term $= 0$ (2))

Assumptions: (1) Incompressible flow (ρ = const)
(2) $w = 0$

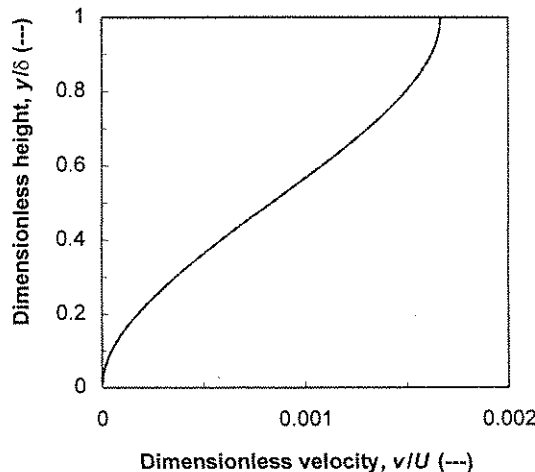
Then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$; $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$; $v = \int_0^y -\frac{\partial u}{\partial x}\,dy + f(x)$ ($f(x) = 0$ (simplest))

From the given profile

$$\frac{\partial u}{\partial x} = 2Uy(-1)\frac{1}{\delta^2}\frac{d\delta}{dx} - Uy^2(-2)\frac{1}{\delta^3}\frac{d\delta}{dx} = 2U\frac{d\delta}{dx}\left(\frac{y^2}{\delta^3} - \frac{y}{\delta^2}\right)$$

Since $\delta = cx^{1/2}$, $\frac{d\delta}{dx} = \frac{1}{2}cx^{-1/2} = \frac{cx^{1/2}}{2x} = \frac{\delta}{2x}$, so $\frac{\partial u}{\partial x} = \frac{U\delta}{x}\left(\frac{y^2}{\delta^3} - \frac{y}{\delta^2}\right)$

Integrating, $\frac{v}{U} = \frac{\delta}{x}\int_0^y \left(\frac{y}{\delta^2} - \frac{y^2}{\delta^3}\right)dy = \frac{\delta}{x}\left[\frac{1}{2}\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right]$ ← $\frac{v}{U}$

Plotting shows:

Maximum occurs at $\left(\frac{y}{\delta}\right) = 1$

$$\left.\frac{v}{U}\right)_{max} = \left.\frac{v}{U}\right)_{\frac{y}{\delta}=1} = \frac{\delta}{x}\left[\frac{1}{2}(1)^2 - \frac{1}{3}(1)^2\right] = \frac{\delta}{6x}$$ ← $\left.\frac{v}{U}\right)_{max}$

Evaluating, $\left.\frac{v}{U}\right)_{max} = \frac{1}{6} \times 0.005\,\text{m} \times \frac{1}{0.5\,\text{m}} = 0.00167$ or 0.167 percent ← $\left.\frac{v}{U}\right)_{max}$

Problem 5.12

Given: Approximation for x component of velocity in laminar boundary layer

$$u = U\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) \qquad \text{where } \delta = cx^{1/2}$$

Show: $\frac{v}{U} = \frac{\delta}{\pi x}\left[\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) + \frac{\pi}{2}\frac{y}{\delta}\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) - 1\right]$ for incompressible flow.

Plot: $\frac{x}{\delta}\frac{v}{U}$ vs. y/δ to locate maximum value of v/U; evaluate at location where $x = 0.5\ m$ and $\delta = 5\ mm$.

Solution: Apply differential continuity for incompressible flow.

Basic equation: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ ($\frac{\partial w}{\partial z} = 0$ (2-D flow))

Thus $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial \delta}\frac{d\delta}{dx} = -\left(\frac{\pi y}{2}\right)\left(-\frac{1}{\delta^2}\right)\cos\left(\frac{\pi y}{2\delta}\right)\frac{U}{2}cx^{-1/2} = \frac{U}{2x}\left(\frac{\pi y}{2\delta}\right)\cos\left(\frac{\pi y}{2\delta}\right)$

Integrating, $v = \int_0^y \frac{\partial v}{\partial y}dy + f(x) = \int_0^y \frac{U}{2x}\left(\frac{\pi}{2}\frac{y}{\delta}\right)\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)dy + f(x)$

$$v = \frac{2\delta}{\pi}\frac{U}{2x}\int_0^{\frac{\pi}{2}\frac{y}{\delta}} \eta\cos\eta\, d\eta + f(x) = \frac{\delta}{\pi}\frac{U}{x}\Big[\cos\eta + \eta\sin\eta\Big]_0^{\frac{\pi}{2}\frac{y}{\delta}} + f(x)^{\,0}$$

$$\frac{v}{U} = \frac{1}{\pi}\frac{\delta}{x}\left[\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) + \left(\frac{\pi}{2}\frac{y}{\delta}\right)\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) - 1\right]$$

This expression is a maximum at $y = \delta$, where

$$\frac{v}{U} = \frac{1}{\pi}\frac{\delta}{x}\left[\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) - 1\right] = \frac{\delta}{\pi x}\left(\frac{\pi}{2} - 1\right)$$

and

$$\left.\frac{v}{U}\right)_{max} = 0.182\frac{\delta}{x}$$

$\left.\frac{v}{U}\right)_{max}$

At the location given

$$\left.\frac{v}{U}\right)_{max} = 0.182 \times 0.005\ m \times \frac{1}{0.5\ m} = 0.00182 \text{ or } 0.182 \text{ percent}$$

$\left.\frac{v}{U}\right)_{max}$

Problem 5.13

5.13 A useful approximation for the x component of velocity in an incompressible laminar boundary layer is a cubic variation from $u = 0$ at the surface ($y = 0$) to the freestream velocity, U, at the edge of the boundary layer ($y = \delta$). The equation for the profile is $u/U = \frac{3}{2}(y/\delta) - \frac{1}{2}(y/\delta)^3$, where $\delta = cx^{1/2}$ and c is a constant. Derive the simplest expression for v/U, the y component of velocity ratio. Plot u/U and v/U versus y/δ, and find the location of the maximum value of the ratio v/U. Evaluate the ratio where $\delta = 5$ mm and $x = 0.5$ m.

Given: Data on boundary layer

Find: y component of velocity ratio; location of maximum value; plot velocity profiles; evaluate at particular point

Solution:

$$u(x,y) = U\cdot\left[\frac{3}{2}\cdot\left(\frac{y}{\delta(x)}\right) - \frac{1}{2}\cdot\left(\frac{y}{\delta(x)}\right)^3\right] \qquad \text{and} \qquad \delta(x) = c\cdot\sqrt{x}$$

so

$$u(x,y) = U\cdot\left[\frac{3}{2}\cdot\left(\frac{y}{c\cdot\sqrt{x}}\right) - \frac{1}{2}\cdot\left(\frac{y}{c\cdot\sqrt{x}}\right)^3\right]$$

For incompressible flow

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$$

Hence

$$v(x,y) = -\int \frac{d}{dx}u(x,y)\,dy \qquad \text{and} \qquad \frac{du}{dx} = \frac{3}{4}\cdot U\cdot\left(\frac{y^3}{c^3\cdot x^{\frac{5}{2}}} - \frac{y}{c\cdot x^{\frac{3}{2}}}\right)$$

so

$$v(x,y) = -\int \frac{3}{4}\cdot U\cdot\left(\frac{y^3}{c^3}\cdot\frac{x^5}{2} - \frac{y}{c}\cdot\frac{x^3}{2}\right)dy$$

$$v(x,y) = \frac{3}{8}\cdot U\cdot\left(\frac{y^2}{c\cdot x^{\frac{3}{2}}} - \frac{y^4}{2\cdot c^3\cdot x^{\frac{5}{2}}}\right) \qquad v(x,y) = \frac{3}{8}\cdot U\cdot\frac{\delta}{x}\cdot\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^4\right]$$

The maximum occurs at $y = \delta$ as seen in the corresponding *Excel* workbook

$$v_{max} = \frac{3}{8}\cdot U\cdot\frac{\delta}{x}\cdot\left(1 - \frac{1}{2}\cdot 1\right)$$

At $\delta = 5\cdot mm$ and $x = 0.5\cdot m$, the maximum vertical velocity is $\frac{v_{max}}{U} = 0.00188$

Problem 5.13

5.13 A useful approximation for the x component of velocity in an incompressible laminar boundary layer is a cubic variation from $u = 0$ at the surface ($y = 0$) to the freestream velocity, U, at the edge of the boundary layer ($y = \delta$). The equation for the profile is $u/U = \frac{3}{2}(y/\delta) - \frac{1}{2}(y/\delta)^3$, where $\delta = cx^{1/2}$ and c is a constant. Derive the simplest expression for v/U, the y component of velocity ratio. Plot u/U and v/U versus y/δ, and find the location of the maximum value of the ratio v/U. Evaluate the ratio where $\delta = 5$ mm and $x = 0.5$ m.

Given: Data on boundary layer

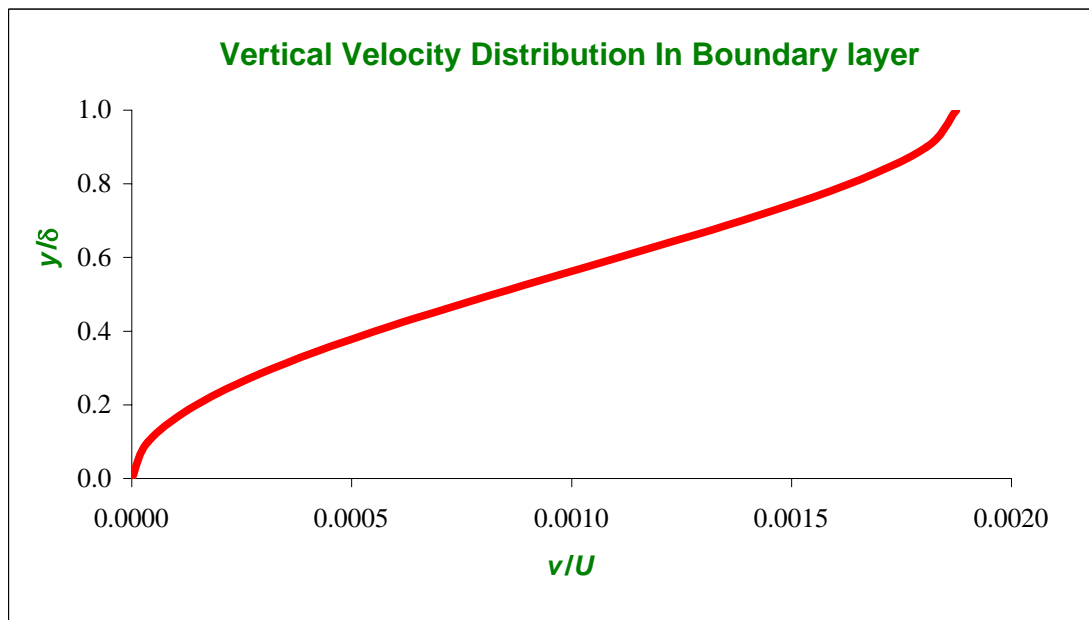
Find: y component of velocity ratio; location of maximum value; plot velocity profiles; evaluate at particular point

Solution:

$$v(x,y) = \frac{3}{8}\cdot U\cdot\frac{\delta}{x}\cdot\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^4\right]$$

To find when v/U is maximum, use *Solver*

v/U	y/d
0.00188	1.0

v/U	y/d
0.000000	0.0
0.000037	0.1
0.000147	0.2
0.000322	0.3
0.000552	0.4
0.00082	0.5
0.00111	0.6
0.00139	0.7
0.00163	0.8
0.00181	0.9
0.00188	1.0

Problem 5.14

Given: Flow in xy plane, $v = -Bxy^3$ where $B = 0.2\ m^{-3}\cdot s^{-1}$ and coordinates are measured in meters; steady, $\rho = c$.

Find: (a) Simplest x component of velocity.
(b) Equation of streamlines.

Plot: streamlines through points (1,4) and (2,4).

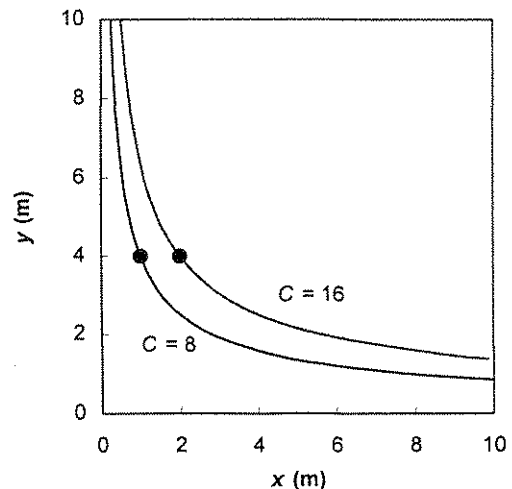
Solution:

Basic equation: $\nabla\cdot\rho\vec{V} + \frac{\partial\rho}{\partial t} = \frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \cancel{\frac{\partial}{\partial z}\rho w}^{=0(1)} + \cancel{\frac{\partial\rho}{\partial t}}^{=0(2)}$

Assumptions: (1) flow in the xy plane (given), $\frac{\partial}{\partial z} = 0$
(2) ρ = constant (given).

Then,
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{or} \quad \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}.$$
and
$$\frac{\partial u}{\partial x} = -\frac{\partial}{\partial y}(-Bxy^3) = 3Bxy^2$$
Integrating,
$$u = \int \frac{\partial u}{\partial x}dx = \int 3Bxy^2 = \frac{3}{2}Bx^2y^2 + f(y).$$
The simplest expression is obtained with $f(y) = 0$
$$\therefore\ u = \frac{3}{2}Bx^2y^2 \longleftarrow u$$
The equation of the streamlines is
$$\left.\frac{dy}{dx}\right)_{sl} = \frac{v}{u} = \frac{-Bxy^3}{\frac{3}{2}Bx^2y^2} = \frac{-2y}{3x}$$
Separating variables & integrating
$$\frac{3}{2}\frac{dy}{y} + \frac{dx}{x} = 0$$
$$\frac{3}{2}\ln y + \ln x = \ln c$$
$$xy^{3/2} = c \quad \text{Streamline}$$

pt (1,4) $\quad xy^{3/2} = 8$

pt (2,4) $\quad xy^{3/2} = 16$

Problem 5.15

Given: Flow in xy plane, $u = Ax^2y^2$ where $A = 0.3\ m^{-3} \cdot s^{-1}$, and coordinates are measured in meters

Find: (a) Possible y component for steady, incompressible flow.
(b) If result is valid for unsteady, incompressible flow
(c) Number of possible y components.
(d) Equation of streamlines for simplest value of v.

Plot: streamlines through points (1,4) and (2,4)

Solution:

Basic equation: $\nabla \cdot \rho\vec{V} + \frac{\partial \rho}{\partial t} = 0 = \frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \frac{\partial}{\partial z}\rho w + \frac{\partial \rho}{\partial t}$ ($=0$ (1), $=0$ (2))

Assumptions: (1) flow in xy plane (given), $\frac{\partial}{\partial z} = 0$
(2) ρ = constant (given)

Then,
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{or} \quad \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{\partial}{\partial x}(Ax^2y^2) = -2Axy^2$$

Integrating
$$v = \int \frac{\partial v}{\partial y}\, dy = -\int 2Axy^2 = -\frac{2}{3}Axy^3 + f(x) \quad \leftarrow v$$

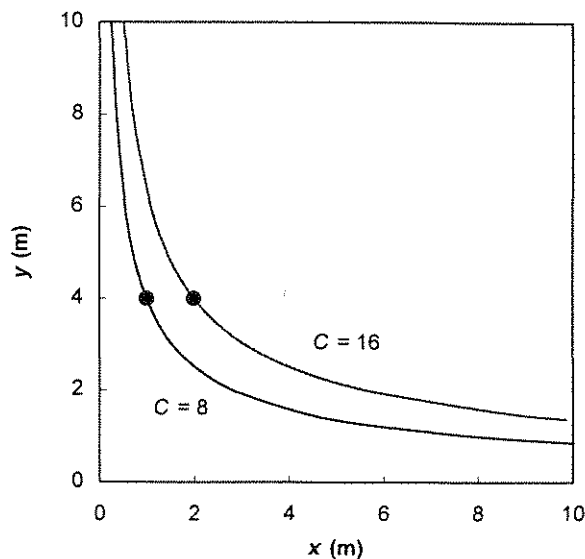
The basic equation reduces to the same form for unsteady flow. Hence the result is also valid for unsteady flow. ← v (b)

There are an infinite number of possible y components, since $f(x)$ is arbitrary. The simplest is obtained with $f(x) = 0$. ← (c)

The equation of the streamline is
$$\left.\frac{dy}{dx}\right)_{s.l} = \frac{v}{u} = \frac{-\frac{2}{3}Axy^3}{Ax^2y^2} = -\frac{2y}{3x}$$

Separating variables & integrating
$$\frac{3}{2}\frac{dy}{y} + \frac{dx}{dx} = 0$$
$$\ln y^{3/2} + \ln x = \ln c$$
$$xy^{3/2} = c \quad \leftarrow \text{Streamline}$$

pt (1,4) $\quad xy^{3/2} = 8$
(2,4) $\quad xy^{3/2} = 16$

Problem 5.16

Given: Conservation of mass.

Find: Identical result to Eq. 5.1a by expanding products of density and velocity in Taylor series.

Solution: Use diagram of Fig. 5.1:

Fig. 5.1 Differential control volume in rectangular coordinates

Apply conservation of mass, using a Taylor series expansion of products. Evaluate derivatives at O.

For the x direction the mass flux is

$$\dot{m}_x = \rho u \, dA = \rho u \, dx \, dy$$

At the right face

$$\dot{m}_{x+dx/2} = \rho u \, dy \, dz + \frac{\partial}{\partial x} \rho u \frac{dx}{2} dy \, dz \quad \text{(out of CV)}$$

At the left face

$$\dot{m}_{x-dx/2} = \rho u \, dy dz + \frac{\partial}{\partial x} \rho u \left(-\frac{dx}{2}\right) dy dz \quad \text{(into CV)}$$

The net mass flux is "out" minus "in," so

$$\dot{m}_x(\text{net}) = \dot{m}_{x+dx/2} - \dot{m}_{x-dx/2} = \frac{\partial}{\partial x} \rho u \, dx dy dz$$

Summing terms for x, y, and z, and including $\frac{\partial \rho}{\partial t} dx dy dz$, we get

$$0 = \frac{\partial}{\partial x} \rho u + \frac{\partial}{\partial y} \rho v + \frac{\partial}{\partial z} \rho w + \frac{\partial \rho}{\partial t}$$

Problem 5.17

[5]

Consider a water stream from a jet of an oscillating lawn sprinkler. Describe the corresponding pathline and streakline.

Open-Ended Problem Statement: Consider a water stream from a jet of an oscillating lawn sprinkler. Describe the corresponding pathline and streakline.

Discussion: Refer back to the discussion of streamlines, pathlines, and streaklines in Section 2-2.

Because the sprinkler jet oscillates, this is an unsteady flow. Therefore pathlines and streaklines need not coincide.

A *pathline* is a line tracing the path of an individual fluid particle. The path of each particle is determined by the jet angle and the speed at which the particle leaves the jet.

Once a particle leaves the jet it is subject to gravity and drag forces. If aerodynamic drag were negligible, the path of each particle would be parabolic. The horizontal speed of the particle would remain constant throughout its trajectory. The vertical speed would be slowed by gravity until reaching peak height, and then it would become increasingly negative until the particle strikes the ground. The effect of aerodynamic drag is to reduce the particle speed. With drag the particle will not rise as high vertically nor travel as far horizontally. At each instant the particle trajectory will be lower and closer to the jet compared to the no-friction case. The trajectory after the particle reaches its peak height will be steeper than in the no-friction case.

A *streamline* is a line drawn in the flow that is tangent everywhere to the velocity vectors of the fluid motion. It is difficult to visualize the streamlines for an unsteady flow field because they move laterally. However, the streamline pattern may be drawn at an instant.

A *streakline* is the locus of the present locations of fluid particles that passed a reference point at previous times. As an example, choose the exit of a jet as the reference point. Imagine marking particles that pass the jet exit at a given instant and at uniform time intervals later. The first particle will travel farthest from the jet exit and on the lowest trajectory; the last particle will be located right at the jet exit. The curve joining the present positions of the particles will resemble a spiral whose radius increases with distance from the jet opening.

Problem 5.18

Given: Velocity fields listed below.

Find: Which are possible incompressible flow cases?

Solution: Apply the continuity equation in differential form.

Basic equation: $\frac{1}{r}\frac{\partial r\rho V_r}{\partial r} + \frac{1}{r}\frac{\partial \rho V_\theta}{\partial \theta} + \frac{\partial \rho V_z}{\partial z} + \frac{\partial \rho}{\partial t} = 0$ (with $\frac{\partial \rho V_z}{\partial z} = 0$ (1) and $\frac{\partial \rho}{\partial t} = 0$ (2))

Assumptions: (1) Two-dimensional flow, so $\frac{\partial}{\partial z} = 0$
(2) Incompressible flow

$$\rho = \text{constant}, \text{ so } \frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial(\text{distance})} = 0$$

Then

$$\frac{1}{r}\frac{\partial rV_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} = 0$$

or

$$\frac{\partial rV_r}{\partial r} + \frac{\partial V_\theta}{\partial \theta} = 0 \quad \text{is the criterion.}$$

Field	V_r	V_θ	$\frac{\partial rV_r}{\partial r}$	$\frac{\partial V_\theta}{\partial \theta}$	$\frac{\partial rV_r}{\partial r} + \frac{\partial V_\theta}{\partial \theta}$	Possible?
(a)	$U\cos\theta$	$-U\sin\theta$	$U\cos\theta$	$-U\cos\theta$	0	Yes
(b)	$-\frac{q}{2\pi r}$	$\frac{K}{2\pi r}$	0	0	0	Yes
(c)	$U\cos\theta\left[1-\left(\frac{a}{r}\right)^2\right]$	$-U\sin\theta\left[1+\left(\frac{a}{r}\right)^2\right]$	$U\cos\theta\left[1+\left(\frac{a}{r}\right)^2\right]$ *	$-U\cos\theta\left[1+\left(\frac{a}{r}\right)^2\right]$	0	Yes

* Note if $V_r = U\cos\theta\left[1-\left(\frac{a}{r}\right)^2\right]$, then $rV_r = U\cos\theta\left[r - \frac{a^2}{r}\right]$

and $\frac{\partial rV_r}{\partial r} = U\cos\theta\left[1 + \frac{a^2}{r^2}\right] = U\cos\theta\left[1 + \left(\frac{a}{r}\right)^2\right]$

Problem 5.19

[3]

5.19 For an incompressible flow in the $r\theta$ plane, the r component of velocity is given as $V_r = -\Lambda \cos\theta / r^2$. Determine a possible θ component of velocity. How many possible θ components are there?

Given: r component of velocity

Find: θ component for incompressible flow; How many θ components

Solution:

Basic equation: $$\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(\rho\cdot r\cdot V_r\right) + \frac{1}{r}\cdot\frac{\partial}{\partial\theta}\left(\rho\cdot V_\theta\right) + \frac{\partial}{\partial z}\left(\rho\cdot V_z\right) + \frac{\partial}{\partial t}\rho = 0$$

Assumption: Incompressible flow; flow in r-θ plane

Hence $$\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\cdot V_r\right) + \frac{1}{r}\cdot\frac{\partial}{\partial\theta}\left(V_\theta\right) = 0$$ or $$\frac{\partial}{\partial\theta}V_\theta = -\frac{\partial}{\partial r}\left(r\cdot V_r\right) = -\frac{\partial}{\partial r}\left(-\frac{\Lambda\cdot\cos(\theta)}{r}\right) = -\frac{\Lambda\cdot\cos(\theta)}{r^2}$$

Integrating $$V_\theta(r,\theta) = -\int \frac{\Lambda\cdot\cos(\theta)}{r^2}\,d\theta = -\frac{\Lambda\cdot\sin(\theta)}{r^2} + f(r)$$

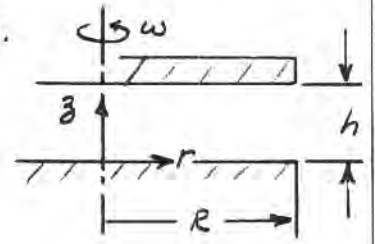
$$V_\theta(r,\theta) = -\frac{\Lambda\cdot\sin(\theta)}{r^2} + f(r)$$

There are an infinite number of solutions as f(r) can be any function of r

The simplest form is $$V_\theta(r,\theta) = -\frac{\Lambda\cdot\sin(\theta)}{r^2}$$

Problem 5.20

Given: Flow between parallel disks as shown.

Velocity is purely tangential.
No-slip condition is satisfied, so velocity varies linearly with z.

Find: Expression for velocity field.

<u>Solution</u>: A general velocity field would be

$$\vec{V} = V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_z \hat{k}$$

but velocity is purely tangential, so $V_r = V_z = 0$. Then we seek

$$V_\theta = V_\theta(r, \theta, z)$$

By symmetry, $\frac{\partial V_\theta}{\partial \theta} = 0$, so

$$V_\theta = V_\theta(r, z)$$

Since the variation with z is <u>linear</u>, $V_\theta = z f(r) + c$ at most, that is

$$\frac{\partial V_\theta}{\partial z} = f(r)$$

at most.

Along the surface $z = 0$, $V_\theta = 0$, so $c = 0$.

Along the surface $z = h$, $V_\theta = \omega r$, so

$$V_\theta(z = h) = \omega r = h f(r)$$

or

$$f(r) = \frac{\omega r}{h}$$

and

$$V_\theta = \omega r \frac{z}{h}$$

Thus

$$\vec{V} = \omega r \frac{z}{h} \hat{e}_\theta$$

$\vec{V}$

Problem 5.21

Given: Definition of ∇ in cylindrical coordinates.

Obtain: $\nabla \cdot \rho\vec{V}$ in cylindrical coordinates (use hint on page 169).

Show result is identical to Eq. 5.2c.

Solution: The definition of ∇ in cylindrical coordinates is

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z} \qquad (3.19)$$

Note $\rho\vec{V} = \rho(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{k} V_z)$

Hint: $\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta$, and $\frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r$ (Page 169)

Substituting $\nabla \cdot \rho\vec{V} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \rho(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{k} V_z)$

$$\nabla \cdot \rho\vec{V} = \hat{e}_r \cdot \frac{\partial}{\partial r} \rho(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{k} V_z)$$
$$+ \hat{e}_\theta \cdot \frac{\partial}{\partial \theta} \rho(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{k} V_z)$$
$$+ \hat{k} \cdot \frac{\partial}{\partial z} \rho(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{k} V_z)$$

$$= \hat{e}_r \cdot \hat{e}_r \frac{\partial}{\partial r}\rho V_r + \hat{e}_\theta \cdot \frac{1}{r}\frac{\partial \hat{e}_r}{\partial \theta}\rho V_r + \hat{e}_\theta \cdot \hat{e}_r \frac{\partial}{\partial \theta}\rho V_r$$
$$+ \hat{e}_\theta \cdot \frac{\partial \hat{e}_\theta}{\partial \theta}\rho V_\theta + \hat{e}_\theta \cdot \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\rho V_\theta + \hat{k} \cdot \hat{k}\frac{\partial}{\partial z}\rho V_z$$

(with $\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta$, $\hat{e}_\theta \cdot \hat{e}_r = 0$, $\frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r$)

$$\nabla \cdot \rho\vec{V} = \frac{\partial}{\partial r}\rho V_r + \rho\frac{V_r}{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\rho V_\theta + \frac{\partial}{\partial z}\rho V_z$$

Combining the first two terms, $\frac{\partial}{\partial r}\rho V_r + \rho\frac{V_r}{r} = \frac{1}{r}\frac{\partial}{\partial r} r\rho V_r$, as may be verified by differentiation. Substituting

$$\nabla \cdot \rho\vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z)$$

This result is identical to the corresponding terms in Eq. 5.2c.

Problem 5.22

5.22 A velocity field in cylindrical coordinates is given as $\vec{V} = \hat{e}_r A/r + \hat{e}_\theta B/r$, where A and B are constants with dimensions of m^2/s. Does this represent a possible incompressible flow? Sketch the streamline that passes through the point $r_0 = 1$ m, $\theta = 90°$ if $A = B = 1\ m^2/s$, if $A = 1\ m^2/s$ and $B = 0$, and if $B = 1\ m^2/s$ and $A = 0$.

Given: The velocity field

Find: Whether or not it is a incompressible flow; sketch various streamlines

Solution:

$$V_r = \frac{A}{r} \qquad V_\theta = \frac{B}{r}$$

For incompressible flow $\quad \frac{1}{r}\cdot\frac{d}{dr}(r\cdot V_r) + \frac{1}{r}\cdot\frac{d}{d\theta}V_\theta = 0 \qquad \frac{1}{r}\cdot\frac{d}{dr}(r\cdot V_r) = 0 \qquad \frac{1}{r}\cdot\frac{d}{d\theta}V_\theta = 0$

Hence $\quad \frac{1}{r}\cdot\frac{d}{dr}(r\cdot V_r) + \frac{1}{r}\cdot\frac{d}{d\theta}V_\theta = 0 \qquad$ Flow is incompressible

For the streamlines $\quad \frac{dr}{V_r} = \frac{r\cdot d\theta}{V_\theta} \qquad \frac{r\cdot dr}{A} = \frac{r^2\cdot d\theta}{B}$

so $\quad \int \frac{1}{r}\,dr = \int \frac{A}{B}\,d\theta \qquad$ Integrating $\quad \ln(r) = \frac{A}{B}\cdot\theta + const$

Equation of streamlines is $\quad r = C\cdot e^{\frac{A}{B}\cdot\theta}$

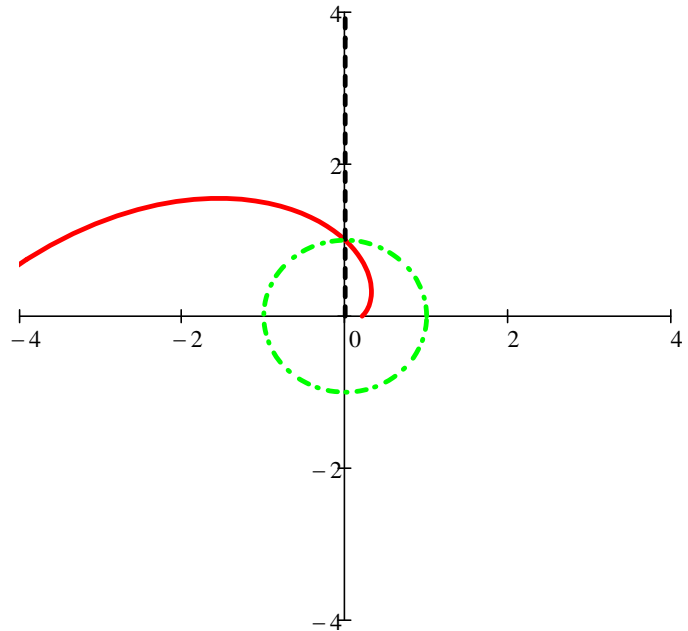
(a) For $A = B = 1\ m^2/s$, passing through point (1m, $\pi/2$)

$$r = e^{\theta - \frac{\pi}{2}}$$

(b) For $A = 1\ m^2/s$, $B = 0\ m^2/s$, passing through point (1m, $\pi/2$)

$$\theta = \frac{\pi}{2}$$

(c) For $A = 0\ m^2/s$, $B = 1\ m^2/s$, passing through point (1m, $\pi/2$)

$$r = 1\cdot m$$

Problem *5.23

Given: Velocity field for viscometric flow of Example Problem 5.7

$$\vec{V} = U\frac{y}{h}\hat{i}$$

Find: (a) Stream function

(b) Locate streamline that divides flow rate equally.

Solution: Flow is incompressible, so stream function can be derived.

$$\frac{\partial \psi}{\partial y} = u = U\frac{y}{h}, \text{ so } \psi = \int \frac{\partial \psi}{\partial y} dy + f(x) = \int \frac{Uy}{h} dy + f(x) = \frac{Uy^2}{2h} + f(x)$$

Let $\psi = 0$ at $y = 0$, so $f(x) = 0$

$$\psi = \frac{Uy^2}{2h}$$

ψ

Stream function is maximum at $y = h$.

$$\psi_{max} = \frac{Uh^2}{2h} = \frac{Uh}{2} \; ; \; Q/w = \psi_{max} - \psi_{min} = \frac{Uh}{2} - 0 = \frac{Uh}{2}$$

$$\psi_{Q/2} = \frac{1}{2}\psi_{max} = \frac{Uh}{4} = \frac{Uy^2}{2h}$$

Thus

$$y^2 = \frac{2h}{U}\frac{Uh}{4} = \frac{h^2}{2} \quad \text{so} \quad y = \frac{h}{\sqrt{2}}$$

$\psi_{Q/2}$

Problem *5.24

*5.24 Determine the family of stream functions ψ that will yield the velocity field $\vec{V} = y(2x+1)\hat{i} + [x(x+1) - y^2]\hat{j}$.

Given: Velocity field

Find: Stream function ψ

Solution:

Basic equation: $\frac{\partial}{\partial x}(\rho \cdot u) + \frac{\partial}{\partial y}(\rho \cdot v) + \frac{\partial}{\partial z}(\rho \cdot w) + \frac{\partial}{\partial t}\rho = 0 \qquad u = \frac{\partial}{\partial y}\psi \qquad v = -\frac{\partial}{\partial x}\psi$

Assumption: Incompressible flow; flow in x-y plane

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ or $\frac{\partial}{\partial x}[y\cdot(2x+2)] + \frac{\partial}{\partial y}\left[x\cdot(x+1) - y^2\right] \rightarrow 0$

Hence $u = y\cdot(2\cdot x+1) = \frac{\partial}{\partial y}\psi \qquad \psi(x,y) = \int y\cdot(2\cdot x+1)\,dy = x\cdot y^2 + \frac{y^2}{2} + f(x)$

and $v = x\cdot(x+1) - y^2 = -\frac{\partial}{\partial x}\psi \qquad \psi(x,y) = -\int \left[x\cdot(x+1) - y^2\right] dx = -\frac{x^3}{3} - \frac{x^2}{2} + x\cdot y^2 + g(y)$

Comparing these $f(x) = -\frac{x^3}{3} - \frac{x^2}{2}$ and $g(y) = \frac{y^2}{2}$

The stream function is $\psi(x,y) = \frac{y^2}{2} + x\cdot y^2 - \frac{x^2}{2} - \frac{x^3}{3}$

Checking $u(x,y) = \frac{\partial}{\partial y}\left(\frac{y^2}{2} + x\cdot y^2 - \frac{x^2}{2} - \frac{x^3}{3}\right) \rightarrow u(x,y) = y + 2\cdot x\cdot y$

$$v(x,y) = -\frac{\partial}{\partial x}\left(\frac{y^2}{2} + x\cdot y^2 - \frac{x^2}{2} - \frac{x^3}{3}\right) \rightarrow v(x,y) = x^2 + x - y^2$$

Problem *5.25

Given: Stream function for an incompressible flow field,

$$\psi = -Ur\sin\theta + \frac{q}{2\pi}\theta$$

Find: (a) An expression for the velocity field.
(b) Points where $|\vec{V}| = 0$.
(c) Show $\psi = 0$ where $|\vec{V}| = 0$.

Solution: The velocity components are given by

$$V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = -U\cos\theta + \frac{q}{2\pi r}$$

$$V_\theta = -\frac{\partial\psi}{\partial r} = U\sin\theta$$

So $\vec{V} = V_r\hat{i}_r + V_\theta\hat{i}_\theta = \left(-U\cos\theta + \frac{q}{2\pi r}\right)\hat{i}_r + U\sin\theta\,\hat{i}_\theta$ $\vec{V}$

Now $|\vec{V}| = (V_r^2 + V_\theta^2)^{1/2} = 0$ only when both V_r and V_θ are zero.

From the component equations, $V_\theta = 0$ for $\theta = 0, \pi$. When $V_r = 0$,

$$r = \frac{q}{2\pi U\cos\theta}$$

For $r > 0$, then $V_r = 0$ for $\theta = 0$, and $r = \frac{q}{2\pi U}$.

Stagnation point ($|\vec{V}| = 0$) occurs at $(r,\theta) = \left(\frac{q}{2\pi U}, 0\right)$ $|\vec{V}| =$

Substituting, $\psi_{stagnation} = -Ur\sin\theta + \frac{q}{2\pi}\theta\Big]_{r=\frac{q}{2\pi U},\ \theta=0}$

or $\psi_{stagnation} = 0$ ψ_{st}

Problem *5.26

*5.26 Does the velocity field of Problem 5.22 represent a possible incompressible flow case? If so, evaluate and sketch the stream function for the flow. If not, evaluate the rate of change of density in the flow field.

Given: The velocity field

Find: Whether or not it is a incompressible flow; sketch stream function

Solution:

$$V_r = \frac{A}{r} \qquad V_\theta = \frac{B}{r}$$

For incompressible flow

$$\frac{1}{r}\cdot\frac{d}{dr}\left(r\cdot V_r\right) + \frac{1}{r}\cdot\frac{d}{d\theta}V_\theta = 0 \qquad \frac{1}{r}\cdot\frac{d}{dr}\left(r\cdot V_r\right) = 0 \qquad \frac{1}{r}\cdot\frac{d}{d\theta}V_\theta = 0$$

Hence

$$\frac{1}{r}\cdot\frac{d}{dr}\left(r\cdot V_r\right) + \frac{1}{r}\cdot\frac{d}{d\theta}V_\theta = 0$$

Flow is incompressible

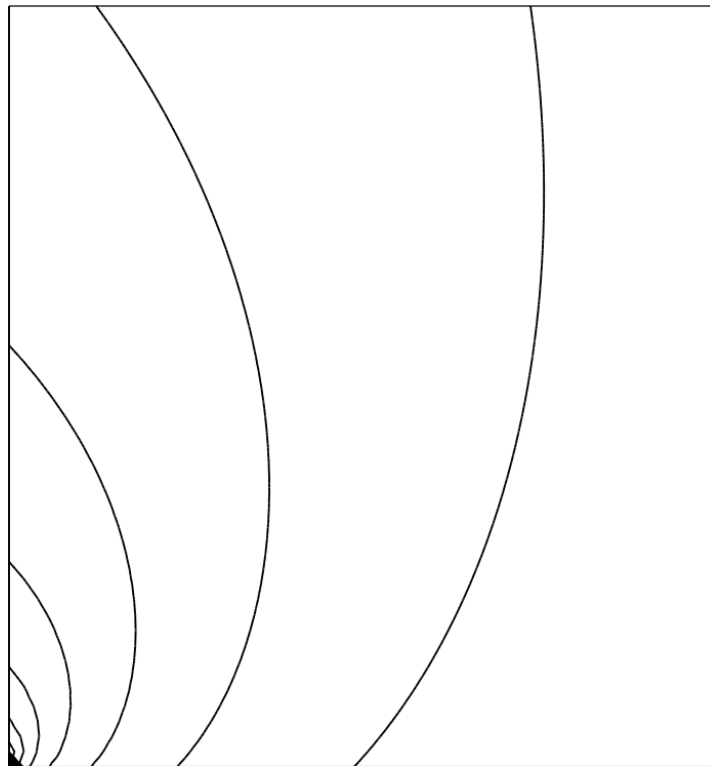
For the stream function

$$\frac{\partial}{\partial\theta}\psi = r\cdot V_r = A \qquad \psi = A\cdot\theta + f(r)$$

Integrating

$$\frac{\partial}{\partial r}\psi = -V_\theta = -\frac{B}{r} \qquad \psi = -B\cdot\ln(r) + g(\theta)$$

Comparing, stream function is $\psi = A\cdot\theta - B\cdot\ln(r)$

ψ

Problem *5.27

*5.27 Consider a flow with velocity components $u = 0$, $v = y(y^2 - 3z^2)$, and $w = z(z^2 - 3y^2)$.

a. Is this a one-, two-, or three-dimensional flow?
b. Demonstrate whether this is an incompressible or compressible flow.
c. If possible, derive a stream function for this flow.

Given: Velocity field

Find: Whether it's 1D, 2D or 3D flow; Incompressible or not; Stream function ψ

Solution:

Basic equation:
$$\frac{\partial}{\partial x}(\rho \cdot u) + \frac{\partial}{\partial y}(\rho \cdot v) + \frac{\partial}{\partial z}(\rho \cdot w) + \frac{\partial}{\partial t}\rho = 0 \qquad v = \frac{\partial}{\partial z}\psi \qquad w = -\frac{\partial}{\partial y}\psi$$

Assumption: Incompressible flow; flow in y-z plane (u = 0)

Velocity field is a function of y and z only, so is 2D

Check for incompressible $\quad \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0$

$$\frac{\partial}{\partial y}\left[y \cdot \left(y^2 - 3 \cdot z^2\right)\right] \rightarrow 3 \cdot y^2 - 3 \cdot z^2 \qquad \frac{\partial}{\partial z}\left[z \cdot \left(z^2 - 3 \cdot y^2\right)\right] \rightarrow 3 \cdot z^2 - 3 \cdot y^2$$

Hence $\quad \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0 \quad$ Flow is INCOMPRESSIBLE

Hence $\quad v = y \cdot \left(y^2 - 3 \cdot z^2\right) = \frac{\partial}{\partial z}\psi \qquad \psi(y,z) = \int y \cdot \left(y^2 - 3 \cdot z^2\right) dz = y^3 \cdot z - y \cdot z^3 + f(y)$

and $\quad w = z \cdot \left(z^2 - 3 \cdot y^2\right) = -\frac{\partial}{\partial y}\psi \qquad \psi(y,z) = -\int \left[z \cdot \left(z^2 - 3 \cdot y^2\right)\right] dy = -y \cdot z^3 + z \cdot y^3 + g(z)$

Comparing these $\quad f(y) = 0 \quad$ and $\quad g(z) = 0$

The stream function is $\quad \psi(y,z) = z \cdot y^3 - z^3 \cdot y$

Checking $\quad u(y,z) = \frac{\partial}{\partial z}\left(z \cdot y^3 - z^3 \cdot y\right) \rightarrow u(y,z) = y^3 - 3 \cdot y \cdot z^2$

$$w(y,z) = -\frac{\partial}{\partial y}\left(z \cdot y^3 - z^3 \cdot y\right) \rightarrow w(y,z) = z^3 - 3 \cdot y^2 \cdot z$$

Problem *5.28

Given: An incompressible, frictionless flow specified by

$\psi = -2Ax - 5Ay$; x, y in meters, $A = 1$ m/s

Find: (a) Sketch streamlines $\psi = 0$ and $\psi = 5\ m^2/s$

(b) Velocity vector at (0,0)

(c) Flow rate between streamlines passing through points (2,2) and (4,1)

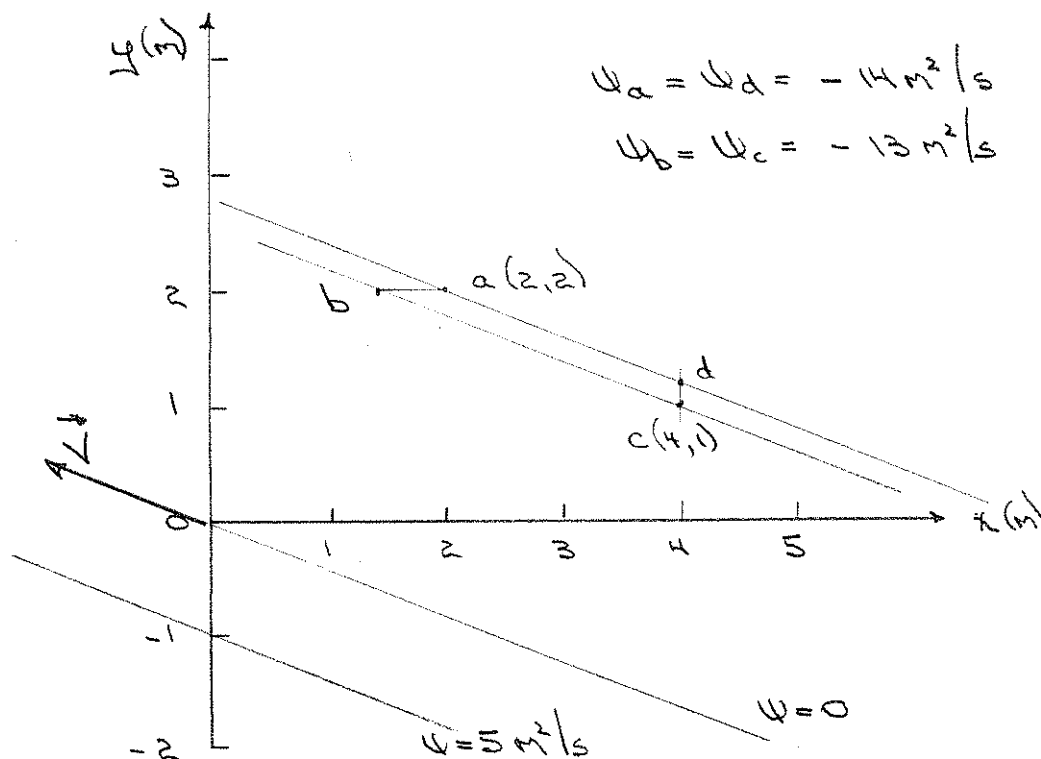
Solution: Streamlines are lines ψ = constant

For $\psi = 0$, $0 = -2Ax - 5Ay$ or $y = -\frac{2}{5}x$

For $\psi = 5$, $5 = -2Ax - 5Ay$ or $y = -\frac{2}{5}x - \frac{1}{5} \times 5\frac{m^2}{s} \times \frac{s}{1m} = -\frac{2}{5}x - 1m$

$\psi_a = \psi_d = -14\ m^2/s$

$\psi_b = \psi_c = -13\ m^2/s$

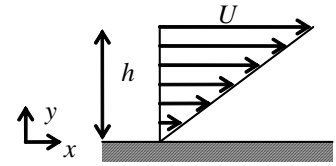
y(m), x(m), a(2,2), b, c(4,1), d, $\vec{V}$, $\psi = 0$, $\psi = 5\ m^2/s$

$u = \frac{\partial\psi}{\partial y} = -5A$; $v = -\frac{\partial\psi}{\partial x} = 2A$, so $\vec{V} = -5\hat{i} + 2\hat{j}$ m/s $\vec{V}$

$Q = \int_{x=b}^{x=a} v\,dx = \int_{x=b}^{x=a} -\frac{\partial\psi}{\partial x}dx = \int_{\psi_b}^{\psi_a} -d\psi = \psi_b - \psi_a = 1\ m^2/s$, i.e. ↑

$Q = \int_{y=c}^{y=d} u\,dy = \int_{y=c}^{y=d} \frac{\partial\psi}{\partial y}dy = \int_{\psi_c}^{\psi_d} d\psi = \psi_d - \psi_c = -1\ m^2/s$, i.e. ←

Thus $Q = 1\ m^3/s$ per meter of depth. Q

Problem *5.29

*5.29 In a parallel one-dimensional flow in the positive x direction, the velocity varies linearly from zero at $y = 0$ to 30 m/s at $y = 1.5$ m. Determine an expression for the stream function, ψ. Also determine the y coordinate above which the volume flow rate is half the total between $y = 0$ and $y = 1.5$ m.

Given: Linear velocity profile

Find: Stream function ψ; y coordinate for half of flow

Solution:

Basic equations: $u = \frac{\partial}{\partial y}\psi$ $\quad v = -\frac{\partial}{\partial x}\psi$ $\quad$ and we have $u = U\cdot\left(\frac{y}{h}\right)$ $\quad v = 0$

Assumption: Incompressible flow; flow in x-y plane

Check for incompressible $\quad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$

$$\frac{\partial}{\partial x}\left(U\cdot\frac{y}{h}\right) \rightarrow 0 \qquad \frac{\partial}{\partial y}0 \rightarrow 0$$

Hence $\quad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0 \quad$ Flow is INCOMPRESSIBLE

Hence $\quad u = U\cdot\frac{y}{h} = \frac{\partial}{\partial y}\psi \qquad \psi(x,y) = \int U\cdot\frac{y}{h}\,dy = \frac{U\cdot y^2}{2\cdot h} + f(x)$

and $\quad v = 0 = -\frac{\partial}{\partial x}\psi \qquad \psi(x,y) = -\int 0\,dx = g(y)$

Comparing these $\quad f(x) = 0 \quad$ and $\quad g(y) = \frac{U\cdot y^2}{2\cdot h}$

The stream function is $\quad \psi(x,y) = \frac{U\cdot y^2}{2\cdot h}$

For the flow $(0 < y < h)$ $\quad Q = \int_0^h u\,dy = \frac{U}{h}\cdot\int_0^h y\,dy = \frac{U\cdot h}{2}$

For half the flow rate $\quad \frac{Q}{2} = \int_0^{h_{half}} u\,dy = \frac{U}{h}\cdot\int_0^{h_{half}} y\,dy = \frac{U\cdot h_{half}{}^2}{2\cdot h} = \frac{1}{2}\cdot\left(\frac{U\cdot h}{2}\right) = \frac{U\cdot h}{4}$

Hence $\quad h_{half}{}^2 = \frac{1}{2}\cdot h^2 \qquad h_{half} = \frac{1}{\sqrt{2}}\cdot h = \frac{1.5\cdot m}{\sqrt{2}\cdot s} = 1.06\cdot\frac{m}{s}$

Problem *5.30

Given: Linear approximation to boundary layer velocity profile

$$u = U\frac{y}{\delta}$$

Find: (a) stream function for the flow field
(b) location of streamlines at one-quarter and one-half the total flow rate in the boundary layer.

Solution: For 2-D incompressible flow, ψ satisfies

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u = \frac{\partial \psi}{\partial y} = U\frac{y}{\delta} \quad \therefore \psi = \int \frac{\partial \psi}{\partial y}dy + f(x) = \int U\frac{y}{\delta}dy + f(x)$$

Thus $\psi = \dfrac{Uy^2}{2\delta} + f(x)$

Let $\psi = 0$ along $y = 0$, so $f(x) = 0$ and $\psi = \dfrac{U}{2\delta}y^2$ ← ψ

The total flow rate within the boundary layer is

$$\frac{Q}{w} = \psi(\delta) - \psi(0) = \frac{1}{2}U\delta$$

At $\frac{1}{4}$ of total, $\psi - \psi_0 = \dfrac{U}{2\delta}y^2 = \dfrac{1}{4}\left(\dfrac{1}{2}U\delta\right)$

$$\therefore \left(\frac{y}{\delta}\right)^2 = \frac{1}{4} \quad \text{and} \quad \frac{y}{\delta} = \frac{1}{2}$$ ← $\frac{1}{4}\frac{Q}{w}$

At $\frac{1}{2}$ of total, $\psi - \psi_0 = \dfrac{U}{2\delta}y^2 = \dfrac{1}{2}\left(\dfrac{1}{2}U\delta\right)$

$$\therefore \left(\frac{y}{\delta}\right)^2 = \frac{1}{2} \quad \text{and} \quad \frac{y}{\delta} = \sqrt{\frac{1}{2}} = 0.707$$ ← $\frac{1}{2}\frac{Q}{w}$

Given: Parabolic approximation to boundary layer velocity profile

$$u = U\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]$$

Find: (a) stream function for the flow field
(b) location of streamlines at one-quarter and one-half the total flow rate in the boundary layer.

Solution: For 2-D incompressible flow, ψ satisfies

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u = \frac{\partial \psi}{\partial y} = U\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]$$

$$\therefore \quad \psi = \int \frac{\partial \psi}{\partial y}\,dy + f(x) = U\int\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] dy + f(x).$$

$$\psi = U\left[\frac{y^2}{\delta} - \frac{y^3}{3\delta^2}\right] + f(x)$$

Let $\psi = 0$ along $y = 0$, so $f(x) = 0$ and $\psi = U\delta\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right]$

The total flow rate within the boundary layer is

$$\frac{Q}{w} = \psi(\delta) - \psi(0) = U\delta\left[1 - \frac{1}{3}\right] = \frac{2}{3}U\delta$$

At $\frac{1}{4}$ of total, $\psi - \psi_0 = U\delta\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right] = \frac{1}{4}\left(\frac{2}{3}U\delta\right)$

$$\therefore \left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3 = \frac{1}{6} = 0.167$$

Trial and error solution gives $\frac{y}{\delta} = 0.442$ ← $\frac{1}{4}\frac{Q}{w}$

At $\frac{1}{2}$ of total, $\psi - \psi_0 = U\delta\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right] = \frac{1}{2}\left(\frac{2}{3}U\delta\right)$

$$\therefore \left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3 = \frac{1}{3} = 0.333$$

Trial and error solution gives $\frac{y}{\delta} = 0.652$ ← $\frac{1}{2}\frac{Q}{w}$

Problem *5.32

Given: Sinusoidal approximation to boundary layer velocity profile

$$u = U\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$

Find: Locate streamlines at quarter and half total flow rate.

Solution: Flow is incompressible so ψ may be derived.

$$u = \frac{\partial\psi}{\partial y} = U\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right);\quad \psi = \int\frac{\partial\psi}{\partial y}dy + f(x) = \int U\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)dy + f(x)$$

Thus $\psi = -\frac{2\delta U}{\pi}\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) + f(x)$

Let $\psi = 0$ along $y = 0$, so $f(x) = 0$ $\qquad \psi = -\frac{2\delta U}{\pi}\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)$ ← ψ

The total flow rate is $\frac{Q}{w} = \psi(\delta) - \psi(0) = -\frac{2\delta U}{\pi}\cos\left(\frac{\pi}{2}\right) + \frac{2\delta U}{\pi}\cos(0) = \frac{2\delta U}{\pi}$

At 1/4 of total, $\psi - \psi_0 = \frac{2\delta U}{\pi}\left[1 - \cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right] = \frac{1}{4}\frac{2\delta U}{\pi} = \frac{\delta U}{2\pi}$

$$1 - \cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) = \frac{\pi}{2\delta U}\frac{\delta U}{2\pi} = \frac{1}{4};\quad \cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) = \frac{3}{4};\quad \frac{y}{\delta} = 0.460$$

← $\frac{1}{4}$ $\frac{\delta}{u}$

At 1/2 of total, $\psi - \psi_0 = \frac{2\delta U}{\pi}\left[1 - \cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right] = \frac{1}{2}\frac{2\delta U}{\pi} = \frac{\delta U}{\pi}$

$$1 - \cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) = \frac{\pi}{2\delta U}\frac{\delta U}{\pi} = \frac{1}{2};\quad \cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) = \frac{1}{2};\quad \frac{y}{\delta} = 0.667$$

← $\frac{1}{2}$ $\frac{\delta}{u}$

Problem *5.33

*5.33 A cubic velocity profile was used to model flow in a laminar incompressible boundary layer in Problem 5.13. Derive the stream function for this flow field. Locate streamlines at one-quarter and one-half the total volume flow rate in the boundary layer.

Given: Data on boundary layer

Find: Stream function; locate streamlines at 1/4 and 1/2 of total flow rate

Solution:

$$u(x,y) = U\cdot\left[\frac{3}{2}\cdot\left(\frac{y}{\delta}\right) - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^3\right] \qquad \text{and} \qquad \delta(x) = c\cdot\sqrt{x}$$

For the stream function $u = \frac{\partial}{\partial y}\psi = U\cdot\left[\frac{3}{2}\cdot\left(\frac{y}{\delta}\right) - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^3\right]$

Hence $\psi = \int U\cdot\left[\frac{3}{2}\cdot\left(\frac{y}{\delta}\right) - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^3\right] dy \qquad \psi = U\cdot\left(\frac{3}{4}\cdot\frac{y^2}{\delta} - \frac{1}{8}\cdot\frac{y^4}{\delta^3}\right) + f(x)$

Let $\psi = 0 = 0$ along $y = 0$, so $f(x) = 0$, so $\qquad \psi = U\cdot\delta\cdot\left[\frac{3}{4}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{8}\cdot\left(\frac{y}{\delta}\right)^4\right]$

The total flow rate in the boundary layer is

$$\frac{Q}{W} = \psi(\delta) - \psi(0) = U\cdot\delta\cdot\left(\frac{3}{4} - \frac{1}{8}\right) = \frac{5}{8}\cdot U\cdot\delta$$

At 1/4 of the total $\qquad \psi - \psi_0 = U\cdot\delta\cdot\left[\frac{3}{4}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{8}\cdot\left(\frac{y}{\delta}\right)^4\right] = \frac{1}{4}\cdot\left(\frac{5}{8}\cdot U\cdot\delta\right)$

$$24\cdot\left(\frac{y}{\delta}\right)^2 - 4\cdot\left(\frac{y}{\delta}\right)^4 = 5 \qquad \text{or} \qquad 4\cdot X^2 - 24\cdot X + 5 = 0 \qquad \text{where} \qquad X^2 = \frac{y}{\delta}$$

The solution to the quadratic is $X = \frac{24 - \sqrt{24^2 - 4\cdot 4\cdot 5}}{2\cdot 4} \qquad X = 0.216$ Note that the other root is $\frac{24 + \sqrt{24^2 - 4\cdot 4\cdot 5}}{2\cdot 4} = 5.784$

Hence $\qquad \frac{y}{\delta} = \sqrt{X} = 0.465$

At 1/2 of the total flow $\qquad \psi - \psi_0 = U\cdot\delta\cdot\left[\frac{3}{4}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{8}\cdot\left(\frac{y}{\delta}\right)^4\right] = \frac{1}{2}\cdot\left(\frac{5}{8}\cdot U\cdot\delta\right)$

$$12\cdot\left(\frac{y}{\delta}\right)^2 - 2\cdot\left(\frac{y}{\delta}\right)^4 = 5 \qquad \text{or} \qquad 2\cdot X^2 - 12\cdot X + 5 = 0 \qquad \text{where} \qquad X^2 = \frac{y}{\delta}$$

The solution to the quadratic is $X = \frac{12 - \sqrt{12^2 - 4\cdot 2\cdot 5.}}{2\cdot 2} \qquad X = 0.450$ Note that the other root is $\frac{12 + \sqrt{12^2 - 4\cdot 2\cdot 5}}{2\cdot 2} = 5.55$

Hence $\qquad \frac{y}{\delta} = \sqrt{X} = 0.671$

Problem *5.34

Given: Rigid-body motion in Example Problem 5.6

$$\vec{V} = r\omega\,\hat{e}_\theta \qquad \omega = 0.5 \text{ rad/s}$$

Find: (a) Obtain the stream function for this flow.
(b) Evaluate the volume flow rate per unit depth between $r_1 = 0.10$ m and $r_2 = 0.12$ m.
(c) Sketch the velocity profile along a line of constant θ.
(d) Check the volume flow rate calculated from the stream function by integrating the velocity profile along this line.

Solution: From the definition of ψ, $\frac{\partial \psi}{\partial r} = -V_\theta = -r\omega$

Thus $\psi = \int \frac{\partial \psi}{\partial r} dr + f(\theta) = \int -r\omega\, dr + f(\theta) = -\frac{1}{2}r^2\omega + f(\theta)$

But $V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{r} f'(\theta) = 0 \quad \therefore f(\theta) = c$

and $\psi = -\frac{1}{2}r^2\omega + c$ ← ψ

The volume flow rate per unit depth is

$$\frac{Q}{b} = \psi(r_2) - \psi(r_1) = -\frac{1}{2}r_2^2\omega + c - \left[-\frac{1}{2}r_1^2\omega + c\right] = \frac{\omega}{2}(r_1^2 - r_2^2)$$

$$\frac{Q}{b} = \frac{1}{2} \times 0.5 \frac{\text{rad}}{\text{s}} \left[(0.10)^2 - (0.12)^2\right] \text{m}^2 = -0.0011 \quad \text{m}^3/\text{s} / \text{m}$$ ← Q/b

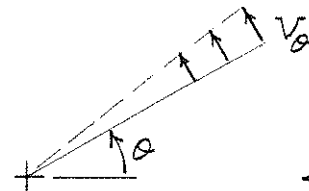
Because $Q/b < 0$, flow is in the direction of $\hat{e}_\theta$.

Along θ = constant, V_θ varies linearly: ← Plot

From the linear velocity variation, $V_\theta = \omega r$

Thus $\frac{Q}{b} = \int_{r_1}^{r_2} V_\theta\, dr = \int_{r_1}^{r_2} r\omega\, dr = \frac{1}{2}r^2\omega\Big]_{r_1}^{r_2} = \frac{\omega}{2}(r_2^2 - r_1^2)$ ← Q/b

From the sketch, this flow is in the direction of $\hat{e}_\theta$.

Comparing the expressions for Q/b shows they are the same except for sign.

Problem *5.35

Given: Velocity field for a free vortex from Example Problem 5.6:

$$\vec{V} = \frac{c}{r}\hat{e}_\theta \qquad c = 0.5\ m^2/sec$$

Find: (a) Obtain the stream function for this flow.
(b) Evaluate the volume flow rate per unit depth between $r_1 = 0.10$ m and $r_2 = 0.12$ m.
(c) Sketch the velocity profile along a line of constant θ.
(d) Check the volume flow rate calculated from the stream function by integrating the velocity profile along this line.

Solution: From the definition of ψ, $\frac{\partial \psi}{\partial r} = -V_\theta = -\frac{c}{r}$

Thus $\psi = \int \frac{\partial \psi}{\partial r} dr + f(\theta) = \int -\frac{c}{r} dr + f(\theta) = -c \ln r + f(\theta)$

But $V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{r} f'(\theta) = 0$. Therefore $f(\theta) = \text{constant} = c_1$, and

$$\psi = -c \ln r + c_1$$ ψ

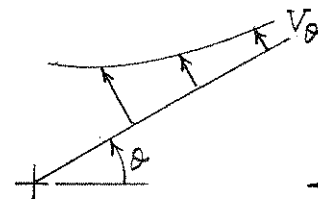
The volume flow rate per unit depth is

$$\frac{Q}{b} = \psi(r_2) - \psi(r_1) = -c \ln r_2 + c_1 - [-c \ln r_1 + c_1] = c(\ln r_1 - \ln r_2) = c \ln\left(\frac{r_1}{r_2}\right)$$

$$\frac{Q}{b} = 0.5\ \frac{m^2}{s} \times \ln\left(\frac{0.10\ m}{0.12\ m}\right) = -0.0912\ m^3/s\ /m$$ Q/b

Because $Q/b < 0$, flow is in the direction of $\hat{e}_\theta$.

Along θ = constant, V_θ varies inversely with r:

Plot

From the expression for $\vec{V}$, $V_\theta = \frac{c}{r}$. Thus

$$\frac{Q}{b} = \int_{r_1}^{r_2} V_\theta dr = \int_{r_1}^{r_2} \frac{c}{r} dr = c \ln\left(\frac{r_2}{r_1}\right)$$ Q/b

From the sketch, this flow is in the direction of $\hat{e}_\theta$.

Comparing shows that the expressions for Q/b are the same except for sign.

Problem 5.36

5.36 Consider the velocity field $\vec{V} = A(x^4 - 6x^2y^2 + y^4)\hat{i} + A(4xy^3 - 4x^3y)\hat{j}$ in the xy plane, where $A = 0.25\ \mathrm{m^{-3} \cdot s^{-1}}$, and the coordinates are measured in meters. Is this a possible incompressible flow field? Calculate the acceleration of a fluid particle at point $(x,y) = (2, 1)$.

Given: Velocity field

Find: Whether flow is incompressible; Acceleration of particle at (2,1)

Solution:

Basic equations

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$$

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\\text{acceleration}}}$$

$$u(x,y) = A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right) \qquad v(x,y) = A\cdot\left(4\cdot x\cdot y^3 - 4\cdot x^3\cdot y\right)$$

For incompressible flow

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$$

Checking

$$\frac{\partial}{\partial x}\left[A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)\right] \rightarrow A\cdot\left(4\cdot x^3 - 12\cdot x\cdot y^2\right) \qquad \frac{\partial}{\partial y}\left[A\cdot\left(4\cdot x\cdot y^3 - 4\cdot x^3\cdot y\right)\right] \rightarrow -A\cdot\left(4\cdot x^3 - 12\cdot x\cdot y^2\right)$$

Hence

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$$

The acceleration is given by

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\\text{acceleration}}}$$

For this flow

$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u$$

$$a_x = A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)\cdot\frac{\partial}{\partial x}\left[A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)\right] + A\cdot\left(4\cdot x\cdot y^3 - 4\cdot x^3\cdot y\right)\cdot\frac{\partial}{\partial y}\left[A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)\right]$$

$$a_x = 4\cdot A^2\cdot x\cdot\left(x^2 + y^2\right)^3$$

$$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v$$

$$a_y = A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)\cdot\frac{\partial}{\partial x}\left[A\cdot\left(4\cdot x\cdot y^3 - 4\cdot x^3\cdot y\right)\right] + A\cdot\left(4\cdot x\cdot y^3 - 4\cdot x^3\cdot y\right)\cdot\frac{\partial}{\partial y}\left[A\cdot\left(4\cdot x\cdot y^3 - 4\cdot x^3\cdot y\right)\right]$$

$$a_y = 4\cdot A^2\cdot y\cdot\left(x^2 + y^2\right)^3$$

Hence at (2,1)

$$a_x = 4\times\left(\frac{1}{4}\cdot\frac{1}{m^3\cdot s}\right)^2\times 2\cdot m\times\left[(2\cdot m)^2 + (1\cdot m)^2\right]^3 \qquad a_x = 62.5\frac{m}{s^2}$$

$$a_y = 4\times\left(\frac{1}{4}\cdot\frac{1}{m^3\cdot s}\right)^2\times 1\cdot m\times\left[(2\cdot m)^2 + (1\cdot m)^2\right]^3 \qquad a_y = 31.3\frac{m}{s^2} \qquad a = \sqrt{{a_x}^2 + {a_y}^2} \qquad a = 69.9\frac{m}{s^2}$$

Problem 5.37

Given: Flow field $\vec{V} = xy^2\hat{i} - \frac{1}{3}y^3\hat{j} + xy\hat{k}$

Find: (a) Dimensions.
(b) If possible incompressible flow.
(c) Acceleration of particle at point $(x,y,z) = (1,2,3)$.

Solution: Apply continuity, use substantial derivative.

Basic equations: $\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0$ (terms 3 and 4 $= 0(1)$, $= 0(2)$)

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$$

(terms 3 and 4 $= 0(1)$, $= 0(2)$)

Assumptions: (1) Two-dimensional flow, $\vec{V} = \vec{V}(x,y)$, so $\partial/\partial z = 0$
(2) Incompressible flow
(3) Steady flow, $\vec{V} \neq \vec{V}(t)$

Then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = y^2 - y^2 = 0$ Flow is a possible incompressible case. $\rho =$

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} \; ; \quad \frac{\partial \vec{V}}{\partial x} = y^2\hat{i} + y\hat{k} \; ; \quad \frac{\partial \vec{V}}{\partial y} = 2xy\hat{i} - y^2\hat{j} + x\hat{k}$$

$$= (xy^2)(y^2\hat{i} + y\hat{k}) + (-\tfrac{1}{3}y^3)(2xy\hat{i} - y^2\hat{j} + x\hat{k})$$

$$= \hat{i}(xy^4 - \tfrac{2}{3}xy^4) + \hat{j}(\tfrac{1}{3}y^5) + \hat{k}(xy^3 - \tfrac{1}{3}xy^3)$$

$$\vec{a}_p = \hat{i}(\tfrac{1}{3}xy^4) + \hat{j}(\tfrac{1}{3}y^5) + \hat{k}(\tfrac{2}{3}xy^3)$$

At $(x,y,z) = (1,2,3)$

$$\vec{a}_p = \hat{i}\left[\tfrac{1}{3}(1)(16)\right] + \hat{j}\left[\tfrac{1}{3}(32)\right] + \hat{k}\left[\tfrac{2}{3}(1)(8)\right] = \frac{16}{3}\hat{i} + \frac{32}{3}\hat{j} + \frac{16}{3}\hat{k}$$

$\vec{a}_p$

($\vec{a}_p$ will be in m/s^2)

Problem 5.38

Given: Flow field $\vec{V} = ax^2y\hat{i} - by\hat{j} + cz^2\hat{k}$; $a = 1/m^2 \cdot s$, $b = 3/s$, $c = 2/m \cdot s$

Find: (a) Dimensions of flow field.
(b) If possible incompressible flow.
(c) Acceleration of a particle at $(x, y, z) = (3, 1, 2)$.

Solution: Apply continuity, use substantial derivative.

Basic equations: $\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0$ (with $\frac{\partial \rho}{\partial t} = 0$)

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$$ (with $\frac{\partial \vec{V}}{\partial t} = 0$)

Assumption: Incompressible flow, ρ = constant

Then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ is criterion.

Note $\vec{V} = \vec{V}(x, y, z)$, so flow is three-dimensional, and ← 3-D.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 2xy - 3 + 4z \neq 0$$

Flow cannot be incompressible. ← $\rho \neq c$

$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$; $\frac{\partial \vec{V}}{\partial x} = 2axy\hat{i}$, $\frac{\partial \vec{V}}{\partial y} = ax^2\hat{i} - b\hat{j}$, $\frac{\partial \vec{V}}{\partial z} = 2cz\hat{k}$

$$= (ax^2y)(2axy\hat{i}) + (-by)(ax^2\hat{i} - b\hat{j}) + (cz^2)(2cz\hat{k})$$

$$\vec{a}_p = \hat{i}(2a^2x^3y^2 - abx^2y) + \hat{j}(b^2y) + \hat{k}(2c^2z^3)$$

At $(x, y, z) = (3, 1, 2)$,

$$\vec{a}_p = \hat{i}\left[2 \times \frac{(1)^2}{m^4 \cdot s^2} \times (3)^3 m^3 \times (1)^2 m^2 - \frac{1}{m^2 \cdot s} \times \frac{3}{s} \times (3)^2 m^2 \times 1\,m\right] + \hat{j}\left[\frac{(3)^2}{s^2} \times 1\,m\right] + \hat{k}\left[2 \times \frac{(2)^2}{m^2 \cdot s^2} \times (2)^3 m^3\right]$$

$$\vec{a}_p = 27\hat{i} + 9\hat{j} + 64\hat{k}\ \frac{m}{s^2}$$ ← $\vec{a}_p$

Given: Velocity field (within a laminar boundary layer) is given by $\vec{V} = A\frac{Uy}{x^{1/2}}\left(\hat{i} + \frac{y}{4x}\hat{j}\right)$

where $A = 141\ m^{-1/2}$

$U = 0.240\ m/s$

Find: (a) Show that this velocity field represents a possible incompressible flow

(b) Calculate $\vec{a}$ of particle at $(x,y) = (0.5m, 5mm)$

(c) Slope of streamline through point $(0.5m, 5mm)$

Solution:

From given velocity field $\vec{V} = \vec{V}(x,y)$, $w=0$, flow is steady

(a) Check conservation of mass for ρ = constant

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}^{\nearrow 0} = 0$$

$$u = A\frac{Uy}{x^{1/2}} \qquad \frac{\partial u}{\partial x} = -\frac{1}{2}AU\frac{y}{x^{3/2}}$$

$$v = AU\frac{y^2}{4x^{3/2}} \qquad \frac{\partial v}{\partial y} = \frac{1}{2}AU\frac{y}{x^{3/2}}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \therefore \text{ incompressible} \quad \text{(Q.E.D.)}$$

(b)
$$\vec{a} = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}^{\nearrow 0} + \frac{\partial \vec{V}}{\partial t}^{\nearrow 0}$$

$$a_{p_x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \ ; \qquad \frac{\partial u}{\partial y} = AU\frac{1}{x^{1/2}}$$

$$a_{p_x} = A\frac{Uy}{x^{1/2}}\left(-\frac{1}{2}AU\frac{y}{x^{3/2}}\right) + AU\frac{y^2}{4x^{3/2}}\left(AU\frac{1}{x^{1/2}}\right)$$

$$a_{p_x} = -\frac{1}{2}A^2U^2\frac{y^2}{x^2} + A^2U^2\frac{y^2}{4x^2} = -\frac{1}{4}\left(\frac{AUy}{x}\right)^2$$

$$a_{p_x} = -\frac{1}{4}\left[\frac{141}{m^{1/2}} \times 0.240\frac{m}{s} \times \frac{0.005m}{0.5m}\right]^2 = -0.0286\ m/s^2$$

$$a_{p_y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \ ; \qquad \frac{\partial v}{\partial x} = -\frac{3}{8}\frac{AUy^2}{x^{5/2}}$$

$$= AU\frac{y}{x^{1/2}}\left(-\frac{3}{8}AU\frac{y^2}{x^{5/2}}\right) + AU\frac{y^2}{4x^{3/2}}\left(\frac{1}{2}AU\frac{y}{x^{3/2}}\right)$$

$$= -\frac{3}{8}A^2U^2\frac{y^3}{x^3} + \frac{1}{8}A^2U^2\frac{y^3}{x^3} = -\frac{1}{4}A^2U^2\frac{y^3}{x^3}$$

$$a_{p_y} = -\frac{1}{4}\left(\frac{141}{m^{1/2}} \times 0.240\frac{m}{s}\right)^2\left(\frac{0.005m}{0.5m}\right)^3 = -2.86 \times 10^{-4}\ m/s^2$$

$$\therefore \vec{a}_p = -2.86\,(10^{-2}\hat{i} + 10^{-4}\hat{j})\ m/s^2 \qquad \vec{a}_p$$

The slope of the streamline is given by

$$\left.\frac{dy}{dx}\right)_s = \frac{v}{u} = \frac{y}{4x} = \frac{5 \times 10^{-3}m}{4 \times 0.5m} = 0.0025 \qquad \left.\frac{dy}{dx}\right)_s$$

Problem 5.40

5.40 The x component of velocity in a steady, incompressible flow field in the xy plane is $u = A(x^5 - 10x^3y^2 + 5xy^4)$, where $A = 2\ \mathrm{m^{-4} \cdot s^{-1}}$ and x is measured in meters. Find the simplest y component of velocity for this flow field. Evaluate the acceleration of a fluid particle at point $(x, y) = (1, 3)$.

Given: x component of velocity field

Find: Simplest y component for incompressible flow; Acceleration of particle at (1,3)

Solution:

Basic equations
$$u = \frac{\partial}{\partial y}\psi \qquad v = -\frac{\partial}{\partial x}\psi \qquad \vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\text{total acceleration of a particle}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective acceleration}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local acceleration}}$$

We are given
$$u(x,y) = A\cdot\left(x^5 - 10\cdot x^3\cdot y^2 + 5\cdot x\cdot y^4\right)$$

Hence for incompressible flow
$$\psi(x,y) = \int u\,dy = \int A\cdot\left(x^5 - 10\cdot x^3\cdot y^2 + 5\cdot x\cdot y^4\right)dy = A\cdot\left(x^5\cdot y - \frac{10}{3}\cdot x^3\cdot y^3 + x\cdot y^5\right) + f(x)$$

$$v(x,y) = -\frac{\partial}{\partial x}\psi(x_y) = -\frac{\partial}{\partial x}\left[A\cdot\left(x^5\cdot y - \frac{10}{3}\cdot x^3\cdot y^3 + x\cdot y^5\right) + f(x)\right] = -A\cdot\left(5\cdot x^4\cdot y - 10\cdot x^2\cdot y^3 + y^5\right) + F(x)$$

Hence
$$v(x,y) = -A\cdot\left(5\cdot x^4\cdot y - 10\cdot x^2\cdot y^3 + y^5\right) + F(x)$$
where F(x) is an arbitrary function of x

The simplest is
$$v(x,y) = -A\cdot\left(5\cdot x^4\cdot y - 10\cdot x^2\cdot y^3 + y^5\right)$$

The acceleration is given by
$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\text{total acceleration of a particle}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective acceleration}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local acceleration}}$$

For this flow
$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u$$

$$a_x = A\cdot\left(x^5 - 10\cdot x^3\cdot y^2 + 5\cdot x\cdot y^4\right)\cdot\frac{\partial}{\partial x}\left[A\cdot\left(x^5 - 10\cdot x^3\cdot y^2 + 5\cdot x\cdot y^4\right)\right] - A\cdot\left(5\cdot x^4\cdot y - 10\cdot x^2\cdot y^3 + y^5\right)\cdot\frac{\partial}{\partial y}\left[A\cdot\left(x^5 - 10\cdot x^3\cdot y^2 + 5\cdot x\cdot y^4\right)\right]$$

$$a_x = 5\cdot A^2\cdot x\cdot\left(x^2 + y^2\right)^4$$

$$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v$$

$$a_y = A\cdot\left(x^5 - 10\cdot x^3\cdot y^2 + 5\cdot x\cdot y^4\right)\cdot\frac{\partial}{\partial x}\left[-A\cdot\left(5\cdot x^4\cdot y - 10\cdot x^2\cdot y^3 + y^5\right)\right] - A\cdot\left(5\cdot x^4\cdot y - 10\cdot x^2\cdot y^3 + y^5\right)\cdot\frac{\partial}{\partial y}\left[-A\cdot\left(5\cdot x^4\cdot y - 10\cdot x^2\cdot y^3 + y^5\right)\right]$$

$$a_y = 5\cdot A^2\cdot y\cdot\left(x^2 + y^2\right)^4$$

Hence at (1,3)
$$a_x = 5\times\left(\frac{1}{2}\cdot\frac{1}{m^4\cdot s}\right)^2\times 1\cdot m\times\left[(1\cdot m)^2 + (3\cdot m)^2\right]^4 \qquad a_x = 1.25\times 10^4\,\frac{m}{s^2}$$

$$a_y = 5\times\left(\frac{1}{2}\cdot\frac{1}{m^4\cdot s}\right)^2\times 3\cdot m\times\left[(1\cdot m)^2 + (3\cdot m)^2\right]^4 \qquad a_y = 3.75\times 10^4\,\frac{m}{s^2} \qquad a = \sqrt{a_x^2 + a_y^2} \qquad a = 3.95\times 10^4\,\frac{m}{s^2}$$

Problem 5.41

5.41 Consider the velocity field $\vec{V} = Ax/(x^2 + y^2)\hat{i} + Ay/(x^2 + y^2)\hat{j}$ in the xy plane, where $A = 10\ \text{m}^2/\text{s}$, and x and y are measured in meters. Is this an incompressible flow field? Derive an expression for the fluid acceleration. Evaluate the velocity and acceleration along the x axis, the y axis, and along a line defined by $y = x$. What can you conclude about this flow field?

Given: Velocity field

Find: Whether flow is incompressible; expression for acceleration; evaluate acceleration along axes and along $y = x$

Solution:

The given data is $A = 10 \cdot \dfrac{m^2}{s}$ $\quad u(x,y) = \dfrac{A \cdot x}{x^2 + y^2}$ $\quad v(x,y) = \dfrac{A \cdot y}{x^2 + y^2}$

For incompressible flow $\quad \dfrac{\partial}{\partial x}u + \dfrac{\partial}{\partial y}v = 0$

Hence, checking $\quad \dfrac{\partial}{\partial x}u + \dfrac{\partial}{\partial y}v = -A \cdot \dfrac{(x^2 - y^2)}{(x^2 + y^2)^2} + A \cdot \dfrac{(x^2 - y^2)}{(x^2 + y^2)^2} = 0$ $\quad$ Incompressible flow

The acceleration is given by

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\text{total acceleration of a particle}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective acceleration}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local acceleration}}$$

For the present steady, 2D flow

$$a_x = u \cdot \frac{du}{dx} + v \cdot \frac{du}{dy} = \frac{A \cdot x}{x^2 + y^2} \cdot \left[-\frac{A \cdot (x^2 - y^2)}{(x^2 + y^2)^2} \right] + \frac{A \cdot y}{x^2 + y^2} \cdot \left[-\frac{2 \cdot A \cdot x \cdot y}{(x^2 + y^2)^2} \right] \qquad a_x = -\frac{A^2 \cdot x}{(x^2 + y^2)^2}$$

$$a_y = u \cdot \frac{dv}{dx} + v \cdot \frac{dv}{dy} = \frac{A \cdot x}{x^2 + y^2} \cdot \left[-\frac{2 \cdot A \cdot x \cdot y}{(x^2 + y^2)^2} \right] + \frac{A \cdot y}{x^2 + y^2} \cdot \left[\frac{A \cdot (x^2 - y^2)}{(x^2 + y^2)^2} \right] \qquad a_y = -\frac{A^2 \cdot y}{(x^2 + y^2)^2}$$

Along the x axis $\quad a_x = -\dfrac{A^2}{x^3} = -\dfrac{100}{x^3}$ $\quad a_y = 0$

Along the y axis $\quad a_x = 0$ $\quad a_y = -\dfrac{A^2}{y^3} = -\dfrac{100}{y^3}$

Along the line $x = y$ $\quad a_x = -\dfrac{A^2 \cdot x}{r^4} = -\dfrac{100 \cdot x}{r^4}$ $\quad a_y = -\dfrac{A^2 \cdot y}{r^4} = -\dfrac{100 \cdot y}{r^4}$

where $\quad r = \sqrt{x^2 + y^2}$

For this last case the acceleration along the line $x = y$ is

$$a = \sqrt{a_x^2 + a_y^2} = -\frac{A^2}{r^4} \cdot \sqrt{x^2 + y^2} = -\frac{A^2}{r^3} = -\frac{100}{r^3} \qquad a = -\frac{A^2}{r^3} = -\frac{100}{r^3}$$

In each case the acceleration vector points towards the origin, proportional to 1/distance^3, so the flow field is a radial decelerating flow

Problem 5.42

Given: Incompressible, two-dimensional flow field with $w = 0$, has a y component of velocity given by

$$v = -Axy$$

where units of v are m/s; x and y are in meters and A is a dimensional constant.

Find: (a) the dimensions of the constant A
(b) the simplest x component of velocity for this flow field
(c) the acceleration of a fluid particle at the point $(x, y) = (1, 2)$

Solution:

(a) Since $v = -Axy$, then the dimensions of A, $[A]$, are given by

$$[A] = \left[\frac{v}{xy}\right] = \frac{L}{t}\cdot\frac{1}{L}\cdot\frac{1}{L} = \frac{1}{Lt}$$ [A]

(b) Apply the continuity equation for the conditions given

Basic equation: $\nabla \cdot \rho\vec{V} + \frac{\partial \rho}{\partial t} = 0$

For incompressible flow, $\frac{\partial \rho}{\partial t} = 0$. Thus with $w = 0$, the basic equation reduces to $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

Then, $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = -\frac{\partial}{\partial y}(-Axy) = Ax$

and

$$u = \int \frac{\partial u}{\partial x}\,dx + f(y) = \int Ax\,dx + f(y) = \frac{1}{2}Ax^2 + f(y)$$

The simplest x component of velocity is obtained with $f(y) = 0$

$$\therefore\ u = \frac{1}{2}Ax^2$$ u

(c) The acceleration of a fluid particle is given by

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$$

(with the last two terms crossed out as ≈ 0)

$$\vec{a}_p = \frac{1}{2}Ax^2\frac{\partial}{\partial x}\left[\frac{1}{2}Ax^2\hat{i} - Axy\hat{j}\right] - Axy\frac{\partial}{\partial y}\left[\frac{1}{2}Ax^2\hat{i} - Axy\hat{j}\right]$$

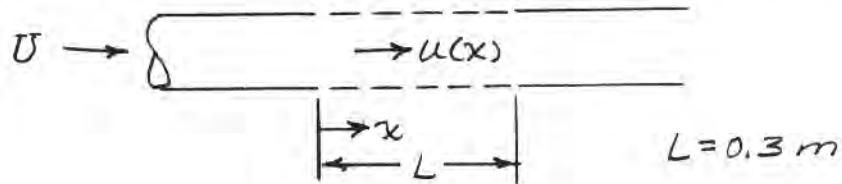
$$\vec{a}_p = \frac{1}{2}Ax^2\left[Ax\hat{i} - Ay\hat{j}\right] - Axy\left[-Ax\hat{j}\right] = \frac{1}{2}A^2x^3\hat{i} + \frac{1}{2}A^2x^2y\hat{j}$$

At the point $(x, y) = (1, 2)$

$$\vec{a}_p = \frac{1}{2}A^2(1)^3\hat{i} + \frac{1}{2}A^2(1)^2(2)\hat{j} = A^2\left[\frac{1}{2}\hat{i} + \hat{j}\right]$$ $\vec{a}_p$

Problem 5.43

Given: Duct flow with inviscid liquid, ρ = constant.

$L = 0.3\ m$

$u(x) = U(1 - x/2L)$ $U = 5\ m/s$

Find: Expression for acceleration along ℄.

Solution: Computing equation

$$a_{px} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

(with $v\frac{\partial u}{\partial y} = 0(1)$, $w\frac{\partial u}{\partial z} = 0(1)$, $\frac{\partial u}{\partial t} = 0(2)$)

Assumptions: (1) Along ℄ $v = w = 0$

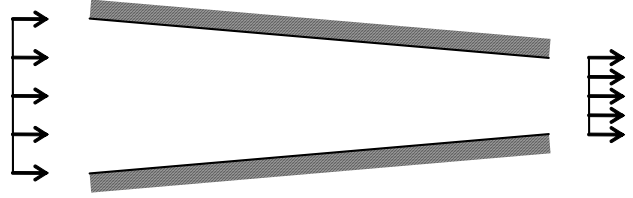
(2) Steady flow

Then

$$a_{px} = u\frac{\partial u}{\partial x} = U\left(1 - \frac{x}{2L}\right)U\left(-\frac{1}{2L}\right) = -\frac{U^2}{2L}\left(1 - \frac{x}{2L}\right)$$

a_{px}

Problem 5.44

5.44 An incompressible liquid with negligible viscosity flows steadily through a horizontal pipe. The pipe diameter linearly varies from a diameter of 10 cm to a diameter of 2.5 cm over a length of 2 m. Develop an expression for the acceleration of a fluid particle along the pipe centerline. Plot the centerline velocity and acceleration versus position along the pipe, if the inlet centerline velocity is 1 m/s.

Given: Flow in a pipe with variable diameter

Find: Expression for particle acceleration; Plot of velocity and acceleration along centerline

Solution:

Assumptions: 1) Incompressible flow 2) Flow profile remains unchanged so centerline velocity can represent average velocity

Basic equations $Q = V\cdot A$

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective acceleration}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local acceleration}}$$

(total acceleration of a particle)

For the flow rate $Q = V\cdot A = V\cdot\frac{\pi\cdot D^2}{4}$

But $D = D_i + \frac{(D_o - D_i)}{L}\cdot x$ where D_i and D_o are the inlet and exit diameters, and x is distance along the pipe of length L: $D(0) = D_i$, $D(L) = D_o$.

Hence
$$V_i\cdot\frac{\pi\cdot D_i^2}{4} = V\cdot\frac{\pi\cdot\left[D_i + \frac{(D_o - D_i)}{L}\cdot x\right]^2}{4}$$

$$V = V_i\cdot\frac{D_i^2}{\left[D_i + \frac{(D_o - D_i)}{L}\cdot x\right]^2} = \frac{V_i}{\left[1 + \frac{\left(\frac{D_o}{D_i} - 1\right)}{L}\cdot x\right]^2} \qquad V(x) = \frac{V_i}{\left[1 + \frac{\left(\frac{D_o}{D_i} - 1\right)}{L}\cdot x\right]^2}$$

Some representative values are $V(0\cdot m) = 1\frac{m}{s}$ $\quad V\left(\frac{L}{2}\right) = 2.56\frac{m}{s}$ $\quad V(L) = 16\frac{m}{s}$

The acceleration is given by

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective acceleration}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local acceleration}}$$

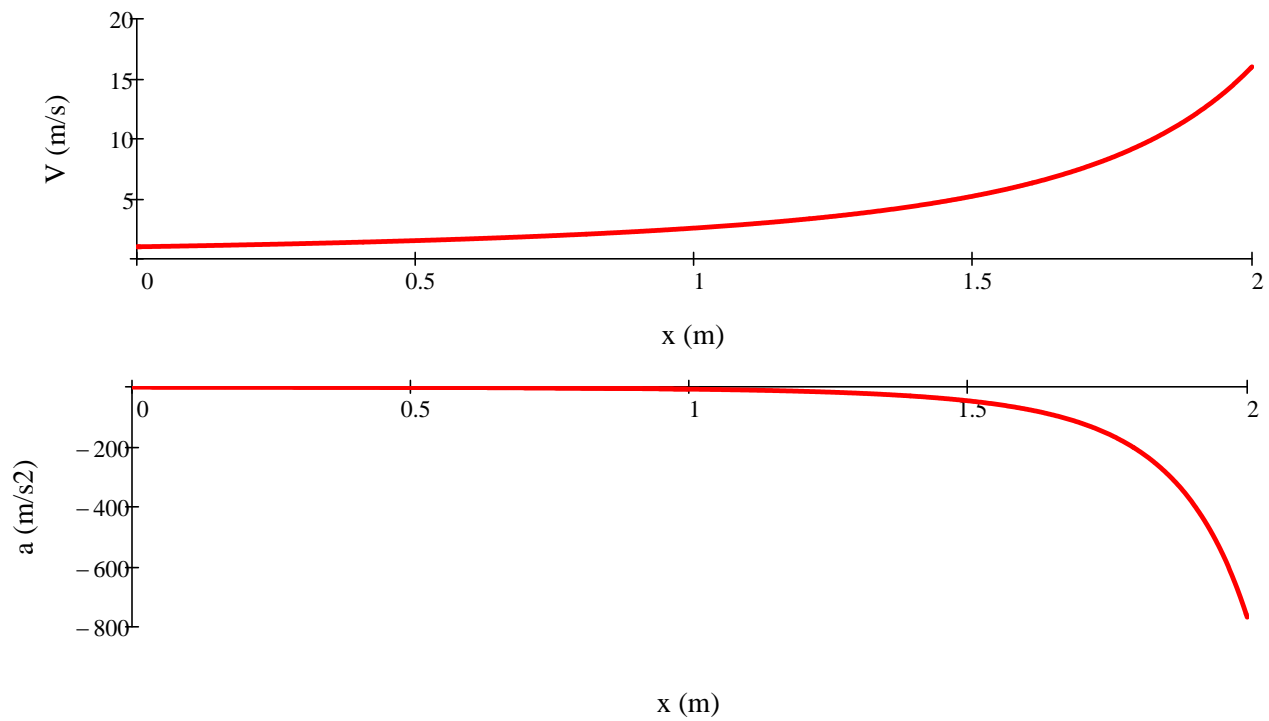
(total acceleration of a particle)

For this flow $a_x = V\cdot\frac{\partial}{\partial x}V$

$$a_x = \frac{V_i}{\left[1 + \frac{\left(\frac{D_o}{D_i} - 1\right)}{L}\cdot x\right]^2}\cdot\frac{\partial}{\partial x}\left[\frac{V_i}{\left[1 + \frac{\left(\frac{D_o}{D_i} - 1\right)}{L}\cdot x\right]^2}\right] = -\frac{2\cdot V_i^2\cdot\left(\frac{D_o}{D_i} - 1\right)}{L\cdot\left[\frac{x\cdot\left(\frac{D_o}{D_i} - 1\right)}{L} + 1\right]^5}$$

$$a_x(x) = \frac{2 \cdot V_i^2 \cdot \left(\frac{D_o}{D_i} - 1\right)}{L \cdot \left[\frac{x \cdot \left(\frac{D_o}{D_i} - 1\right)}{L} + 1\right]^5}$$

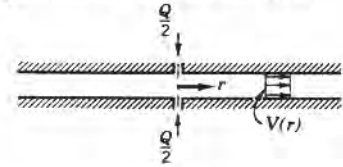
Some representative values are $a_x(0 \cdot m) = -0.75 \frac{m}{s^2}$ $a_x\left(\frac{L}{2}\right) = -7.864 \frac{m}{s^2}$ $a_x(L) = -768 \frac{m}{s^2}$

The following plots can be done in *Excel*

Problem 5.45

Given: Incompressible flow between parallel plates as shown.

Find: (a) Show $V_r = \frac{Q}{2\pi r h}$

(b) Acceleration in gap.

Q/2
r
V(r)
Q/2

Solution: Apply conservation of mass

Basic equation: $\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(V_\theta) + \frac{\partial}{\partial z}V_z = 0$ (terms $=0(1)$, $=0(2)$)

Assumptions: (1) $V_\theta = 0$

(2) $V_z = 0$

Then

$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r) = 0 \quad \text{or} \quad rV_r = C \quad \text{or} \quad V_r = \frac{C}{r} \text{ is form of solution.}$$

The volume flow rate is $Q = 2\pi r h V_r$, so $V_r = \frac{Q}{2\pi r h}$ ← V_r

Because $V_\theta = 0$, $a_\theta = 0$. The radial acceleration is

$$a_r = V_r \frac{\partial V_r}{\partial r} = \frac{Q}{2\pi r h}\left[(-1)\frac{Q}{2\pi r^2 h}\right] = -\left(\frac{Q}{2\pi h}\right)^2 \frac{1}{r^3}$$

Thus

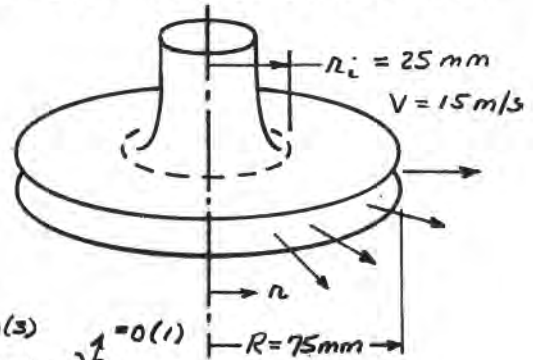
$$\vec{a}_p = -\left(\frac{Q}{2\pi h}\right)^2 \frac{1}{r^3}\hat{e}_r$$ ← $\vec{a}_p$

The above expressions are valid only for $r > 0$.

Problem 5.46

Given: Incompressible, inviscid flow of air between parallel disks.

Find: (a) Simplify continuity.
(b) Show $\vec{V} = V(R/r)\hat{e}_r$, $r_i < r < R$
(c) Calculate acceleration of a particle at $r = r_i, R$.

$r_i = 25$ mm
$V = 15$ m/s
$R = 75$mm

Solution: Apply continuity equation and substantial derivative

Basic equations:
$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) + \frac{\partial \rho}{\partial t} = 0$$
(terms marked $=0(2)$, $=0(3)$, $=0(1)$)

$$a_r = V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + \frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r} + \frac{\partial V_r}{\partial t}$$
(terms marked $=0(2)$, $=0(3)$, $=0(2)$, $=0(4)$)

Assumptions: (1) Incompressible flow, ρ = constant
(2) Radial flow, $V_\theta = 0$
(3) Uniform flow at each radial location, $\partial/\partial z = 0$
(4) Steady flow

Then
$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r) = 0 \quad \text{or} \quad rV_r = \text{constant} = RV\ ; \quad V_r = V\frac{R}{r}$$

So that $\vec{V} = V\frac{R}{r}\hat{e}_r$ ← $\vec{V}$

The radial acceleration of a fluid particle is

$$a_r = V_r\frac{\partial V_r}{\partial r} = V\frac{R}{r}(VR)\left(-\frac{1}{r^2}\right) = -\frac{V^2R^2}{r^3} = -\frac{V^2}{R}\left(\frac{R}{r}\right)^3$$

At $r = r_i = 25$ mm,

$$a_r = -(15)^2\frac{m^2}{s^2}\times\frac{1}{0.075\,m}\left(\frac{75}{25}\right)^3 = -81.0\ \frac{km}{s^2}$$ ← $a_r(r_i)$

At $r = R = 75$ mm

$$a_r = -(15)^2\frac{m^2}{s^2}\times\frac{1}{0.075\,m}\left(\frac{75}{75}\right)^3 = -3.00\ \frac{km}{s^2}$$ ← $a_r(R)$

Problem 5.47

5.47 As part of a pollution study, a model concentration c as a function of position x has been developed,

$$c(x) = A(e^{-x/a} - e^{-x/2a})$$

where $A = 10^{-5}$ ppm (parts per million) and $a = 1$ m. Plot this concentration from $x = 0$ to $x = 10$ m. If a vehicle with a pollution sensor travels through this atmosphere at $u = U(U = 20$ m/s), develop an expression for the measured concentration rate of change of c with time, and plot using given data. At what location will the sensor indicate the most rapid rate of change, and what is the value of this rate of change?

Given: Data on pollution concentration

Find: Plot of concentration; Plot of concentration over time for moving vehicle; Location and value of maximum rate change

Solution:

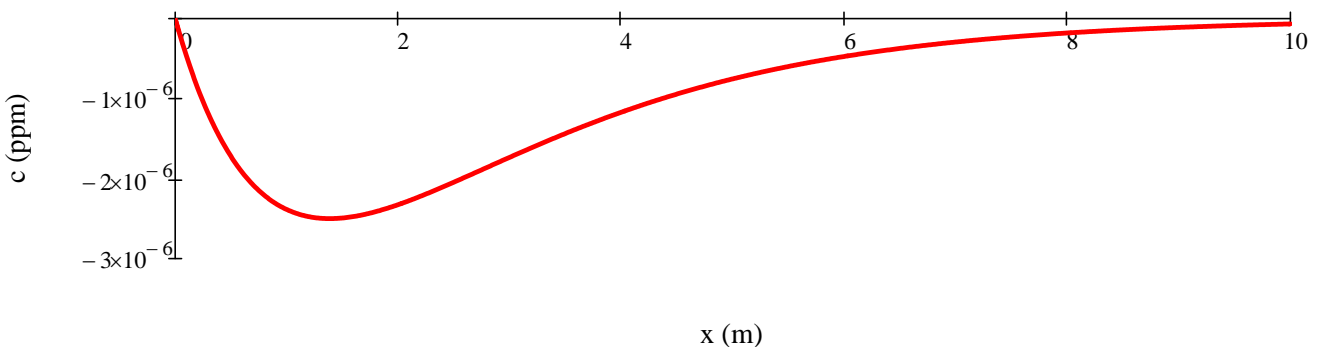
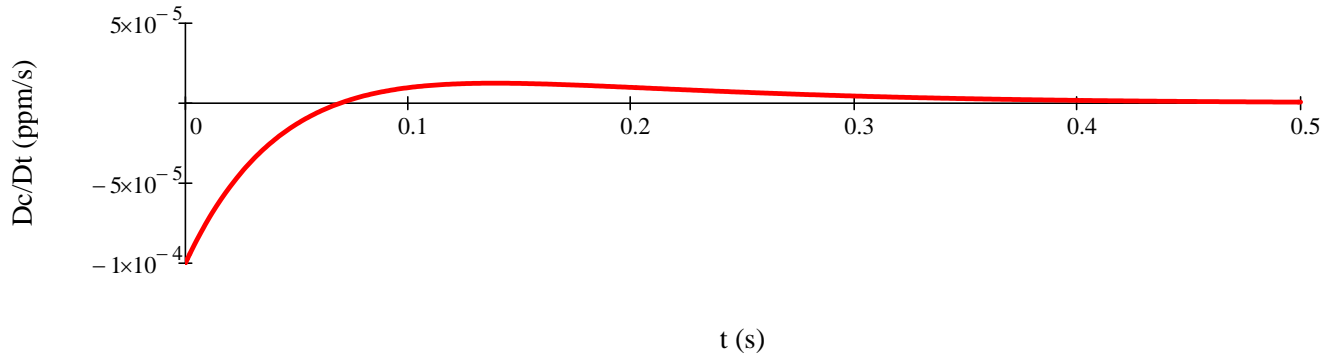
Basic equation: Material derivative $\dfrac{D}{Dt} = u\dfrac{\partial}{\partial x} + v\dfrac{\partial}{\partial y} + w\dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial t}$

For this case we have $u = U \quad v = 0 \quad w = 0 \quad c(x) = A\cdot\left(e^{-\frac{x}{a}} - e^{-\frac{x}{2\cdot a}}\right)$

Hence
$$\frac{Dc}{Dt} = u\cdot\frac{dc}{dx} = U\cdot\frac{d}{dx}\left[A\cdot\left(e^{-\frac{x}{a}} - e^{-\frac{x}{2\cdot a}}\right)\right] = \frac{U\cdot A}{a}\cdot\left(\frac{1}{2}\cdot e^{-\frac{x}{2\cdot a}} - e^{-\frac{x}{a}}\right)$$

We need to convert this to a function of time. For this motion u = U so $x = U\cdot t$

$$\frac{Dc}{Dt} = \frac{U\cdot A}{a}\cdot\left(\frac{1}{2}\cdot e^{-\frac{U\cdot t}{2\cdot a}} - e^{-\frac{U\cdot t}{a}}\right)$$

The following plots can be done in *Excel*

The maximum rate of change is when

$$\frac{d}{dx}\left(\frac{Dc}{Dt}\right) = \frac{d}{dx}\left[\frac{U \cdot A}{a}\cdot\left(\frac{1}{2}\cdot e^{-\frac{x}{2\cdot a}} - e^{-\frac{x}{a}}\right)\right] = 0$$

$$\frac{U \cdot A}{a^2}\cdot\left(e^{-\frac{x}{a}} - \frac{1}{4}\cdot e^{-\frac{x}{2\cdot a}}\right) = 0 \qquad \text{or} \qquad e^{-\frac{x}{2\cdot a}} = \frac{1}{4}$$

$$x_{max} = 2\cdot a\cdot \ln(4) = 2 \times 1\cdot m \times \ln\left(\frac{1}{4}\right) \qquad x_{max} = 2.77\cdot m$$

$$t_{max} = \frac{x_{max}}{U} = 2.77\cdot m \times \frac{s}{20\cdot m} \qquad t_{max} = 0.138\cdot s$$

$$\frac{Dc_{max}}{Dt} = \frac{U \cdot A}{a}\cdot\left(\frac{1}{2}\cdot e^{-\frac{x_{max}}{2\cdot a}} - e^{-\frac{x_{max}}{a}}\right)$$

$$\frac{Dc_{max}}{Dt} = 20\cdot\frac{m}{s} \times 10^{-5}\cdot ppm \times \frac{1}{1\cdot m} \times \left(\frac{1}{2} \times e^{-\frac{2.77}{2\cdot 1}} - e^{-\frac{2.77}{1}}\right) \qquad \frac{Dc_{max}}{Dt} = 1.25 \times 10^{-5}\cdot\frac{ppm}{s}$$

Note that there is another maximum rate, at t = 0 (x = 0)

$$\frac{Dc_{max}}{Dt} = 20\cdot\frac{m}{s} \times 10^{-5}\cdot ppm \times \frac{1}{1\cdot m}\cdot\left(\frac{1}{2} - 1\right) \qquad \frac{Dc_{max}}{Dt} = -1 \times 10^{-4}\cdot\frac{ppm}{s}$$

Given: Aircraft flying north with velocity component $u = 300$ mph is climbing at rate, $v = 3000$ ft/min. The rate of temperature change with vertical distance y is $\partial T/\partial y = -3°F/1000$ ft. The variation of temperature with position x is $\partial T/\partial x = -1°F/\text{mile}$

Find: the rate of temperature change shown by a recorder on board the aircraft

Solution: Apply the substantial derivative concept

Basic equation: $\frac{DT}{Dt} = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \frac{\partial T}{\partial t}$ (with $\frac{\partial T}{\partial t} \to 0$)

Substituting numerical values,

$$\frac{DT}{Dt} = 300\,\frac{\text{mile}}{\text{hr}} \times -\frac{1°F}{\text{mile}} \times \frac{\text{hr}}{60\,\text{min}} + 3000\,\frac{\text{ft}}{\text{min}} \times -\frac{3°F}{1000\,\text{ft}}$$

$$\frac{DT}{Dt} = (-5-9)°F/\text{min} = -14°F/\text{min}$$

$\frac{DT}{Dt}$

Problem 5.49

[2]

Given: Instruments on board an aircraft flying through a cold front give the following information:

- rate of change of temperature is −0.5 F/min
- air speed = 300 knots
- rate of climb = 3500 ft/min

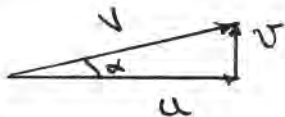
Front is stationary and vertically uniform

Find: rate of change of temperature with respect to horizontal distance through the cold front

Solution: Apply the substantial derivative concept

Basic equation: $\frac{DT}{Dt} = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \frac{\partial T}{\partial t}$ ($\frac{\partial T}{\partial y} = 0$, vertically uniform; $\frac{\partial T}{\partial t} = 0$, stationary front)

$\frac{DT}{Dt} = -0.5\text{ F/min}$. Need to find $\frac{\partial T}{\partial x}$

Velocity picture

$$V = 300\frac{\text{nm}}{\text{hr}} \times 6080\frac{\text{ft}}{\text{nm}} \times \frac{\text{hr}}{3600\text{ s}} = 507\frac{\text{ft}}{\text{s}}$$

$$v = 3500\frac{\text{ft}}{\text{min}} \times \frac{\text{min}}{60\text{ s}} = 58.3\text{ ft/s}$$

Then $\alpha = \sin^{-1}\frac{v}{V} = \sin^{-1}\frac{58.3}{507} = 6.60^\circ$

and $u = V\cos\alpha = 507\frac{\text{ft}}{\text{sec}}\cos 6.60^\circ = 504\text{ ft/s}$

$$\therefore \frac{\partial T}{\partial x} = \frac{1}{u}\frac{DT}{Dt} = \frac{\text{s}}{504\text{ ft}} \times \frac{-0.5^\circ\text{F}}{\text{min}} \times \frac{\text{min}}{60\text{ s}} \times \frac{5280\text{ ft}}{\text{mi}}$$

$$\frac{\partial T}{\partial x} = -0.0873^\circ\text{F/mile} \qquad \frac{\partial T}{\partial x}$$

Problem 5.50

Given: Sediment concentration rates in a river after a rainfall are:

$$\frac{\partial c}{\partial t} = 100 \frac{ppm}{hr}, \quad \frac{\partial c}{\partial x} = 50 \frac{ppm}{mi} \quad \text{(downstream)}$$

Stream speed is $u_s = 0.5$ mph, where a boat is used to survey concentration.

Boat speed is $V_b = 2.5$ mph.

Find: (a) Calculate rates of change of sediment concentration observed when boat travels upstream, drifts with the current, or travels downstream.

(b) Explain physically why the observed rates differ.

Solution: Apply substantial derivative concept

Basic equation: $\frac{Dc}{Dt} = u\frac{\partial c}{\partial x} + \frac{\partial c}{\partial t}$

To obtain rate of change seen from boat, set $u = u_B$.

(i) For travel upstream, $u_B = u_s - V_b = 0.5 - 2.5 = -2.0$ mph

$$\frac{Dc}{Dt}(up) = -2.0 \frac{mi}{hr} \times 50 \frac{ppm}{mi} + 100 \frac{ppm}{hr} = 0.00 \text{ ppm/hr}$$

up

(ii) For drifting, $u_B = u_s + 0 = 0.5$ mph

$$\frac{Dc}{Dt}(drift) = 0.5 \frac{mi}{hr} \times 50 \frac{ppm}{mi} + 100 \frac{ppm}{hr} = 125 \text{ ppm/hr}$$

drift

(iii) For travel downstream, $u_B = u_s + V_b = 0.5 + 2.5 = 3.0$ mph

$$\frac{Dc}{Dt}(down) = 3.0 \frac{mi}{hr} \times 50 \frac{ppm}{mi} + 100 \frac{ppm}{hr} = 250 \text{ ppm/hr}$$

down

Physically the observed rates of change differ because the observer is <u>convected</u> through the flow. The convective change may add to or subtract from the local rate of change.

Problem 5.51

Expand $(\vec{V}\cdot\nabla)\vec{V}$ in rectangular coordinates to obtain the convective acceleration of a fluid particle. Verify the results given in Eqs 5.11

Solution:

In rectangular coordinates $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$, $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$

$$(\vec{V}\cdot\nabla)\vec{V} = \left[(u\hat{i} + v\hat{j} + w\hat{k})\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\right] u\hat{i} + v\hat{j} + w\hat{k}$$

$$= \left[u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right] u\hat{i} + v\hat{j} + w\hat{k}$$

$$(\vec{V}\cdot\nabla)\vec{V} = \left\{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right\}^{①}\hat{i} + \left\{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right\}^{②}\hat{j} + \left\{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right\}^{③}\hat{k}$$

Term ① is x component of convective acceleration

Eq 5.11a $\quad a_{x_p} = \left\{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right\} + \frac{\partial u}{\partial t}$

Term ② is the y component of convective acceleration

Eq 5.11b $\quad a_{y_p} = \left\{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right\} + \frac{\partial v}{\partial t}$

Term ③ is the z component of convective acceleration

Eq 5.11c $\quad a_{z_p} = \left\{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right\} + \frac{\partial w}{\partial t}$

Problem 5.52

Given: Velocity field represented by

$$\vec{V} = (Ax - B)\hat{i} + Cy\hat{j} + Dt\hat{k} \qquad (x, y \text{ in m})$$

where $A = 2\ s^{-1}$, $B = 4\ m/s$, and $D = 5\ m/s^2$

Find: (a) Proper value of C for incompressible flow.
(b) Acceleration of particle at $(x,y) = (3,2)$.
(c) Sketch streamlines in xy plane.

Solution: For incompressible flow, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$. Since $w = Dt$, $\partial w/\partial z = 0$, and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial y} = C = -\frac{\partial u}{\partial x} = -A = -2\ s^{-1}$$

C

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$$

$$\vec{a}_p = (Ax - B)(A\hat{i}) + (Cy)(C\hat{j}) + (Dt)(0) + D\hat{k}$$

$$\vec{a}_p(3,2) = \left(\frac{2}{s} \times 3m - \frac{4m}{s}\right)\left(\frac{2}{s}\right)\hat{i} + \left(\frac{-2}{s} \times 2m\right)\left(-\frac{2}{s}\right)\hat{j} + 5\frac{m}{s^2}\hat{k}$$

$$\vec{a}_p(3,2) = 4\hat{i} + 8\hat{j} + 5\hat{k} \quad m/s^2$$

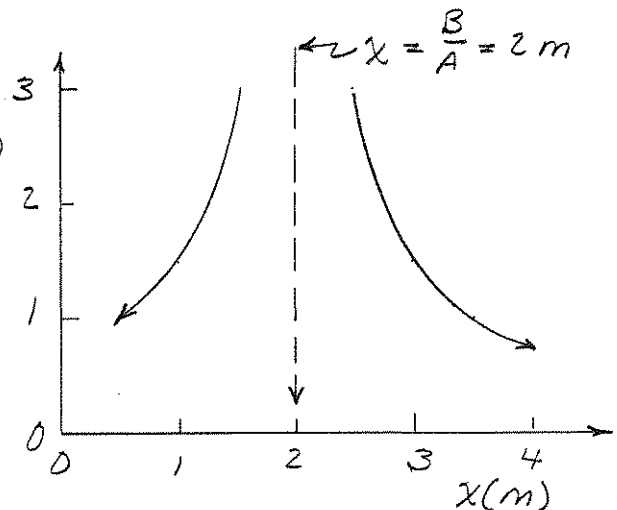
$\vec{a}_p(3,2)$

In the xy plane, streamlines are $\frac{dy}{dx} = \frac{v}{u} = \frac{Cy}{Ax - B}$. Thus

$$\frac{dx}{Ax - B} = \frac{dy}{Cy} \quad \text{or} \quad \frac{dx}{Ax - B} = -\frac{dy}{Ay} \quad \text{or} \quad \frac{dx}{x - B/A} + \frac{dy}{y} = 0$$

Integrating

$$\ln(x - B/A) + \ln y = \ln c_0$$

$$\left(x - \frac{B}{A}\right)y = \text{const}$$

Problem 5.53

Given: Steady, two-dimensional velocity field, $\vec{V} = Ax\hat{i} - Ay\hat{j}$; $A = 1\ s^{-1}$, coordinates measured in meters

Show: that streamlines are hyperbolas, $xy = C$

Find: (a) Expression for acceleration.
(b) Particle acceleration at $(x,y) = (\frac{1}{2}, 2)$, $(1,1)$ and $(2, \frac{1}{2})$

Plot: streamlines corresponding to $C = 0, 1$, and $2\ m^2$; show acceleration vectors on the plot

Solution:

Along a streamline, $\frac{dy}{dx} = \frac{v}{u} = \frac{-y}{x}$ or $\frac{dy}{y} + \frac{dx}{x} = 0$

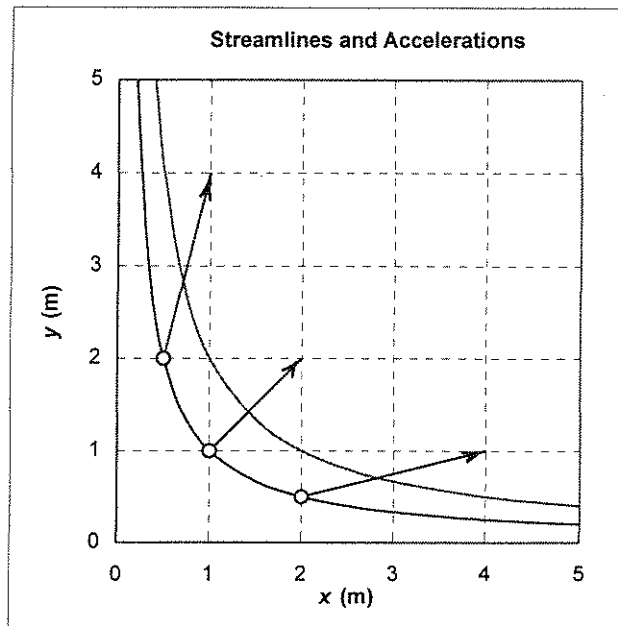
Integrating we obtain $\ln y + \ln x = \ln C$ and $xy = C$ ← Streamline

The acceleration of a particle is

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t} \quad \{w = 0 \text{ and steady flow}\}$$

$$\vec{a}_p = Ax(A\hat{i}) - (Ay)(-A\hat{j}) = A^2(x\hat{i} + y\hat{j}) \quad \leftarrow \vec{a}_p$$

$$\left.\vec{a}_p\right)_{\frac{1}{2},2} = \frac{1}{2}\hat{i} + 2\hat{j}\ m/s^2 \qquad \left.\vec{a}_p\right)_{1,1} = \hat{i} + \hat{j}\ m/s^2$$

$$\left.\vec{a}_p\right)_{2,\frac{1}{2}} = 2\hat{i} + \frac{1}{2}\hat{j}\ m/s^2 \quad \leftarrow \vec{a}_p$$

Plot:

Problem 5.54

Given: Velocity field $\vec{V} = (Ax - B)\hat{i} - Ay\hat{j}$; $A = 0.2\ s^{-1}$; $B = 0.6\ s^{-1}$; x in m.

Find: (a) General expression for acceleration of a fluid particle.
(b) Acceleration at $(x,y) = (0, 4/3), (1,2)$, and $(2,4)$.
(c) Plot of streamlines.
(d) Acceleration vectors on plot.

Solution: Note $w = 0$ and flow is steady. Then

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = (Ax - B)A\hat{i} + (-Ay)(-A)\hat{j} = (A^2x - AB)\hat{i} + A^2y\hat{j}$$ $\vec{a}_p$

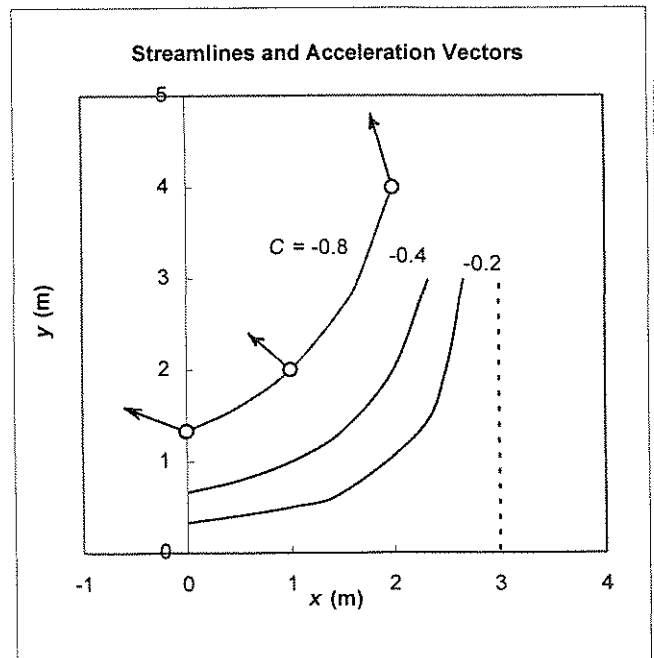
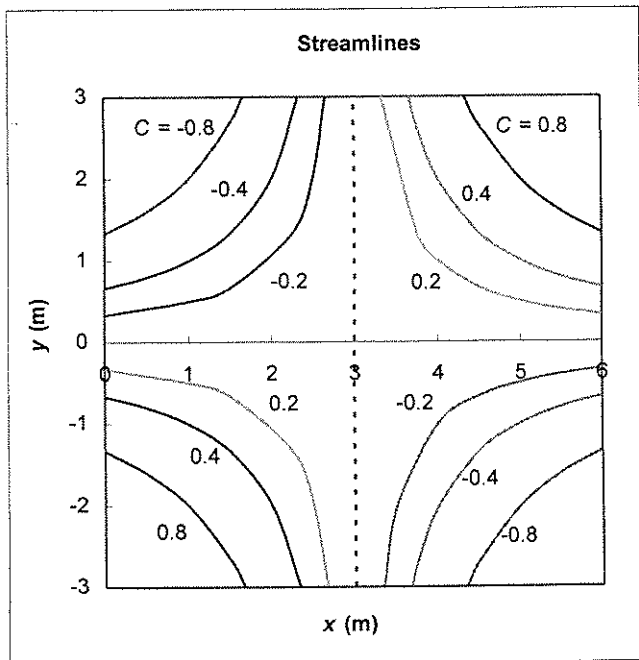
At $(x,y) = (0, 4/3)$, $\vec{a}_p = -0.12\hat{i} + 0.0533\hat{j}$ m/s²

$(1,2)$, $\vec{a}_p = -0.08\hat{i} + 0.0800\hat{j}$ m/s²

$(2,4)$, $\vec{a}_p = -0.04\hat{i} + 0.160\hat{j}$ m/s² $\vec{a}_p$

streamlines are $\frac{dx}{u} = \frac{dy}{v} = \frac{dx}{Ax - B} = \frac{dy}{-Ay}$. Integrating,

$$\frac{1}{A}\ln(Ax - B) + \frac{1}{A}\ln y = \frac{1}{A}\ln c \quad \text{or} \quad (Ax - B)y = c$$

The plots are:

Problem 5.55

Given: Air flowing downward toward infinite horizontal flat plate. Velocity field is

$$\vec{V} = (ax\hat{i} - ay\hat{j})(2 + \cos\omega t)\ ;\ a = 3\ s^{-1},\ \omega = \pi\ s^{-1}$$

Find: (a) Expression for streamline at $t = 1.5$ s.
(b) Plot of streamline through $(x,y) = (2,4)$ at this instant.
(c) Velocity vector
(d) Vectors representing local, convective, and total acceleration.

Solution: Streamline is $\frac{dx}{u} = \frac{dy}{v}$, or $\frac{dx}{x} + \frac{dy}{y} = 0$ or $xy = c$

At point $(x,y) = (2,4)$, $c = 2m \times 4m = 8m^2$; $xy = 8\ m^2$ — Streamline

The plot is shown below. Note $u = ax\hat{i}[2+\cos\omega t]$, $v = -ay\hat{j}[2+\cos\omega t]$

At $(x,y,t) = (2\,m, 4\,m, 1.5\,s)$, $\vec{V} = (6\hat{i} - 12\hat{j})(2+0) = 12\hat{i} - 24\hat{j}$ — $\vec{V}$

The local acceleration components at $(x,y,t) = (2m, 4m, 1.5s)$ are

$$a_{x,local} = \frac{\partial u}{\partial t} = ax\hat{i}(-\omega\sin\omega t) = \frac{3}{s} \times 2m \times \left(\frac{-\pi}{s}\right) \times \sin\left(\frac{3\pi}{2}\right) = 6\pi\hat{i}\ m/s^2$$

$$a_{y,local} = \frac{\partial v}{\partial t} = -ay\hat{j}(-\omega\sin\omega t) = \frac{3}{s} \times 4m \times \left(\frac{-\pi}{s}\right) \times \sin\left(\frac{3\pi}{2}\right) = -12\pi\hat{j}\ m/s^2$$ — Local

The convective acceleration components at $(x,y,t) = (2m, 4m, 1.5s)$ are

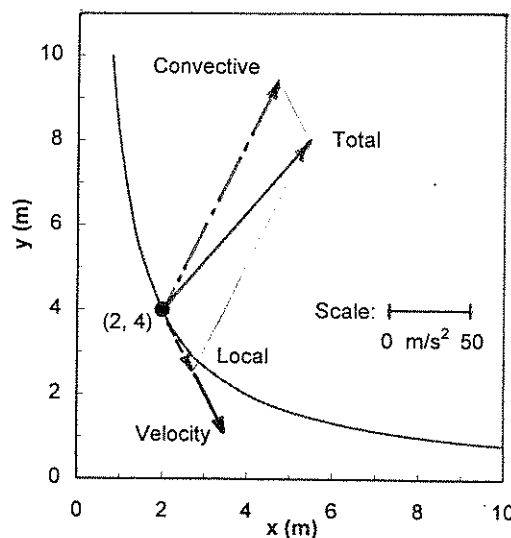
$$a_{x,conv} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = ax(a\hat{i})\left[2 + \cos\frac{3\pi}{2}\right]^2 = (3)(2)(3)[2]^2\hat{i} = 72\hat{i}$$

$$a_{y,conv} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = (-ay)(-a\hat{j})\left[2 + \cos\frac{3\pi}{2}\right]^2 = 4a^2y\hat{j} = 4(3)^2 4\hat{j} = 144\hat{j}$$ — Convective

The total acceleration is the sum of the convective and local values:

$$a_{x,total} = a_{x,conv} + a_{x,local} = (72 + 6\pi)\hat{i} = 90.8\hat{i}\ m/s^2$$

$$a_{y,total} = a_{y,conv} + a_{y,local} = (144 - 12\pi)\hat{j} = 106\ \hat{j}\ m/s^2$$ — Total

The plot is

Problem 5.56

Given: Laminar boundary layer, linear approximate profile.

$$\frac{u}{U} = \frac{y}{\delta} \qquad \delta = cx^{1/2}$$

From Problem 5.7, $v = \frac{uy}{4x} = U\frac{y^2}{4x\delta}$

Find: (a) x and y components of acceleration of a fluid particle.
(b) Locate maximum values.
(c) Ratio, $a_{x,max}/a_{y,max}$.

Solution: Basic equations: $a_{px} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$ (=0(1), =0(2))

$$a_{py} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$ (=0(1), =0(2))

Assumptions: (1) w and $\partial/\partial z$ zero, (2) steady flow. $\frac{d\delta}{dx} = \frac{1}{2}cx^{-1/2} = \frac{\delta}{2x}$

$$u = U\frac{y}{\delta}\ ;\ \frac{\partial u}{\partial x} = Uy\left(-\frac{1}{\delta^2}\right)\frac{d\delta}{dx} = -Uy\frac{1}{\delta^2}\frac{\delta}{2x} = -\frac{Uy}{2x\delta}\ ;\ \frac{\partial u}{\partial y} = \frac{U}{\delta}$$

$$v = U\frac{y^2}{4x\delta}\ ;\ \frac{\partial v}{\partial x} = \frac{Uy^2}{4}\left(-\frac{1}{x^2\delta} - \frac{1}{x\delta^2}\frac{d\delta}{dx}\right) = -\frac{3Uy^2}{8x^2\delta}\ ;\ \frac{\partial v}{\partial y} = \frac{Uy}{2x\delta}$$

Thus

$$a_{px} = \left(U\frac{y}{\delta}\right)\left(-\frac{Uy}{2x\delta}\right) + \left(U\frac{y^2}{4x\delta}\right)\left(\frac{U}{\delta}\right) = -\frac{U^2}{2x}\left(\frac{y}{\delta}\right)^2 + \frac{U^2}{4x}\left(\frac{y}{\delta}\right)^2 = -\frac{U^2}{4x}\left(\frac{y}{\delta}\right)^2$$

a_{px}

$$a_{py} = \left(U\frac{y}{\delta}\right)\left(-\frac{3Uy^2}{8x^2\delta}\right) + \left(U\frac{y^2}{4x\delta}\right)\left(U\frac{y}{2x\delta}\right) = -\frac{3U^2}{8x}\left(\frac{y}{x}\right)\left(\frac{y}{\delta}\right)^2 + \frac{U^2}{8x}\left(\frac{y}{x}\right)\left(\frac{y}{\delta}\right)^2$$

$$a_{py} = -\frac{U^2}{4x}\left(\frac{y}{x}\right)\left(\frac{y}{\delta}\right)^2$$

a_{py}

Maximum values are at $y = \delta$

$$a_{px,max} = -\frac{U^2}{4x}$$

(max: a_{px}

$$a_{py,max} = -\frac{U^2}{4x}\left(\frac{\delta}{x}\right)$$

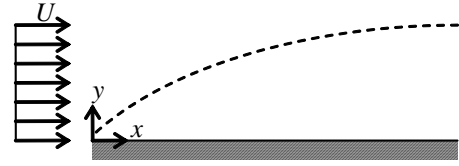
(max a_{py}

Thus $\frac{a_{px,max}}{a_{py,max}} = \frac{x}{\delta}$

At $x = 0.5\,m$, $\delta = 5\,mm$, $\frac{a_{px,max}}{a_{py,max}} = \frac{0.5\,m}{0.005\,m} = 100$

Ratio

Problem 5.57

5.57 A parabolic approximate velocity profile was used in Problem 5.11 to model flow in a laminar incompressible boundary layer on a flat plate. For this profile, find the x component of acceleration, a_x, of a fluid particle within the boundary layer. Plot a_x at location $x = 0.8$ m, where $\delta = 1.2$ mm, for a flow with $U = 6$ m/s. Find the maximum value of a_x at this x location.

Given: Flow in boundary layer

Find: Expression for particle acceleration a_x; Plot acceleration and find maximum at x = 0.8 m

Solution:

Basic equations
$$\frac{u}{U} = 2\cdot\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \qquad \frac{v}{U} = \frac{\delta}{x}\cdot\left[\frac{1}{2}\cdot\left(\frac{y}{\delta}\right) - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^3\right] \qquad \delta = c\cdot\sqrt{x}$$

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = \underbrace{u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}}_{\text{convective acceleration}} + \frac{\partial\vec{V}}{\partial t}$$

total acceleration of a particle; convective acceleration; local acceleration

We need to evaluate
$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u$$

First, substitute
$$\lambda(x,y) = \frac{y}{\delta(x)} \quad \text{so} \quad \frac{u}{U} = 2\cdot\lambda - \lambda^2 \qquad \frac{v}{U} = \frac{\delta}{x}\cdot\left(\frac{1}{2}\cdot\lambda - \frac{1}{3}\cdot\lambda^3\right)$$

Then
$$\frac{\partial}{\partial x}u = \frac{du}{d\lambda}\cdot\frac{d\lambda}{dx} = U\cdot(2-2\cdot\lambda)\cdot\left(-\frac{y}{\delta^2}\right)\cdot\frac{d\delta}{dx} \qquad \frac{d\delta}{dx} = \frac{1}{2}\cdot c\cdot x^{-\frac{1}{2}}$$

$$\frac{\partial}{\partial x}u = U\cdot(2-2\cdot\lambda)\cdot\left(-\frac{\lambda}{\delta}\right)\cdot\frac{1}{2}\cdot c\cdot x^{-\frac{1}{2}} = U\cdot(2-2\cdot\lambda)\cdot\left(-\frac{\lambda}{c\cdot x^{\frac{1}{2}}}\right)\cdot\frac{1}{2}\cdot c\cdot x^{-\frac{1}{2}}$$

$$\frac{\partial}{\partial x}u = -U\cdot(2-2\cdot\lambda)\cdot\frac{\lambda}{2\cdot x} = -\frac{U\cdot\left(\lambda-\lambda^2\right)}{x}$$

$$\frac{\partial}{\partial y}u = U\cdot\left(\frac{2}{\delta} - 2\cdot\frac{y}{\delta^2}\right) = \frac{2\cdot U}{\delta}\cdot\left[\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right] = \frac{2\cdot U\cdot\left(\lambda-\lambda^2\right)}{y}$$

Hence
$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = U\cdot\left(2\cdot\lambda-\lambda^2\right)\left[\frac{U\cdot\left(\lambda-\lambda^2\right)}{x}\right] + U\cdot\frac{\delta}{x}\cdot\left(\frac{1}{2}\cdot\lambda - \frac{1}{3}\cdot\lambda^3\right)\cdot\left[\frac{2\cdot U\cdot\left(\lambda-\lambda^2\right)}{y}\right]$$

Collecting terms
$$a_x = \frac{U^2}{x}\cdot\left(-\lambda^2 + \frac{4}{3}\cdot\lambda^3 - \frac{1}{3}\cdot\lambda^4\right) = \frac{U^2}{x}\cdot\left[-\left(\frac{y}{\delta}\right)^2 + \frac{4}{3}\cdot\left(\frac{y}{\delta}\right)^3 - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^4\right]$$

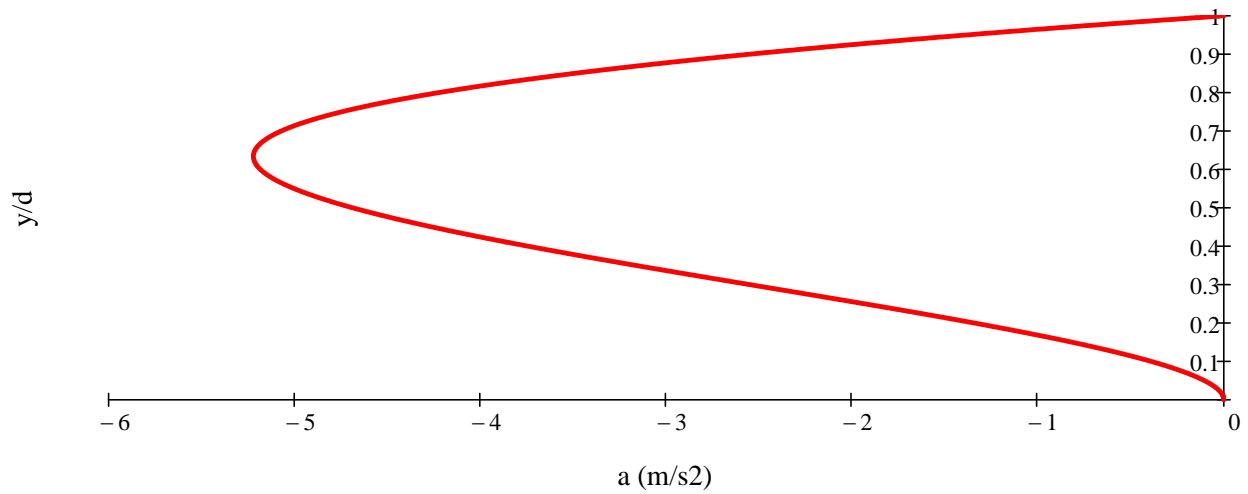
To find the maximum
$$\frac{da_x}{d\lambda} = 0 = \frac{U^2}{x}\cdot\left(-2\cdot\lambda + 4\cdot\lambda^2 - \frac{4}{3}\cdot\lambda^3\right) \quad \text{or} \quad -1 + 2\cdot\lambda - \frac{2}{3}\cdot\lambda^2 = 0$$

The solution of this quadratic ($\lambda < 1$) is
$$\lambda = \frac{3-\sqrt{3}}{2} \qquad \lambda = 0.634 \qquad \frac{y}{\delta} = 0.634$$

At λ = 0.634

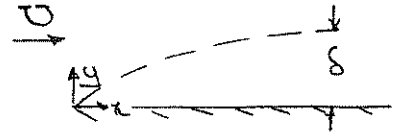
$$a_x = \frac{U^2}{x}\cdot\left(-0.634^2 + \frac{4}{3}\cdot 0.634^3 - \frac{1}{3}\cdot 0.634^4\right) = -0.116\cdot\frac{U^2}{x}$$

$$a_x = -0.116 \times \left(6\cdot\frac{m}{s}\right)^2 \times \frac{1}{0.8\cdot m} \qquad a_x = -5.22\frac{m}{s^2}$$

The following plot can be done in *Excel*

Problem 5.58

Given: Laminar boundary layer on a flat plate. (Problem 5.11)

$$\frac{u}{U} = \sin\frac{\pi y}{2\delta}, \quad \delta = c x^{1/2}$$

$$\frac{v}{U} = \frac{1}{\pi}\frac{\delta}{x}\left[\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) + \left(\frac{\pi}{2}\frac{y}{\delta}\right)\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) - 1\right]$$

Find: Expression for a_{xp} and a_{yp}

Plot: a_x and a_y as functions of y/δ for $U = 5$ m/s, $x = 1$ m & $\delta = 1$ mm; determine maximum values at locations at which maxima occur.

Solution:

Basic equations:

$$a_{px} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \quad \text{---- (1)}$$

$$a_{py} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \quad \text{---- (2)}$$

(the $w\,\partial/\partial z$ terms are crossed out, marked 0(2); the $\partial/\partial t$ terms are crossed out, marked 0(1))

Assumptions: (1) steady flow
(2) w and $\partial/\partial z = 0$

Let $\eta = \frac{\pi}{2}\frac{y}{\delta}$; $\eta = \eta(x,y)$; $\frac{\partial \eta}{\partial y} = \frac{\pi}{2\delta}$; $\delta = cx^{1/2}$, $\frac{d\delta}{dx} = \frac{1}{2}cx^{-1/2} = \frac{\delta}{2x}$

$$\frac{\partial \eta}{\partial x} = \frac{\partial \eta}{\partial \delta}\frac{d\delta}{dx} = \frac{\pi y}{2}\left(-\frac{1}{\delta^2}\right)\frac{\delta}{2x} = -\frac{\pi}{4x}\left(\frac{y}{\delta}\right)$$

$u = U\sin\eta$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = U\cos\eta\left(-\frac{\pi}{4x}\frac{y}{\delta}\right) = -\frac{U}{2x}\left(\frac{\pi}{2}\frac{y}{\delta}\right)\cos\eta = -\frac{U}{2x}\eta\cos\eta \quad \text{---- (3)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = U\cos\eta\frac{\pi}{2\delta} = \frac{U\pi}{2\delta}\cos\eta \quad \text{---- (4)}$$

$v = U\frac{1}{\pi}\frac{\delta}{x}(\cos\eta + \eta\sin\eta - 1)$. Differentiating using product rule

$$\frac{\partial v}{\partial x} = \frac{U}{\pi}\left(\frac{1}{x}\frac{d\delta}{dx} - \frac{\delta}{x^2}\right)(\cos\eta + \eta\sin\eta - 1) + \frac{U}{\pi}\frac{\delta}{x}(-\sin\eta + \eta\cos\eta + \sin\eta)\frac{\partial\eta}{\partial x}$$

$$= \frac{U}{\pi}\left(\frac{1}{x}\frac{\delta}{2x} - \frac{\delta}{x^2}\right)(\cos\eta + \eta\sin\eta - 1) + \frac{U}{\pi}\frac{\delta}{x}\eta\cos\eta\left(-\frac{\pi}{4x}\right)\frac{y}{\delta}$$

$$\frac{\partial v}{\partial x} = -\frac{U}{\pi}\frac{\delta}{2x^2}(\cos\eta + \eta\sin\eta - 1) - \frac{U\delta}{4x^2}\left(\frac{y}{\delta}\right)\eta\cos\eta \quad \text{---- (5)}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{U}{\pi}\frac{\delta}{x}(-\sin\eta + \eta\cos\eta + \sin\eta)\frac{\pi}{2\delta} = \frac{U}{2x}\eta\cos\eta \quad \text{---- (6)}$$

Substituting into Eq. 1,

$$a_x = U\sin\eta\left(-\frac{U}{2x}\eta\cos\eta\right) + \frac{U}{\pi}\frac{\delta}{x}(\cos\eta - \eta\sin\eta - 1)\frac{U\pi}{2\delta}\cos\eta$$

$$a_x = -\frac{U^2}{2x}\eta\sin\eta\cos\eta + \frac{U^2}{2x}(\cos\eta - \eta\sin\eta - 1)\cos\eta$$

$$a_x = \frac{U^2}{2x}\cos\eta\left[\cos\eta - \eta\sin\eta - 1 - \eta\sin\eta\right]$$

(the $\eta\sin\eta$ terms in the bracket are struck through)

Problem 5.58

$$a_x = \frac{U^2}{2x}\cos\eta\,(\cos\eta - 1) = -\frac{U^2}{2x}\cos\eta\,(1-\cos\eta) \quad \longleftarrow a_x$$

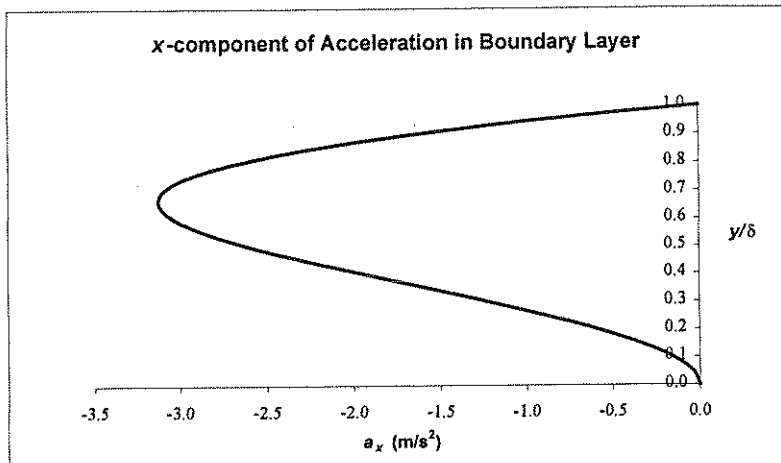
Substituting into Eq. 2.

$$a_y = U\sin\eta\left[-\frac{U}{\pi}\frac{\delta}{2x^2}(\cos\eta + \eta\sin\eta - 1) - \frac{U\delta}{4x^2}\left(\frac{y}{\delta}\right)\eta\cos\eta\right] + \frac{U}{\pi}\frac{\delta}{x}(\cos\eta + \eta\sin\eta - 1)\frac{U}{2x}\eta\cos\eta$$

$$a_y = \frac{U^2\delta}{\pi\, 2x^2}\left\{\left[-\sin\eta(\cos\eta + \eta\sin\eta - 1) - \frac{\pi}{2}\left(\frac{y}{\delta}\right)\eta\cos\eta\sin\eta\right] + \eta\cos\eta(\cos\eta + \eta\sin\eta - 1)\right\}$$

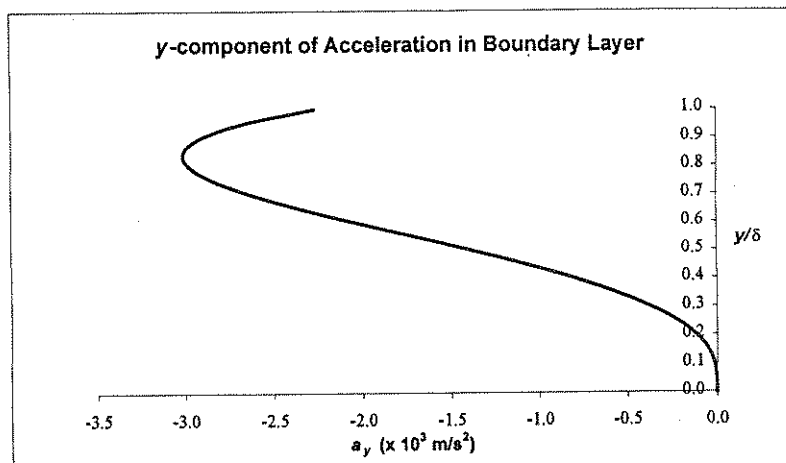
$$a_y = \frac{U^2\delta}{\pi\, 2x^2}\left\{-\sin\eta(\cos\eta + \eta\sin\eta - 1) - \frac{\pi}{2}\left(\frac{y}{\delta}\right)\eta\cos\eta\sin\eta + \eta\cos\eta(\cos\eta + \eta\sin\eta - 1)\right\} \quad \longleftarrow a_y$$

x **component**

y/δ	η	a_x (m/s²)
0.00	0.000	0.000
0.05	0.0785	-0.0384
0.10	0.157	-0.152
0.15	0.236	-0.336
0.20	0.314	-0.582
0.25	0.393	-0.879
0.30	0.471	-1.21
0.35	0.550	-1.57
0.40	0.628	-1.93
0.45	0.707	-2.28
0.50	0.785	-2.59
0.55	0.864	-2.85
0.60	0.942	-3.03
0.65	1.02	-3.12
0.70	1.10	-3.10
0.75	1.18	-2.95
0.80	1.26	-2.67
0.85	1.34	-2.24
0.90	1.41	-1.65
0.95	1.49	-0.904
1.00	1.57	0.000

y/δ	η	a_x (m/s²)
0.667	1.05	-3.12

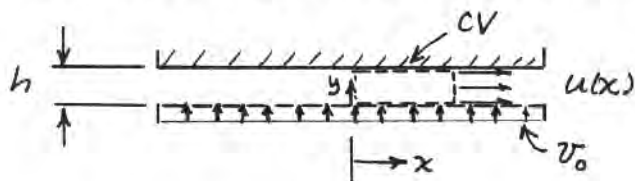
(Maximum absolute value using *Solver*)

y **component**

y/δ	η	a_y (x 10³ m/s²)
0.00	0.000	0.0000
0.05	0.0785	-0.00192
0.10	0.157	-0.0152
0.15	0.236	-0.0506
0.20	0.314	-0.117
0.25	0.393	-0.223
0.30	0.471	-0.372
0.35	0.550	-0.566
0.40	0.628	-0.803
0.45	0.707	-1.08
0.50	0.785	-1.39
0.55	0.864	-1.71
0.60	0.942	-2.04
0.65	1.02	-2.35
0.70	1.10	-2.62
0.75	1.18	-2.84
0.80	1.26	-2.98
0.85	1.34	-3.01
0.90	1.41	-2.91
0.95	1.49	-2.67
1.00	1.57	-2.27

y/δ	η	a_y (x 10³ m/s²)
0.839	1.32	-3.01

(Maximum absolute value using *Solver*)

Note: a_y is normalized with x^2/δ and a_x is normalized with x. Thus $a_y = O\left(\frac{\delta}{x}a_x\right) \approx 0.001\,a_x$.

Given: Air flow through porous surface into narrow gap.

Find: (a) Show $u(x) = v_0 x/h$
(b) Component, v
(c) Acceleration of a fluid particle in the gap.

Solution: Apply conservation of mass to CV shown.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$ (first term $= 0$ (1))

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section

Then

$$0 = \{-x w v_0\} + \{h w u(x)\} \quad \text{or} \quad u(x) = v_0 \frac{x}{h}$$

$u(x)$

Apply differential form to find v:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \frac{\partial u}{\partial x} = \frac{v_0}{h}$$

$$v - v_0 = \int_0^y \frac{\partial v}{\partial y} dy + f(x) = \int_0^y -\frac{v_0}{h} dy + f(x) = -\frac{v_0 y}{h} + f(x)$$

or

$$v = v_0\left(1 - \frac{y}{h}\right) \quad [f(x) = 0 \text{ since } v = v_0 = \text{const along } y = 0]$$

$v(y)$

$$a_{p_x} = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

($v\frac{\partial u}{\partial y} = 0$ (3), $w\frac{\partial u}{\partial z} = 0$, $\frac{\partial u}{\partial t} = 0$ (1))

$$a_{p_x} = \left(v_0 \frac{x}{h}\right)\left(\frac{v_0}{h}\right) = \frac{v_0^2 x}{h^2}$$

a_{p_x}

$$a_{p_y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_{p_y} = v_0\left(1 - \frac{y}{h}\right)\left(-\frac{v_0}{h}\right) = \frac{v_0^2}{h}\left(\frac{y}{h} - 1\right)$$

a_{p_y}

Thus

$$\vec{a}_p = a_{p_x}\hat{i} + a_{p_y}\hat{j} = \frac{v_0^2 x}{h^2}\hat{i} + \frac{v_0^2}{h}\left(\frac{y}{h} - 1\right)\hat{j}$$

$\vec{a}_p$

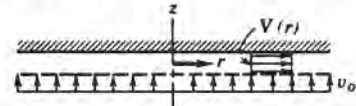
Problem 5.60

[3]

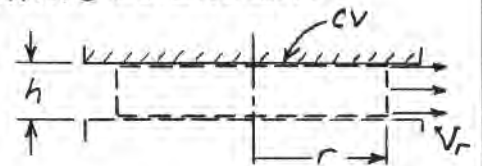
Given: Flow between parallel disks through porous surface.

Find: (a) Show $V_r = v_0 r/2h$

(b) V_z, if $v_0 \ll V_r$

(c) Components of acceleration for a fluid particle in the gap.

Solution: Apply CV form of continuity to finite CV shown.

Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} \quad (=0(1) \text{ on first term})$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section

Then

$$0 = \left\{-|\rho v_0 \pi r^2|\right\} + \left\{+|\rho V_r 2\pi r h|\right\} \quad \text{or} \quad V_r = \frac{v_0 r}{2h}$$

V_r

Apply differential form of conservation of mass for incompressible flow.

Basic equation: $\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(V_\theta) + \frac{\partial}{\partial z}V_z = 0$ (second term $=0(1)$)

Assumptions: (4) $V_\theta = 0$ by symmetry
(5) $V_r = v_0 r/2h$ from above

Then

$$\frac{\partial V_z}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}(rV_r) = -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{v_0 r^2}{2h}\right) = -\frac{1}{r}\left(\frac{v_0 r}{h}\right) = -\frac{v_0}{h}$$

Integrating,

$$V_z = -\frac{v_0 z}{h} + f(r)$$

Boundary conditions are $V_z = v_0$ at $z = 0$, $V_z = 0$ at $z = h$

Thus from first BC, $f(r) = v_0$ = constant, so

$$V_z = v_0\left(1 - \frac{z}{h}\right)$$

V_z

The r component of acceleration is

$$a_r = V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z} + \frac{\partial V_r}{\partial t} = \left(\frac{v_0 r}{2h}\right)\left(\frac{v_0}{2h}\right) = \left(\frac{v_0}{2h}\right)^2 r$$

a_r

The z component is

$$a_z = V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z} + \frac{\partial V_z}{\partial t} = v_0\left(1 - \frac{z}{h}\right)\left(-\frac{v_0}{h}\right) = \frac{v_0^2}{h}\left(\frac{z}{h} - 1\right)$$

a_z

Given: Steady, inviscid flow over a circular cylinder of radius R.

$$\vec{V} = U\cos\theta\left[1-\left(\frac{R}{r}\right)^2\right]\hat{e}_r - U\sin\theta\left[1+\left(\frac{R}{r}\right)^2\right]\hat{e}_\theta$$

Find: (a) Expression for acceleration of particle moving along $\theta=\pi$
(b) Expression for acceleration of particle moving along $r=R$
(c) Locations at which accelerations a_r and a_θ reach maximum and minimum values.

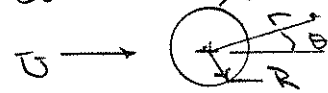
Plot: a_r as a function of R/r for $\theta=\pi$ and as a function of θ for $r=R$; plot a_θ as a function of θ for $r=R$

Solution:

Basic equations: $a_r = V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} + \frac{\partial V_r}{\partial t}$ $= 0(1)$

$a_\theta = V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} + \frac{\partial V_\theta}{\partial t}$ $= 0(1)$

Assumptions: (1) steady flow

$U \longrightarrow$ (cylinder, r, θ, R)

Along $\theta = \pi$

- $\cos\theta = -1$, $\sin\theta = 0$, so $V_\theta = 0$ and $V_r = -U\left[1-\left(\frac{R}{r}\right)^2\right]$

Then

$$a_r = V_r\frac{\partial V_r}{\partial r} = -U\left[1-\left(\frac{R}{r}\right)^2\right](-U)(-2)\left(-\frac{R^2}{r^3}\right) = \frac{2U^2}{R}\left[1-\left(\frac{R}{r}\right)^2\right]\left(\frac{R}{r}\right)^3 \quad a_r$$

$$a_\theta = 0 \quad a_\theta$$

$\Big\}\ \theta=\pi$

To determine location of maximum a_r, let $\frac{R}{r}=\eta$ and evaluate $\frac{da_r}{d\eta}$

$$a_r = \frac{2U^2}{R}\left[1-\eta^2\right]\eta^3 = \frac{2U^2}{R}\left[\eta^3-\eta^5\right]$$

$$\frac{da_r}{d\eta} = \frac{2U^2}{R}\left[3\eta^2-5\eta^4\right]. \text{ Thus } \frac{da_r}{d\eta}=0 \text{ at } \eta^2=\frac{3}{5} \text{ or } \eta = 0.775$$

Thus, $a_{r\,max}$ occurs at $r = R/0.775 = 1.29R$ $\quad r_{a\,max}$

$$a_{r\,max} = \frac{2U^2}{R}(0.775)^3\left[1-(0.775)^2\right] = 0.372\frac{U^2}{R} \text{ @ } r=1.29R.$$

Since $a_\theta=0$, $\vec{a}_{max} = a_{r\,max}\hat{e}_r = 0.372\frac{U^2}{R}\hat{e}_r$ @ $r=1.29R$

Along $r=R$

- $r=R$, $V_r=0$ and $V_\theta = -2U\sin\theta$

$$a_r = -\frac{V_\theta^2}{r} = -\frac{(-2U\sin\theta)^2}{R} = -\frac{4U^2}{R}\sin^2\theta \quad a_r$$

$$a_\theta = \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} = \left(-\frac{2U\sin\theta}{R}\right)\left(-\frac{2U\cos\theta}{R}\right) = \frac{4U^2}{R^2}\sin\theta\cos\theta \quad a_\theta$$

$\Big\}\ r=R$

a_r has maximum negative value at $\theta = \pm\pi/2$
has minimum value (of zero) at $\theta = 0, \pi$

a_θ has maximum values at $\theta = \pm\pi/4$, $3\pi/4$
has minimum values at $\theta = 0, \pm\pi/2, \pi$

Problem 5.61

The acceleration magnitude is

$$|\vec{a}| = \left[a_r^2 + a_\theta^2\right]^{1/2} = \left[\left(-4\frac{U^2}{R}\right)^2 \sin^4\theta + \left(\frac{4U^2}{R}\right)^2 \sin^2\theta \cos^2\theta\right]^{1/2} = \frac{4U^2}{R}\sin\theta$$

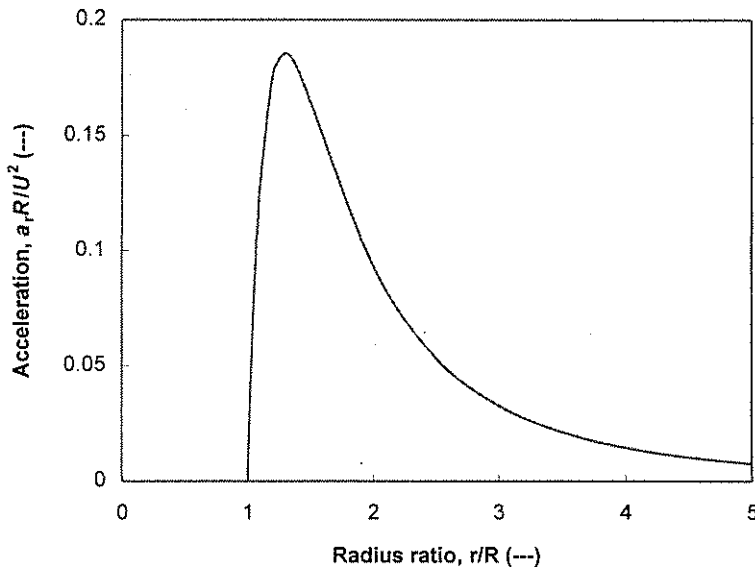
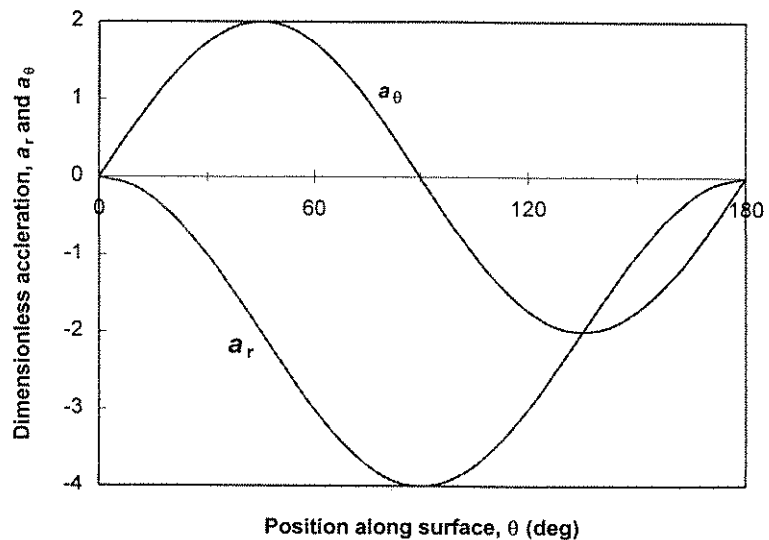
. This is a maximum at $\theta = \pm \pi/2$.

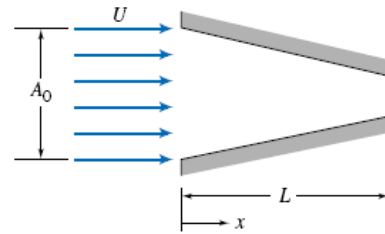
Thus $\vec{a}_{max} = \pm 4\frac{U^2}{R}$ at $\theta = \pm \pi/2$.

Plots:

(1) $\theta = \pi$ $\quad a_r = \frac{2U^2}{R}\left(\frac{R}{r}\right)^3\left[1-\left(\frac{R}{r}\right)^2\right]$; $\quad \frac{a_r}{U^2/R} = \left(\frac{R}{r}\right)^3\left[1-\left(\frac{R}{r}\right)^2\right]$

(2) $r = R$. $\quad a_r = -4\frac{U^2}{R}\sin^2\theta$; $\quad \frac{a_r}{U^2/R} = -4\sin^2\theta$

$a_\theta = 4\frac{U^2}{R}\sin\theta\cos\theta$; $\quad \frac{a_\theta}{U^2/R} = 4\sin\theta\cos\theta$

Problem 5.62

5.62 Consider the incompressible flow of a fluid through a nozzle as shown. The area of the nozzle is given by $A = A_0(1 - bx)$ and the inlet velocity varies according to $U = U_0(1 - e^{-\lambda t})$, where $A_0 = 0.5$ m^2, $L = 5$ m, $b = 0.1$ m^{-1}, $\lambda = 0.2$ s^{-1}, and $U_0 = 5$ m/s. Find and plot the acceleration on the centerline, with time as a parameter.

Given: Velocity field and nozzle geometry

Find: Acceleration along centerline; plot

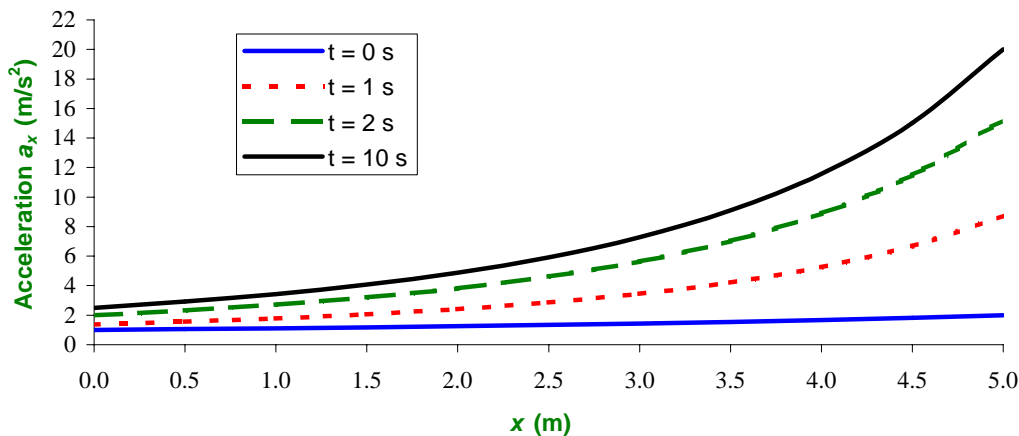
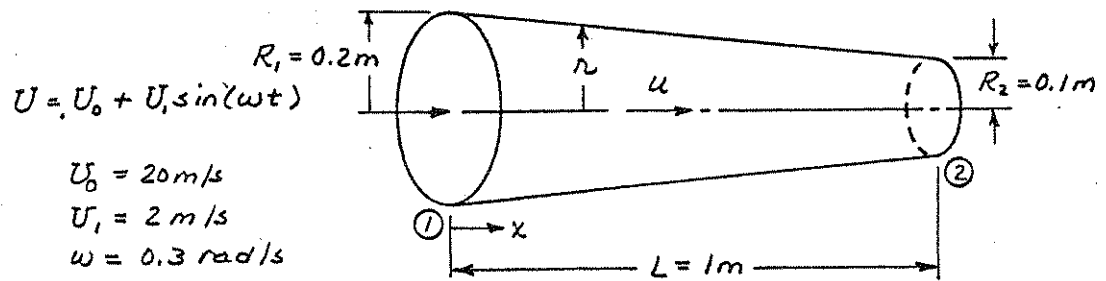
Solution:

$$a_x = \frac{U_0}{(1 - b \cdot x)} \cdot \left[\lambda \cdot e^{-\lambda \cdot t} + \frac{b \cdot U_0}{(1 - b \cdot x)^2} \cdot \left(1 - e^{-\lambda \cdot t}\right)^2\right]$$

$A_0 =$ 0.5 m^2
$L =$ 5 m
$b =$ 0.1 m^{-1}
$\lambda =$ 0.2 s^{-1}
$U_0 =$ 5 m/s

t =	0	5	10	60
x (m)	a_x (m/s^2)	a_x (m/s^2)	a_x (m/s^2)	a_x (m/s^2)
0.0	1.00	1.367	2.004	2.50
0.5	1.05	1.552	2.32	2.92
1.0	1.11	1.78	2.71	3.43
1.5	1.18	2.06	3.20	4.07
2.0	1.25	2.41	3.82	4.88
2.5	1.33	2.86	4.61	5.93
3.0	1.43	3.44	5.64	7.29
3.5	1.54	4.20	7.01	9.10
4.0	1.67	5.24	8.88	11.57
4.5	1.82	6.67	11.48	15.03
5.0	2.00	8.73	15.22	20.00

For large time (> 30 s) the flow is essentially steady-state

Problem 5.63

Given: One-dimensional, incompressible flow through circular channel.

Find: (a) The acceleration of a particle at the channel exit.
(b) Plot as a function of time for a complete cycle.
(c) On same plot, show acceleration if channel is constant area; explain

Solution: The acceleration of a particle in one-dimensional flow is

$$a_x = u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}$$

From continuity, $u = U\frac{A_1}{A} = U\frac{R_1^2}{r^2}$

From geometry, $r = R_1 - (R_1 - R_2)\frac{x}{L} = R_1 - \Delta R\frac{x}{L}$, so

$$u = U\frac{R_1^2}{(R_1 - \Delta R\frac{x}{L})^2} = \left[U_0 + U_1\sin(\omega t)\right]\frac{1}{\left[1 - \frac{\Delta R}{R_1}(\frac{x}{L})\right]^2}$$

Thus

$$a_x = \left[U_0 + U_1\sin(\omega t)\right]\frac{1}{\left[1 - \frac{\Delta R}{R_1}(\frac{x}{L})\right]^2}\left[U_0 + U_1\sin(\omega t)\right](-2)\left(-\frac{\Delta R}{R_1 L}\right)\frac{1}{\left[1 - \frac{\Delta R}{R_1}(\frac{x}{L})\right]^3}$$

$$+ \frac{\omega U_1\cos(\omega t)}{\left[1 - \frac{\Delta R}{R_1}(\frac{x}{L})\right]^2}$$

$$a_x = \frac{2\Delta R}{R_1 L}\frac{\left[U_0 + U_1\sin(\omega t)\right]^2}{\left[1 - \frac{\Delta R}{R_1}(\frac{x}{L})\right]^5} + \frac{\omega U_1\cos(\omega t)}{\left[1 - \frac{\Delta R}{R_1}(\frac{x}{L})\right]^2}$$

At $x/L = 1$, $\left[1 - \frac{\Delta R}{R_1}(\frac{x}{L})\right] = 1 - \frac{0.1m}{0.2m} = 0.5$, so

$$a_x = 2 \times 0.1m \times \frac{1}{0.2m} \times \frac{1}{1m}\left[20 + 2\sin(\omega t)\right]^2\frac{m^2}{s^2} \times \frac{1}{(0.5)^5} + 0.3\frac{rad}{s} \times 2\frac{m}{s} \times \cos(\omega t) \times \frac{1}{(0.5)^2}$$

or

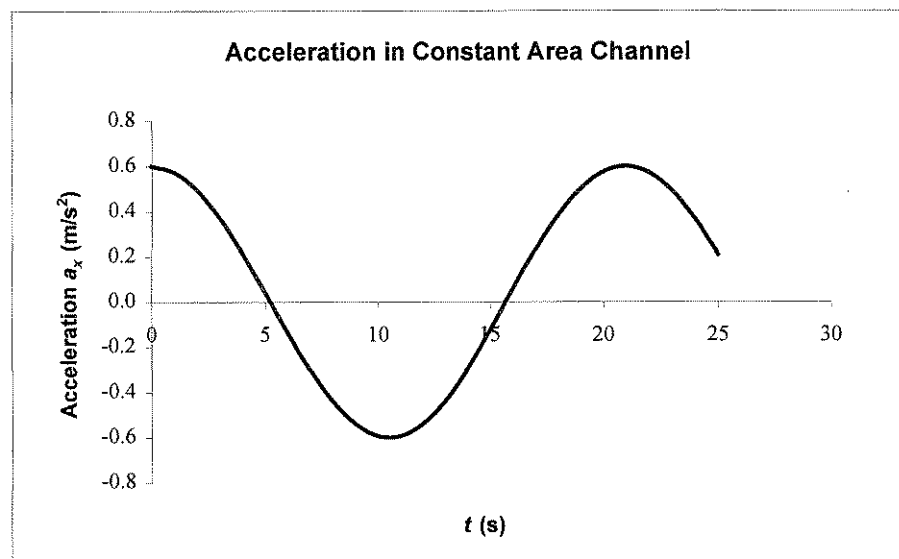
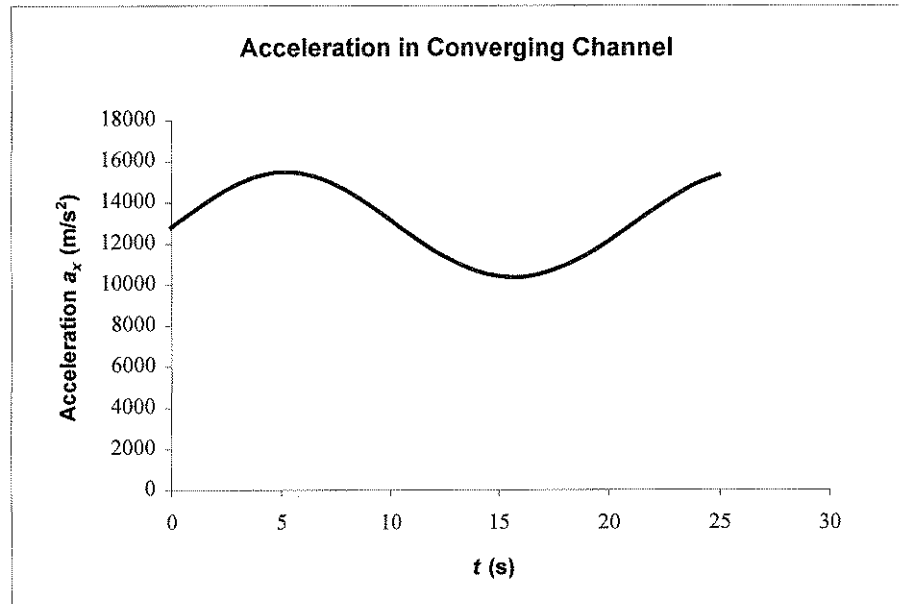
$$a_x\,(m/sec^2) = 32\left[20 + 2\sin(\omega t)\right]^2 + 2.4\cos(\omega t) \quad (\text{at } x = L)$$

a_x

(See next page for plots)

Problem 5.63

The acceleration in the channel and in a constant area are calculated and plotted below

t (s)	a_x (m/s^2) (Convergent)	a_x (m/s^2) (A = const.)
0	12802	0.600
1	13570	0.573
2	14288	0.495
3	14885	0.373
4	15298	0.217
5	15481	0.042
6	15414	-0.136
7	15104	-0.303
8	14586	-0.442
9	13915	-0.542
10	13161	-0.594
11	12397	-0.592
12	11690	-0.538
13	11098	-0.436
14	10665	-0.294
15	10419	-0.126
16	10377	0.052
17	10541	0.227
18	10900	0.381
19	11431	0.501
20	12097	0.576
21	12845	0.600
22	13612	0.570
23	14326	0.489
24	14914	0.365
25	15315	0.208

The acceleration in the convergent channel is massively larger than that in the constant area channel because very large convective acceleration is generated by the convergence (the constant area channel only has local acceleration)

Problem 5.64

Given: Steady, two-dimensional velocity field of Problem 5.53,

$$\vec{V} = Ax\hat{i} - Ay\hat{j} \quad ; \quad A = 1\,s^{-1}$$

Find: (a) Expressions for particle coordinates, $x_p = f_1(t)$ and $y_p = f_2(t)$.
(b) Time required for particle to travel from $(x_0, y_0) = (\frac{1}{2}, 2)$ to $(x, y) = (1, 1)$ and $(2, \frac{1}{2})$.
(c) Compare acceleration determined from $f_1(t)$ and $f_2(t)$ with those found in Problem 5.53.

Solution: For the given flow, $u = Ax$ and $v = -Ay$. Thus

$$u_p = \frac{df_1}{dt} = Ax_p = Af_1 \;, \text{ or } \quad \frac{df_1}{f_1} = A\,dt$$

Integrating from x_0 to f_1,

$$\int_{x_0}^{f_1} \frac{df_1}{f_1} = \ln f_1 \Big]_{x_0}^{f_1} = \ln\left(\frac{f_1}{x_0}\right) = At \;, \text{ or } \; f_1 = x_0 e^{At}$$

$f_1(t)$

Likewise $\quad v_p = \frac{df_2}{dt} = -Ay_p = -Af_2$, or $\frac{df_2}{f_2} = -A\,dt$

Integrating from y_0 to f_2,

$$\int_{y_0}^{f_2} \frac{df_2}{f_2} = \ln f_2 \Big]_{y_0}^{f_2} = \ln\left(\frac{f_2}{y_0}\right) = -At \;\text{ or } f_2 = y_0 e^{-At}$$

$f_2(t)$

For a particle initially at $(\frac{1}{2}, 2)$, $x_0 = \frac{1}{2}$ and $y_0 = 2$

To reach the point $(x, y) = (1, 1)$, $e^{At} = \frac{x}{x_0} = 2$, so $t = \frac{\ln 2}{A} = 0.693$ sec

$$e^{-At} = \frac{y}{y_0} = \frac{1}{2}, \text{ so } t = \frac{-\ln\frac{1}{2}}{A} = 0.693 \text{ sec}$$

$t(1,1)$

To reach the point $(x,y) = (2, \frac{1}{2})$, $e^{At} = \frac{x}{x_0} = 4$, so $t = \frac{\ln 4}{A} = 1.39$ sec

$$e^{-At} = \frac{y}{y_0} = \frac{1}{4}, \text{ so } t = \frac{-\ln\frac{1}{4}}{A} = 1.39 \text{ sec}$$

$t(2,\frac{1}{2})$

The acceleration components are

$$a_{p_x} = \frac{d^2 f_1}{dt^2} = x_0 A^2 e^{At} = x_0 A^2 \frac{f_1}{x_0} = A^2 f_1$$

$$a_{p_y} = \frac{d^2 f_2}{dt^2} = y_0 A^2 e^{-At} = y_0 A^2 \frac{f_2}{y_0} = A^2 f_2$$

At $(x, y) = (1, 1)$

$$\vec{a}_p = a_{p_x}\hat{i} + a_{p_y}\hat{j} = \frac{(1)^2}{s^2} \times 1\,m\,\hat{i} + \frac{(1)^2}{s^2} \times 1\,m\,\hat{j} = (\hat{i} + \hat{j})\frac{m}{s^2}$$

$\vec{a}_p(1,1)$

At $(x, y) = (2, \frac{1}{2})$

$$\vec{a}_p = \frac{(1)^2}{s^2} \times 2m\,\hat{i} + \frac{(1)^2}{s^2} \times \frac{1}{2}m\,\hat{j} = (2\hat{i} + \tfrac{1}{2}\hat{j})\frac{m}{s^2}$$

$\vec{a}_p(2,\frac{1}{2})$

These are identical to the accelerations found in Problem 5.53.

Problem 5.65

Expand $(\vec{V}\cdot\nabla)\vec{V}$ in cylindrical coordinates to obtain the convective acceleration of a fluid particle. Verify the results given in Eqs. 5.12.

Recall $\partial \hat{e}_r / \partial\theta = \hat{e}_\theta$ and $\partial \hat{e}_\theta / \partial\theta = -\hat{e}_r$

Solution:

In cylindrical coordinates $\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z}$

$$\vec{V} = V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_z \hat{k}$$

$$(\vec{V}\cdot\nabla)\vec{V} = \left[V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}\right]\cdot\left[\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{k}\frac{\partial}{\partial z}\right](V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k})$$

$$= \left[V_r\frac{\partial}{\partial r} + \frac{V_\theta}{r}\frac{\partial}{\partial\theta} + V_z\frac{\partial}{\partial z}\right](V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k})$$

$$= V_r\frac{\partial}{\partial r}V_r\hat{e}_r + \frac{V_\theta}{r}\frac{\partial}{\partial\theta}V_r\hat{e}_r + V_z\frac{\partial}{\partial z}(V_r\hat{e}_r)$$

$$+ V_r\frac{\partial}{\partial r}V_\theta\hat{e}_\theta + \frac{V_\theta}{r}\frac{\partial}{\partial\theta}V_\theta\hat{e}_\theta + V_z\frac{\partial}{\partial z}V_\theta\hat{e}_\theta$$

$$+ V_r\frac{\partial}{\partial r}V_z\hat{k} + \frac{V_\theta}{r}\frac{\partial}{\partial\theta}V_z\hat{k} + V_z\frac{\partial}{\partial z}V_z\hat{k}$$

$$= \hat{e}_r\left\{V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial\theta} + V_z\frac{\partial V_r}{\partial z}\right\}^{①} + \frac{V_\theta V_r}{r}\left(\frac{\partial\hat{e}_r}{\partial\theta}\right) = \hat{e}_\theta$$

$$+ \hat{e}_\theta\left\{V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial\theta} + V_z\frac{\partial V_\theta}{\partial z}\right\}^{②} + \frac{V_\theta^2}{r}\left(\frac{\partial\hat{e}_\theta}{\partial\theta}\right) = -\hat{e}_r$$

$$+ \hat{k}\left\{V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial\theta} + V_z\frac{\partial V_z}{\partial z}\right\}^{③}$$

$$(\vec{V}\cdot\nabla)\vec{V} = \hat{e}_r\left\{V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial\theta} - \frac{V_\theta^2}{r} + V_z\frac{\partial V_r}{\partial z}\right\}$$

$$+ \hat{e}_\theta\left\{V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_\theta V_r}{r} + V_z\frac{\partial V_\theta}{\partial z}\right\}$$

$$+ \hat{k}\left\{V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial\theta} + V_z\frac{\partial V_z}{\partial z}\right\}$$

Term ① is the r component of convective acceleration

Eq. 5.12a $\quad a_{r_p} = \left\{V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial\theta} - \frac{V_\theta^2}{r} + V_z\frac{\partial V_r}{\partial z}\right\} + \frac{\partial V_r}{\partial t}$

Term ② is the θ component of convective acceleration

Eq. 5.12b $\quad a_{\theta_p} = \left\{V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_r V_\theta}{r} + V_z\frac{\partial V_\theta}{\partial z}\right\} + \frac{\partial V_\theta}{\partial t}$

Term ③ is the z component of convective acceleration

Eq. 5.12c $\quad a_{z_p} = \left\{V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial\theta} + V_z\frac{\partial V_z}{\partial z}\right\} + \frac{\partial V_z}{\partial t}$

Problem 5.66

5.66 Which, if any, of the flow fields of Problem 5.1 are irrotational?

Given: Velocity components

Find: Which flow fields are irrotational

Solution:

a. $u = 2x^2 + y^2 - x^2y;\ v = x^3 + x(y^2 - 2y)$
b. $u = 2xy - x^2 + y;\ v = 2xy - y^2 + x^2$
c. $u = xt + 2y;\ v = xt^2 - yt$
d. $u = (x + 2y)xt;\ v = -(2x + y)yt$

For a 2D field, the irrotionality the test is $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

(a) $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = \left[3\cdot x^2 + \left(y^2 - 2\cdot y\right)\right] - \left(2\cdot y - x^2\right) = 4\cdot x^2 + y^2 - 4\cdot y \neq 0$ Not irrotional

(b) $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = (2\cdot y + 2\cdot x) - (2\cdot y - 2\cdot x) = 4\cdot x \neq 0$ Not irrotional

(c) $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = \left(t^2\right) - (2) = t^2 - 2 \neq 0$ Not irrotional

(d) $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = (-2\cdot y\cdot t) - (2\cdot x\cdot t) = -2\cdot x\cdot t - 2\cdot y\cdot t \neq 0$ Not irrotional

Problem 5.67

5.67 A flow is represented by the velocity field $\vec{V} = (x^7 - 21x^5y^2 + 35x^3y^4 - 7xy^6)\hat{i} + (7x^6y - 35x^4y^3 + 21x^2y^5 - y^7)\hat{j}$. Determine if the field is (a) a possible incompressible flow and (b) irrotational.

Given: Flow field

Find: If the flow is incompressible and irrotational

Solution:

Basic equations: Incompressibility $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ Irrotationality $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

a) $u(x,y) = x^7 - 21\cdot x^5\cdot y^2 + 35\cdot x^3\cdot y^4 - 7\cdot x\cdot y^6$ $v(x,y) = 7\cdot x^6\cdot y - 35\cdot x^4\cdot y^3 + 21\cdot x^2\cdot y^5 - y^7$

$\frac{\partial}{\partial x}u(x,y) \rightarrow 7\cdot x^6 - 105\cdot x^4\cdot y^2 + 105\cdot x^2\cdot y^4 - 7\cdot y^6$ $\frac{\partial}{\partial y}v(x,y) \rightarrow 7\cdot x^6 - 105\cdot x^4\cdot y^2 + 105\cdot x^2\cdot y^4 - 7\cdot y^6$

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v \neq 0$ COMPRESSIBLE

b) $u(x,y) = x^7 - 21\cdot x^5\cdot y^2 + 35\cdot x^3\cdot y^4 - 7\cdot x\cdot y^6$ $v(x,y) = 7\cdot x^6\cdot y - 35\cdot x^4\cdot y^3 + 21\cdot x^2\cdot y^5 - y^7$

$\frac{\partial}{\partial x}v(x,y) \rightarrow 42\cdot x^5\cdot y - 140\cdot x^3\cdot y^3 + 42\cdot x\cdot y^5$ $-\frac{\partial}{\partial y}u(x,y) \rightarrow 42\cdot x^5\cdot y - 140\cdot x^3\cdot y^3 + 42\cdot x\cdot y^5$

Hence $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u \neq 0$ ROTATIONAL

Note that if we define $v(x,y) = -\left(7\cdot x^6\cdot y - 35\cdot x^4\cdot y^3 + 21\cdot x^2\cdot y^5 - y^7\right)$ then the flow is incompressible and irrotational!

Problem 5.68

Given: Sinusoidal approximation to boundary-layer velocity profile,

$$u = U\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) \quad \text{where } \delta = 5\text{ mm at } x = 0.5\text{ m (Problem 5.12)}$$

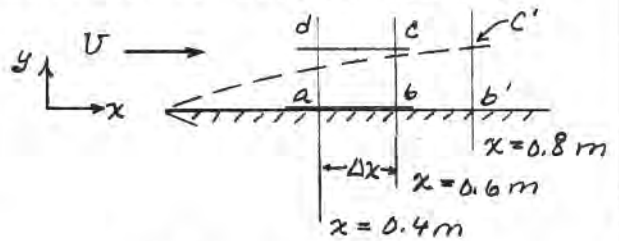
Neglect vertical component of velocity. $U = 0.5$ m/s.

Find: (a) Circulation about contour bounded by $x = 0.4$ m, $x = 0.6$ m, $y = 0$, and $y = 8$ mm.

(b) Result if evaluated $\Delta x = 0.2$ m further downstream?

Solution: Evaluate circulation

Defining equation:

$$\Gamma = \oint \vec{V}\cdot d\vec{s}$$

From the definition

$$\Gamma = \int_{ab} \vec{V}\cdot d\vec{s} + \int_{bc} \vec{V}\cdot d\vec{s} + \int_{cd} \vec{V}\cdot d\vec{s} + \int_{da} \vec{V}\cdot d\vec{s} = \int_0^{\Delta x} U\hat{i}\cdot dx(-\hat{i})$$

($u = 0$ on ab; $u \perp dy$ on bc and da)

$$\Gamma = -U\Delta x = -5\,\frac{m}{sec} \times 0.2\text{ m} = -0.100\text{ m}^2/\text{sec}$$

Γ

At the downstream location, since $\delta = cx^{1/2}$

$$\delta' = \delta\left(\frac{x}{x'}\right)^{1/2} = 5\text{ mm}\left(\frac{0.8}{0.5}\right)^{1/2} = 6.32\text{ mm}$$

Point c' is also outside the boundary layer. Consequently the integral along c'c will be the same as along cd. Thus

$$\Gamma_{bb'c'c} = \Gamma_{abcd}$$

Γ

Problem 5.69

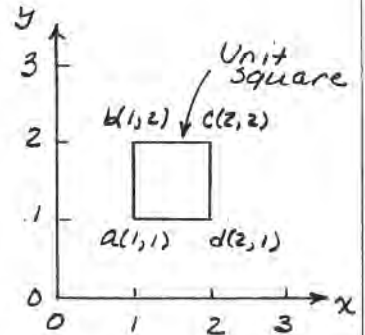
Given: Velocity field for flow in a rectangular "corner,"

$$\vec{V} = Ax\hat{i} - Ay\hat{j} \quad \text{with } A = 0.3\,s^{-1}$$

as in Example Problem 5.8.

Find: Circulation about unit square shown.

Solution: Evaluate circulation

Defining equation:
$$\Gamma = \oint \vec{V} \cdot d\vec{s}$$

The dot product is $\vec{V}\cdot d\vec{s} = (Ax\hat{i} - Ay\hat{j})\cdot(dx\hat{i} + dy\hat{j}) = Ax\,dx - Ay\,dy$.

For the contour shown, $dy = 0$ along ad and cb, and $dx = 0$ along ba and dc. Thus

$$\Gamma = \int_a^d Ax\,dx + \int_d^c -Ay\,dy + \int_c^b Ax\,dx + \int_b^a -Ay\,dy$$

$$= \frac{Ax^2}{2}\Big]_{x_a}^{x_d} - \frac{Ay^2}{2}\Big]_{y_d}^{y_c} + \frac{Ax^2}{2}\Big]_{x_c}^{x_b} - \frac{Ay^2}{2}\Big]_{y_b}^{y_a}$$

$$= \frac{A}{2}\left(x_d^2 - x_a^2 + x_b^2 - x_c^2\right) - \frac{A}{2}\left(y_c^2 - y_d^2 + y_a^2 - y_b^2\right)$$

$\Gamma = 0$ (since $x_a = x_b$ and $x_c = x_d$

$y_a = y_d$ and $y_b = y_c$) ← Γ

{ This result is to be expected, since flow is irrotational ($\nabla \times \vec{V} = 0$). From Stokes' Theorem (Eq. 5.18),

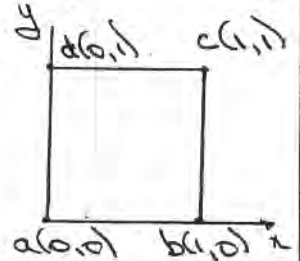
$$\Gamma = \int_A (\nabla \times \vec{V})_z\, dA = 0$$ }

Problem 5.70

Given: Two dimensional flow field $\vec{V} = Axy\hat{i} + By^2\hat{j}$, where $A = 1\,m^{-1}\cdot s^{-1}$, $B = -\frac{1}{2}\,m^{-1}\cdot s^{-1}$ and coordinates are measured in meters

Show: velocity field represents a possible incompressible flow

Find: (a) Rotation at point $(x,y) = (1,1)$
(b) Circulation about curve shown

(Sketch: square with corners a(0,0), b(1,0), c(1,1), d(0,1); axes x, y)

Solution:

For incompressible flow $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

For given flow field.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}(Axy) + \frac{\partial}{\partial y}(By^2) = Ay + 2By = (1)y + 2\left(-\frac{1}{2}\right)y = 0$$ ✓

The fluid rotation is defined as $\vec{\omega} = \frac{1}{2}\nabla\times\vec{V}$

$$\vec{\omega} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Axy & By^2 & 0 \end{vmatrix} = -\frac{1}{2}Ax\hat{k}$$

$$\vec{\omega}_{1,1} = -\frac{1}{2}\times\frac{1}{m\cdot s}\times 1m\,\hat{k} = -0.5\,\hat{k}\ \text{rad/s}$$ ($\vec{\omega}_{1,1}$)

The circulation is defined as $\Gamma = \oint \vec{V}\cdot d\vec{s}$

For the contour shown with $\vec{V} = Axy\hat{i} + By^2\hat{j}$

$$\Gamma = \int_a^b u\,dx + \int_b^c v\,dy + \int_c^d u(-dx) + \int_d^a v(-dy)$$

$u = 0$ along ab.

$$\Gamma = \int_0^1 By^2\,dy + \int_c^d Axy\,dx + \int_1^0 By^2\,dy \qquad \{y=1 \text{ along } cd\}$$

$$\Gamma = \frac{1}{3}By^3\Big]_0^1 + \frac{1}{2}Ax^2\Big]_1^0 + \frac{1}{3}By^3\Big]_1^0$$

$$\Gamma = \frac{1}{3}B - \frac{1}{2}A - \frac{1}{3}B = -\frac{1}{2}A = -\frac{1}{2}\,m^2/s$$ (Γ)

Problem *5.71

*5.71 Consider the flow field represented by the stream function $\psi = x^6 - 15x^4y^2 + 15x^2y^4 - y^6$. Is this a possible two-dimensional, incompressible flow? Is the flow irrotational?

Given: Stream function

Find: If the flow is incompressible and irrotational

Solution:

Basic equations: Incompressibility $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ Irrotationality $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

Note: The fact that ψ exists means the flow is incompressible, but we check anyway

$$\psi(x,y) = x^6 - 15\cdot x^4\cdot y^2 + 15\cdot x^2\cdot y^4 - y^6$$

Hence $u(x,y) = \frac{\partial}{\partial y}\psi(x,y) \rightarrow 60\cdot x^2\cdot y^3 - 30\cdot x^4\cdot y - 6\cdot y^5$ $v(x,y) = -\frac{\partial}{\partial x}\psi(x,y) \rightarrow 60\cdot x^3\cdot y^2 - 6\cdot x^5 - 30\cdot x\cdot y^4$

For incompressibility

$\frac{\partial}{\partial x}u(x,y) \rightarrow 120\cdot x\cdot y^3 - 120\cdot x^3\cdot y$ $\frac{\partial}{\partial y}v(x,y) \rightarrow 120\cdot x^3\cdot y - 120\cdot x\cdot y^3$

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ INCOMPRESSIBLE

For irrotationality

$\frac{\partial}{\partial x}v(x,y) \rightarrow 180\cdot x^2\cdot y^2 - 30\cdot x^4 - 30\cdot y^4$ $-\frac{\partial}{\partial y}u(x,y) \rightarrow 30\cdot x^4 - 180\cdot x^2\cdot y^2 + 30\cdot y^4$

Hence $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$ IRROTATIONAL

Problem *5.72

*5.72 Consider a flow field represented by the stream function $\psi = 3x^5y - 10x^3y^3 + 3xy^5$. Is this a possible two-dimensional incompressible flow? Is the flow irrotational?

Given: Stream function

Find: If the flow is incompressible and irrotational

Solution:

Basic equations: Incompressibility $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ Irrotationality $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

Note: The fact that ψ exists means the flow is incompressible, but we check anyway

$$\psi(x,y) = 3 \cdot x^5 \cdot y - 10 \cdot x^3 \cdot y^3 + 3 \cdot x \cdot y^5$$

Hence $u(x,y) = \frac{\partial}{\partial y}\psi(x,y) \rightarrow 3 \cdot x^5 - 30 \cdot x^3 \cdot y^2 + 15 \cdot x \cdot y^4$ $v(x,y) = -\frac{\partial}{\partial x}\psi(x,y) \rightarrow 30 \cdot x^2 \cdot y^3 - 15 \cdot x^4 \cdot y - 3 \cdot y^5$

For incompressibility

$\frac{\partial}{\partial x}u(x,y) \rightarrow 15 \cdot x^4 - 90 \cdot x^2 \cdot y^2 + 15 \cdot y^4$ $\frac{\partial}{\partial y}v(x,y) \rightarrow 90 \cdot x^2 \cdot y^2 - 15 \cdot x^4 - 15 \cdot y^4$

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ INCOMPRESSIBLE

For irrotationality

$\frac{\partial}{\partial x}v(x,y) \rightarrow 60 \cdot x \cdot y^3 - 60 \cdot x^3 \cdot y$ $-\frac{\partial}{\partial y}u(x,y) \rightarrow 60 \cdot x^3 \cdot y - 60 \cdot x \cdot y^3$

Hence $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$ IRROTATIONAL

Problem *5.73

[2]

*5.73 Consider a flow field represented by the stream function $\psi = -A/2(x^2 + y^2)$, where A = constant. Is this a possible two-dimensional incompressible flow? Is the flow irrotational?

Given: The stream function

Find: Whether or not the flow is incompressible; whether or not the flow is irrotational

Solution:

The stream function is

$$\psi = -\frac{A}{2\cdot\pi\left(x^2 + y^2\right)}$$

The velocity components are

$$u = \frac{d\psi}{dy} = \frac{A\cdot y}{\pi\left(x^2 + y^2\right)^2} \qquad v = -\frac{d\psi}{dx} = -\frac{A\cdot x}{\pi\left(x^2 + y^2\right)^2}$$

Because a stream function exists, the flow is: Incompressible

Alternatively, we can check with

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$$

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = -\frac{4\cdot A\cdot x\cdot y}{\pi\left(x^2 + y^2\right)^3} + \frac{4\cdot A\cdot x\cdot y}{\pi\left(x^2 + y^2\right)^3} = 0$$ Incompressible

For a 2D field, the irrotionality the test is

$$\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$$

$$\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = \frac{A\cdot\left(x^2 - 3\cdot y^2\right)}{\pi\cdot\left(x^2 + y^2\right)^3} - \frac{A\cdot\left(3\cdot x^2 - y^2\right)}{\pi\cdot\left(x^2 + y^2\right)^3} = -\frac{2\cdot A}{\pi\cdot\left(x^2 + y^2\right)^2} \neq 0$$ Not irrotational

Problem *5.74

Given: Velocity field for motion in x direction with constant shear. The shear rate is

$$\frac{\partial u}{\partial y} = A \quad \text{where } A = 0.1\ s^{-1}$$

Find: (a) Expression for $\vec{V}$
(b) Rate of rotation
(c) Stream function.

Solution: The velocity field is

$$\vec{V} = u\hat{i} = \left[\int \frac{\partial u}{\partial y}\,dy + f(x)\right]\hat{i} = \left[Ay + f(x)\right]\hat{i}$$

$\vec{V}$

Fluid rotation is given by

$$\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} = \frac{1}{2}\left(\cancelto{0}{\frac{\partial v}{\partial x}} - \frac{\partial u}{\partial y}\right)\hat{k} = -\frac{1}{2}\frac{\partial u}{\partial y}\hat{k} = -\frac{A}{2}\hat{k} = -0.05\ s^{-1}\ \hat{k}$$

$\vec{\omega}$

From the definition of the stream function,

$$u = \frac{\partial \psi}{\partial y} \quad \text{so} \quad \frac{\partial \psi}{\partial y} = Ay + f(x) \quad \text{and} \quad \psi = \frac{1}{2}Ay^2 + f(x)y + g(x)$$

$$v = -\frac{\partial \psi}{\partial x} = f'(x)y + g'(x) = 0$$

Thus $f'(x) = 0$ and $g'(x) = 0$, and

$$\psi = \frac{1}{2}Ay^2 + c$$

c

Problem *5.75

Given: Flow field represented by $\psi = x^2 - y^2$

Find: corresponding velocity field

Show: that flow field is irrotational

Plot: several streamlines and illustrate the velocity field

Solution:

Apply definition of ψ and irrotationality condition:

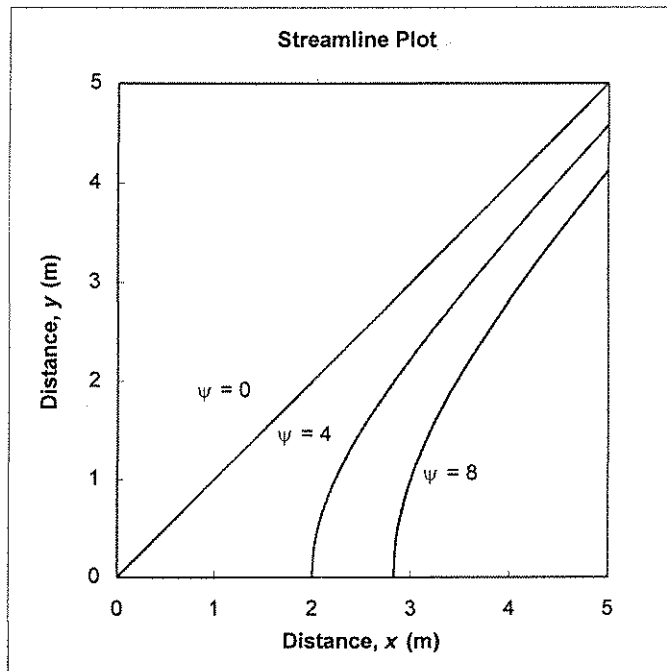
Computing equations: $u = \frac{\partial \psi}{\partial y}$ $\quad v = -\frac{\partial \psi}{\partial x}$

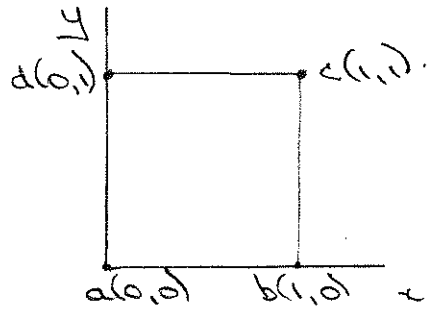
$\vec{\omega} = \frac{1}{2} \nabla \times \vec{V} = 0$

From the given $\psi = x^2 - y^2$

$$\left.\begin{aligned} u &= \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(x^2 - y^2) = -2y \\ v &= -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}(x^2 - y^2) = -2x \end{aligned}\right\} \vec{V} = u\hat{i} + v\hat{j} = -2y\hat{i} - 2x\hat{j} \quad \leftarrow \vec{V}$$

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2y & -2x & 0 \end{vmatrix} = \hat{k}(-2-(-2)) = 0$$

Since $\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} = 0$ flow is irrotational $\leftarrow \vec{\omega} = 0$

Given: Velocity field $\vec{V} = Axy\hat{i} + By^2\hat{j}$, where $A = 4\ \text{m}^{-1}\text{s}^{-1}$, $B = -2\ \text{m}^{-1}\text{s}^{-1}$ and coordinates are in meters.

d(0,1) c(1,1) a(0,0) b(1,0)

Find: (a) Fluid rotation
(b) Circulation about "curve" shown
(c) Stream function

Plot: several streamlines in first quadrant.

Solution:

(a) The fluid rotation is given by

$$\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Axy & By^2 & 0 \end{vmatrix} = \hat{k}\,\frac{1}{2}(-Ax) = -\frac{1}{2} \times \frac{4}{\text{m}\cdot\text{s}} x\,\hat{k} = -2x(\text{m})\,\hat{k}\ \frac{\text{rad}}{\text{s}}$$ $\vec{\omega}$

(b) The circulation is defined as $\Gamma = \oint \vec{V}\cdot d\vec{s}$

For the contour shown with $\vec{V} = Axy\hat{i} + By^2\hat{j}$

$$\Gamma = \int_a^b Axy\,dx \;(y=0) + \int_b^c By^2\,dy + \int_c^d Axy\,dx \;(y=1) + \int_d^a By^2\,dy$$

$$\Gamma = \int_0^1 By^2\,dy + \int_1^0 Ax\,dx + \int_1^0 By^2\,dy = B\frac{y^3}{3}\Big]_0^1 + A\frac{x^2}{2}\Big]_1^0 + B\frac{y^3}{3}\Big]_1^0$$

$$\Gamma = \frac{1}{3}B - \frac{1}{2}A - \frac{1}{3}B = -\frac{1}{2}A = -2\ \text{m}^2/\text{s}$$ Γ

(c) For incompressible flow $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = Ay + 2By = 4y + 2(-2)y = 0 \checkmark \quad \therefore \text{incompressible}$$

Thus $u = Axy = \frac{\partial \psi}{\partial y}$ and

$$\psi = \int Axy\,dy + f(x)$$

$$\psi = \frac{1}{2}Axy^2 + f(x)$$

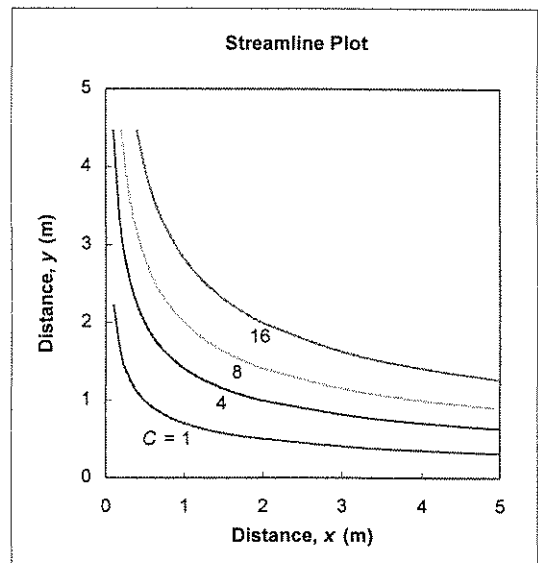
Then,

$$v = -\frac{\partial \psi}{\partial x} = -\frac{1}{2}Ay^2 - \frac{df}{dx} = By^2$$

$$\therefore \frac{df}{dx} = -\frac{1}{2}Ay^2 - By^2 = -2y^2 + 2y^2 = 0$$

Hence $f =$ constant.

Taking $f = 0$ gives

$$\psi = \frac{1}{2}Axy^2 = 2xy^2$$ ψ

Problem *5.77

Given: Flow field represented by $\psi = Axy + Ay^2$; $A = 1\,s^{-1}$

Find: (a) Show that this represents a possible incompressible flow field.
(b) Evaluate the rotation of the flow.
(c) Plot a few streamlines in the upper half plane.

Solution: For incompressible flow, $\nabla \cdot \vec{V} = 0$

The velocity field is determined from the stream function

$$\left.\begin{aligned} u &= \partial\psi/\partial y = Ax + 2Ay \\ v &= -\partial\psi/\partial x = -Ay \end{aligned}\right\} \quad \vec{V} = A\{(x+2y)\hat{i} - y\hat{j}\}$$

Then

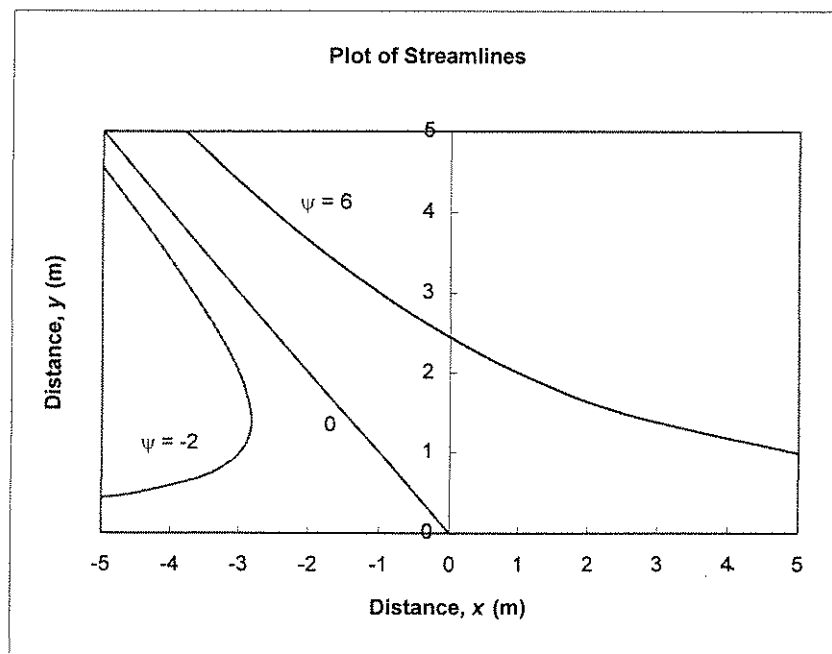
$$\nabla \cdot \vec{V} = \frac{\partial}{\partial x}A(x+2y) - \frac{\partial}{\partial y}(Ay) = A - A = 0 \quad \text{Q.E.D.}$$

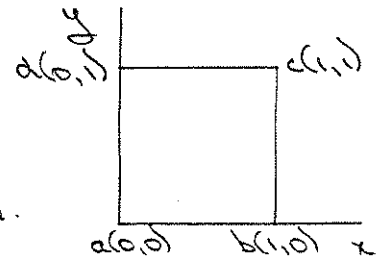
The rotation is given by $\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$

$$\vec{\omega} = \frac{1}{2}\left[\frac{\partial}{\partial x}(-Ay) - \frac{\partial}{\partial y}A(x+2y)\right]\hat{k} = \frac{1}{2}\left[0 - 2A\right]\hat{k} = -A\hat{k}$$

$$\vec{\omega} = -\hat{k}\ \text{rad/s} \qquad \vec{\omega}$$

To plot a few streamlines, $\psi = Axy + Ay^2$, note that for a given streamline

$$x = \frac{\psi}{y} - y.$$

Given: Velocity field, $\vec{V} = (Ay+B)\hat{i} + Ax\hat{j}$, where $A = 6\,s^{-1}$, $B = 3\,m\,s^{-1}$ and coordinates are in meters.

d(0,1) c(1,1) a(0,0) b(1,0) x y

Find: (a) An expression for the stream function.
(b) Circulation about "curve" shown.

Plot: several streamlines (including stagnation streamline) in the first quadrant.

<u>Solution</u>

For incompressible flow $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}(Ay+B) + \frac{\partial}{\partial y}(Ax) = 0+0 = 0 \quad \therefore \text{incompressible.}$$

Then

$$u = Ay+B = \frac{\partial \psi}{\partial y} \quad \text{and } \psi = \int (Ay+B)\,dy + f(x) = \frac{1}{2}Ay^2 + By + f(x)$$

and

$$v = -\frac{\partial \psi}{\partial x} = -\frac{df}{dx} = Ax \quad \text{and } f(x) = -\frac{1}{2}Ax^2 + \cancel{\text{constant}} \text{ set} = 0$$

$$\therefore \psi = \frac{1}{2}A(y^2 - x^2) + By \qquad \psi$$

Several streamlines are plotted below. The stagnation point (where $\vec{V} = 0$) is at $x = 0$, $y = -B/A = -0.5\,m$.

The circulation is defined as $\Gamma = \oint \vec{V} \cdot d\vec{s}$

For the contour shown with $\vec{V} = (Ay+B)\hat{i} + Ax\hat{j}$

$$\Gamma = \int_a^b u\,dx + \int_b^c v\,dy + \int_c^d u\,dx + \cancel{\int_d^a v\,dy} = 0$$

$$\Gamma = \int_0^1 B\,dx + \int_0^1 A\,dy + \int_1^0 (A+B)\,dx$$

$\left\{ \begin{array}{l} x = 1 \text{ from } b \text{ to } c \\ y = 1 \quad " \quad c \text{ to } d \end{array} \right\}$

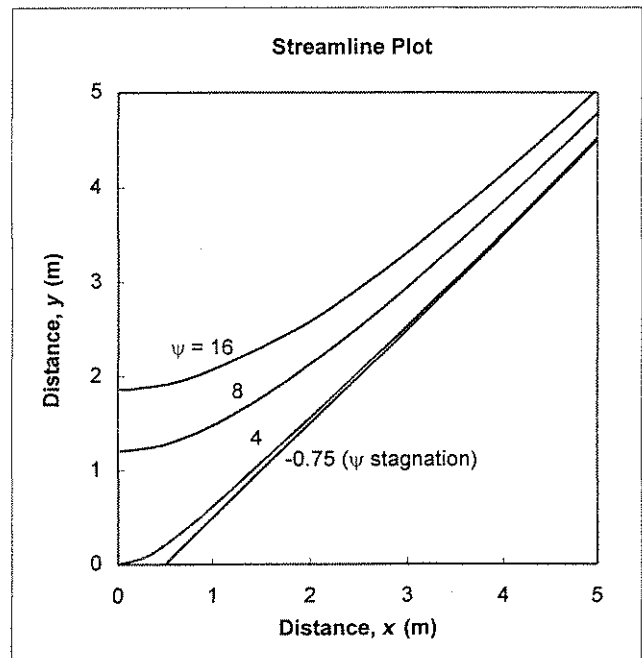
$$\Gamma = Bx\Big]_0^1 + Ay\Big]_0^1 + (A+B)\Big]_1^0$$

$$\Gamma = B + A - (A+B)$$

$$\Gamma = 0 \qquad \Gamma$$

Note: The flow is irrotational, i.e. $\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} = 0$ and hence we would expect $\Gamma = 0$

At stagnation, $\psi(x,y) = \psi(0,-0.5)$

$$\psi(x,y) = 3[(-0.5)^2 - 0] + 3(-0.5) = -3/4$$

Problem 5.79

Given: Viscometric flow of Example Problem 5.7, $\vec{V} = U(y/h)\hat{i}$, where $U = 4$ mm/s and $h = 4$ mm.

Find: (a) Average rate of rotation of two line segments at ± 45°
(b) Show that this is the same as in the Example.

Solution: Consider lines shown:

$$u_c = u_a + \frac{\partial u}{\partial y}(\ell \sin\theta_1)$$

$$-\omega_{ac} = \frac{(u_c - u_a)\sin\theta_1}{\ell} \quad \left\{\begin{matrix}\text{Component } \perp \\ \text{to } \ell \text{ is } u\sin\theta_1.\end{matrix}\right\}$$

$$-\omega_{ac} = \frac{\frac{\partial u}{\partial y}(\ell\sin\theta_1)\sin\theta_1}{\ell} = \frac{\partial u}{\partial y}\sin^2\theta_1 = \frac{U}{h}\sin^2\theta_1$$

Sketch showing θ_2:

$$u_b = u_d + \frac{\partial u}{\partial y}(\ell \sin\theta_2)$$

$$-\omega_{bd} = \frac{(u_b - u_d)\sin\theta_2}{\ell} \quad \left\{\begin{matrix}\text{Component } \perp \\ \text{to } \ell \text{ is } u\sin\theta_2.\end{matrix}\right\}$$

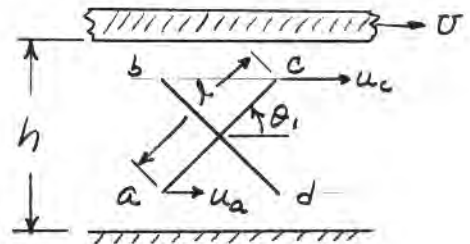
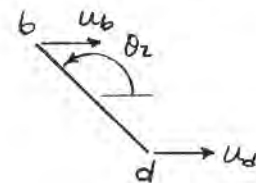
$$-\omega_{bd} = \frac{\frac{\partial u}{\partial y}(\ell\sin\theta_2)\sin\theta_2}{\ell} = \frac{\partial u}{\partial y}\sin^2\theta_2 = \frac{U}{h}\sin^2\theta_2$$

$$\omega\,(+\circlearrowleft) = \frac{1}{2}(\omega_{ac} + \omega_{bd}) = -\frac{1}{2}\frac{U}{h}(\sin^2\theta_1 + \sin^2\theta_2) = -\frac{1}{2}\frac{U}{h}(\sin^2 45° + \sin^2 135°)$$

$$= -\frac{1}{2}\frac{U}{h}\left[\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2\right] = -\frac{1}{2}\frac{U}{h} \qquad \omega$$

$$\omega = -\frac{1}{2} \times \frac{4\text{ mm}}{\text{sec}} \times \frac{1}{4\text{ mm}} = -0.5\ \text{s}^{-1} \qquad \omega$$

Problem *5.80

[3]

Given: Velocity field $\vec{V} = -\frac{q}{2\pi r}\hat{e}_r + \frac{K}{2\pi r}\hat{e}_\theta$ approximates a tornado.

Is it irrotational? Obtain the stream function.

Solution: Apply irrotationality condition.

Basic equation: $\nabla \times \vec{V} = 0$ (if irrotational)

It makes sense to work in cylindrical coordinates, where

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z}$$

But flow is in the $r\theta$ plane, so $\frac{\partial}{\partial z} = 0$. Then

$$\nabla \times \vec{V} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right) \times \left(V_r \hat{e}_r + V_\theta \hat{e}_\theta\right)$$

$$= \hat{e}_r \times \left(\frac{\partial V_r}{\partial r}\hat{e}_r + \frac{\partial V_\theta}{\partial r}\hat{e}_\theta\right)$$

$$+ \hat{e}_\theta \frac{1}{r} \times \left(\frac{\partial V_r}{\partial \theta}\hat{e}_r + V_r \frac{\partial \hat{e}_r}{\partial \theta} + \frac{\partial V_\theta}{\partial \theta}\hat{e}_\theta + V_\theta \frac{\partial \hat{e}_\theta}{\partial \theta}\right)$$

(with $\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta$, $\frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r$)

$$\nabla \times \vec{V} = \hat{k}\left(\frac{\partial V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{V_\theta}{r}\right) = \hat{k}\frac{1}{r}\left(\frac{\partial r V_\theta}{\partial r} - \frac{\partial V_r}{\partial \theta}\right)$$

For the given flow field, $\vec{V} = \vec{V}(r)$, so

$$\nabla \times \vec{V} = \hat{k}\,\frac{1}{r}\frac{\partial r V_\theta}{\partial r} = \hat{k}\,\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{K}{2\pi}\right) \equiv 0$$

Flow is irrotational. ←

$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$; $\frac{1}{r}\frac{\partial \psi}{\partial \theta} = -\frac{q}{2\pi r}$; $\frac{\partial \psi}{\partial \theta} = \frac{-q}{2\pi}$; $\psi = \frac{-q}{2\pi}\theta + f(r)$

$V_\theta = -\frac{\partial \psi}{\partial r}$; $\frac{\partial \psi}{\partial r} = -\frac{K}{2\pi r}$; $\psi = -\frac{K}{2\pi}\ln r + g(\theta)$

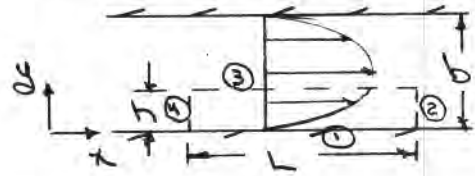
Comparing,

$$\psi = \frac{-q}{2\pi}\theta - \frac{K}{2\pi}\ln r$$

← ψ

Problem 5.81

Given: Flow between parallel plates. Velocity field given by

$$u = U\left(\frac{y}{b}\right)\left[1 - \frac{y}{b}\right]$$

Find: (a) expression for circulation about a closed contour of height h and length L.
(b) evaluate for $h = b/2$ and $h = b$.
(c) show that same result is obtained from area integral of Stokes Theorem (Eq. 5.18).

Solution:

Basic equations: $\Gamma = \oint \vec{V} \cdot d\vec{s} = \int_A (\nabla \times \vec{V})_z \, dA$

Then, $\Gamma = \int_1 \vec{V} \cdot d\vec{s} + \int_2 \vec{V} \cdot d\vec{s} + \int_3 \vec{V} \cdot d\vec{s} + \int_4 \vec{V} \cdot d\vec{s}$ (integrals 1, 2, 4 crossed out, = 0)

$$= \int_L^0 U \frac{y}{b}\left(1 - \frac{y}{b}\right) dx$$

$$\Gamma = -UL \frac{h}{b}\left(1 - \frac{h}{b}\right) \longleftarrow \Gamma$$

For $h = y = \frac{b}{2}$, $\Gamma = -\frac{UL}{4}$

$h = y = b$, $\Gamma = 0$

From Stokes Theorem,

$$\Gamma = \int_A (\vec{\nabla} \times \vec{V})_z \, dA = \int_A \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA = \int_A -U\left(\frac{1}{b} - \frac{2y}{b^2}\right) dA$$

$$\Gamma = -U \int \left(\frac{1}{b} - \frac{2y}{b^2}\right) L \, dy = -UL\left[\frac{y}{b} - \frac{y^2}{b^2}\right]_0^h$$

$$\Gamma = -UL\left[\frac{h}{b} - \frac{h^2}{b^2}\right] = -UL \frac{h}{b}\left(1 - \frac{h}{b}\right) \longleftarrow \Gamma$$

Given: Velocity profile for fully developed flow in a circular tube is

$$V_z = V_{max}\left[1-\left(r/R\right)^2\right]$$

Find: (a) rates of linear and angular deformation for this flow.
(b) expression for the vorticity vector, $\vec{\zeta}$

Solution:

Computing equations: B.1 and B.2 of Appendix B

Volume dilation rate $= \nabla\cdot\vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{\partial}{\partial z}V_z = 0$

Rates of linear deformation in each of the three coordinate directions r, θ, z are zero ← Linear Def

Angular deformation in the:

$r\theta$ plane is $r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) + \frac{1}{r}\frac{\partial V_r}{\partial\theta} = 0$

θz plane is $\frac{\partial V_\theta}{\partial z} + \frac{1}{r}\frac{\partial V_z}{\partial\theta} = 0$

zr plane is $\frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r} = -V_{max}\frac{2r}{R^2}$ ← Angular Def.

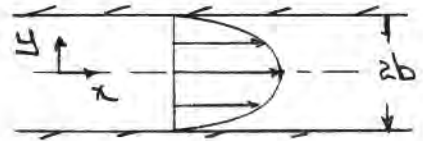
The vorticity vector is given by $\vec{\zeta} = \nabla\times\vec{V}$

In cylindrical coordinates,

$$\nabla\times\vec{V} = \hat{e}_r\left(\frac{1}{r}\frac{\partial V_z}{\partial\theta} - \frac{\partial V_\theta}{\partial z}\right) + \hat{e}_\theta\left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right) + \hat{k}\left(\frac{1}{r}\frac{\partial rV_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial\theta}\right) \quad (5.16)$$

$$\vec{\zeta} = \nabla\times\vec{V} = \hat{e}_\theta V_{max}\frac{2r}{R^2}$$ ← $\vec{\zeta}$

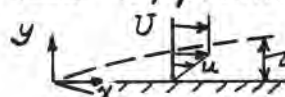
Given: Flow between parallel plates. Velocity field given by

$$u = U_{max}\left[1 - \left(\frac{y}{b}\right)^2\right]$$

Find: (a) rates of linear and angular deformation
(b) expression for the vorticity vector, $\vec{\zeta}$
(c) location of maximum vorticity

Solution:

The rate of linear deformation is zero since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = 0$

The rate of angular deformation in the xy plane is

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = -\frac{2y\,U_{max}}{b^2}$$

The vorticity vector is given by $\vec{\zeta} = \nabla \times \vec{V}$

$$\vec{\zeta} = \hat{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\vec{\zeta} = -\frac{\partial u}{\partial y}\hat{k} = \frac{2y\,U_{max}}{b^2}\hat{k}$$ $\vec{\zeta}$

The vorticity is a maximum at $y = \pm b$

Problem 5.84

Given: Linear approximate velocity profile in boundary layer.

$\delta = cx^{1/2}$; $u = U\frac{y}{\delta} = \frac{Uy}{cx^{1/2}}$; $v = \frac{uy}{4x} = \frac{Uy^2}{4cx^{3/2}}$

Find: (a) Express rotation, find maximum.
(b) Express angular deformation, locate maximum.
(c) Express linear deformation, locate maximum.
(d) Express shear force per unit volume, locate maximum.

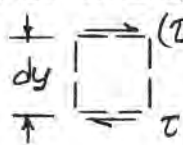
Solution: Work in xy plane.

Computing equations: $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ $\quad -\frac{d\gamma}{dt} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$

Linear def: $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}$

Evaluating partial derivatives,

$$\frac{\partial u}{\partial x} = -\frac{1}{2}\frac{Uy}{cx^{3/2}} \quad \frac{\partial u}{\partial y} = \frac{U}{cx^{1/2}} \quad \frac{\partial v}{\partial x} = -\frac{3}{8}\frac{Uy^2}{cx^{5/2}} \quad \frac{\partial v}{\partial y} = \frac{1}{2}\frac{Uy}{cx^{3/2}}$$

Then

$$\omega_z = \frac{1}{2}\left[-\frac{3}{8}\frac{Uy^2}{cx^{5/2}} - \frac{U}{cx^{1/2}}\right] = -\frac{U}{2cx^{1/2}}\left[1 + \frac{3}{8}\left(\frac{y}{x}\right)^2\right] \quad (\text{max at } y = \delta)$$

ω_z

$$-\frac{d\gamma}{dt} = -\frac{3}{8}\frac{Uy^2}{cx^{5/2}} + \frac{U}{cx^{1/2}} = \frac{U}{cx^{1/2}}\left[1 - \frac{3}{8}\left(\frac{y}{x}\right)^2\right] \quad (\text{max at } y = 0)$$

$-\frac{d\gamma}{dt}$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}\frac{Uy}{cx^{3/2}} = -\frac{U}{2cx^{1/2}}\left(\frac{y}{x}\right) \quad (\text{max at } y = \delta)$$

$$\frac{\partial v}{\partial y} = +\frac{1}{2}\frac{Uy}{cx^{3/2}} = +\frac{U}{2cx^{1/2}}\left(\frac{y}{x}\right) \quad (\text{max at } y = \delta)$$

sum = 0

Lin Def

Shear stress is $\tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \mu\left(\frac{U}{cx^{1/2}} - \frac{3}{8}\frac{Uy^2}{cx^{5/2}}\right) = \frac{\mu U}{cx^{1/2}}\left[1 - \frac{3}{8}\left(\frac{y}{x}\right)^2\right]$

Net shear force on a fluid element is $d\tau\, dx\, dz$

$(\tau + d\tau)\, dx\, dz$; dy ; $\tau\, dx\, dz$

$$d\tau = \frac{\partial \tau}{\partial y}dy = \frac{\mu U}{cx^{1/2}}\left(-\frac{3}{8}\frac{2y}{x^2}\right)dy = -\frac{3\mu Uy}{4cx^{5/2}}dy$$

Shear stress per volume is $\frac{dF}{dV} = -\frac{3\mu U}{4cx^{3/2}}\left(\frac{y}{x}\right)$ (max at $y = \delta$)

$\frac{dF}{dV}$

Problem 5.85

Given: x component of velocity in laminar boundary layer in water

$$u = U\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) \qquad U = 3\ m/s, \quad \delta = 2\ mm$$

y component is much smaller than u.

Find: (a) Expression for net shear force per unit volume in x direction.
(b) Maximum value for this flow

Solution: Consider a small element of fluid

$(\tau + d\tau)\,dx\,dz$

$\tau\,dx\,dz$

Then

$$dF_{shear,x} = (\tau + d\tau)dx\,dz - \tau\,dx\,dz$$

$$= d\tau\,dx\,dz = \frac{d\tau}{dy}dx\,dy\,dz$$

and

$$\frac{dF_{s,x}}{d\forall} = \frac{d\tau}{dy} = \frac{d}{dy}\left(\mu\frac{du}{dy}\right) = \mu\frac{d^2u}{dy^2}$$

From the given profile,

$$\frac{du}{dy} = \frac{\pi U}{2\delta}\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$

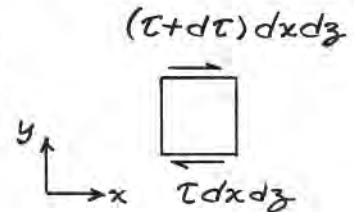
and

$$\frac{d^2u}{dy^2} = U\left(\frac{\pi}{2\delta}\right)^2\left(-\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right)$$

The maximum value occurs when $y = \delta$, when

$$\frac{dF_{sx,max}}{d\forall} = -\mu U\left(\frac{\pi}{2\delta}\right)^2 \longleftarrow \frac{dF_{sx}}{d\forall}$$

$$= -1\times10^{-3}\frac{N\cdot sec}{m^2}\times 3\frac{m}{sec}\left(\frac{\pi}{2}\frac{1}{0.002\ m}\right)^2 = -1.85\times10^{3}\ N/m^3$$

$$\frac{dF_{sx,max}}{d\forall} = -1.85\ kN/m^3 \longleftarrow \frac{dF_{sx}}{d\forall}$$

Given: Velocity profile for fully developed laminar flow in a tube

$$\frac{u}{u_{max}} = 1 - \left(\frac{r}{R}\right)^2$$

where $u_{max} = 10$ ft/s, $R = 3$ in., fluid is water.

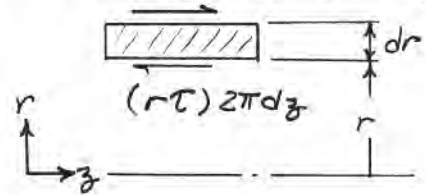
Find: (a) Expression for shear force per unit volume in z direction.
(b) Maximum value for these conditions.

Solution: Consider a differential element: $\left[r\tau + \frac{d}{dr}(r\tau)dr\right]2\pi dz$

Then

$$dF_{shear,z} = \left[r\tau + \frac{d}{dr}(r\tau)dr\right]2\pi dz - r\tau\, 2\pi dz$$

$$= \frac{d}{dr}(r\tau)\, 2\pi\, dr\, dz$$

$(r\tau)\,2\pi dz$ dr r z

Since $d\forall = 2\pi r\,dr\,dz$, then

$$\frac{dF_{sz}}{d\forall} = \frac{1}{2\pi r\,dr\,dz}\frac{d}{dr}(r\tau)\,2\pi\,dr\,dz = \frac{1}{r}\frac{d}{dr}(r\tau)$$

In cylindrical coordinates, $\tau_{rz} = \mu\frac{du}{dr}$. For the given profile

$$\tau = \tau_{rz} = \mu\frac{du}{dr} = -\mu u_{max}\frac{2r}{R^2}$$

Substituting

$$\frac{dF_{sz}}{d\forall} = \frac{1}{r}\frac{d}{dr}\left[r\left(-\frac{2\mu u_{max} r}{R^2}\right)\right] = \frac{1}{r}\frac{d}{dr}\left[-\frac{2\mu u_{max} r^2}{R^2}\right] = \frac{1}{r}\left[\frac{-4\mu u_{max} r}{R^2}\right]$$

$$\frac{dF_{sz}}{d\forall} = -\frac{4\mu u_{max}}{R^2} = \text{constant}$$

$\frac{dF_{sz}}{d\forall}$

Evaluating,

$$\frac{dF_{sz}}{d\forall} = -4\times10^{-3}\frac{N\cdot s}{m^2}\times 10\frac{ft}{s}\times\frac{1}{(0.25)^2 ft^2}\times(0.3048)^2\frac{m^2}{ft^2}\times\frac{lbf}{4.448\,N}$$

$$\frac{dF_{sz}}{d\forall} = -0.0134\ lbf/ft^3$$

Problem 5.87

5.87 Use *Excel* to generate the solution of Eq. 5.28 for $m = 1$ shown in Fig. 5.16. To do so, you need to learn how to perform linear algebra in Excel. For example, for $N = 4$ you will end up with the matrix equation of Eq. 5.34. To solve this equation for the u values, you will have to compute the inverse of the 4×4 matrix, and then multiply this inverse into the 4×1 matrix on the right of the equation. In *Excel*, to do *array operations*, you must use the following rules: Pre-select the cells that will contain the result; use the appropriate *Excel array function* (look at *Excel's* Help for details); press Ctrl+Shift+Enter, not just Enter. For example, to invert the 4×4 matrix you would: Pre-select a blank 4×4 array that will contain the inverse matrix; type = *minverse*([array containing matrix to be inverted]); press Ctrl+Shift+Enter. To multiply a 4×4 matrix into a 4×1 matrix you would: Pre-select a blank 4×1 array that will contain the result; type = *mmult*([array containing 4×4 matrix], [array containing 4×1 matrix]); press Ctrl+Shift+Enter.

$$\frac{du}{dx} + u^m = 0; \qquad 0 \le x \le 1; \qquad u(0) = 1$$

N = **4**

Δx = **0.333**

	Eq. 5.34 (LHS)				**(RHS)**
	1.000	0.000	0.000	0.000	1
	-1.000	1.333	0.000	0.000	0
	0.000	-1.000	1.333	0.000	0
	0.000	0.000	-1.000	1.333	0

x	**Inverse Matrix**				**Result**	**Exact**	**Error**
0.000	1.000	0.000	0.000	0.000	1.000	1.000	0.000
0.333	0.750	0.750	0.000	0.000	0.750	0.717	0.000
0.667	0.563	0.563	0.750	0.000	0.563	0.513	0.001
1.000	0.422	0.422	0.563	0.750	0.422	0.368	0.001
							0.040

N = **8**

Δx = **0.143**

	Eq. 5.34 (LHS)								**(RHS)**
	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1
	-1.000	1.143	0.000	0.000	0.000	0.000	0.000	0.000	0
	0.000	-1.000	1.143	0.000	0.000	0.000	0.000	0.000	0
	0.000	0.000	-1.000	1.143	0.000	0.000	0.000	0.000	0
	0.000	0.000	0.000	-1.000	1.143	0.000	0.000	0.000	0
	0.000	0.000	0.000	0.000	-1.000	1.143	0.000	0.000	0
	0.000	0.000	0.000	0.000	0.000	-1.000	1.143	0.000	0
	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.143	0

	Inverse Matrix										
x	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**Result**	**Exact**	**Error**
0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000	1.000	0.000
0.143	0.875	0.875	0.000	0.000	0.000	0.000	0.000	0.000	0.875	0.867	0.000
0.286	0.766	0.766	0.875	0.000	0.000	0.000	0.000	0.000	0.766	0.751	0.000
0.429	0.670	0.670	0.766	0.875	0.000	0.000	0.000	0.000	0.670	0.651	0.000
0.571	0.586	0.586	0.670	0.766	0.875	0.000	0.000	0.000	0.586	0.565	0.000
0.714	0.513	0.513	0.586	0.670	0.766	0.875	0.000	0.000	0.513	0.490	0.000
0.857	0.449	0.449	0.513	0.586	0.670	0.766	0.875	0.000	0.449	0.424	0.000
1.000	0.393	0.393	0.449	0.513	0.586	0.670	0.766	0.875	0.393	0.368	0.000
											0.019

N = **16**

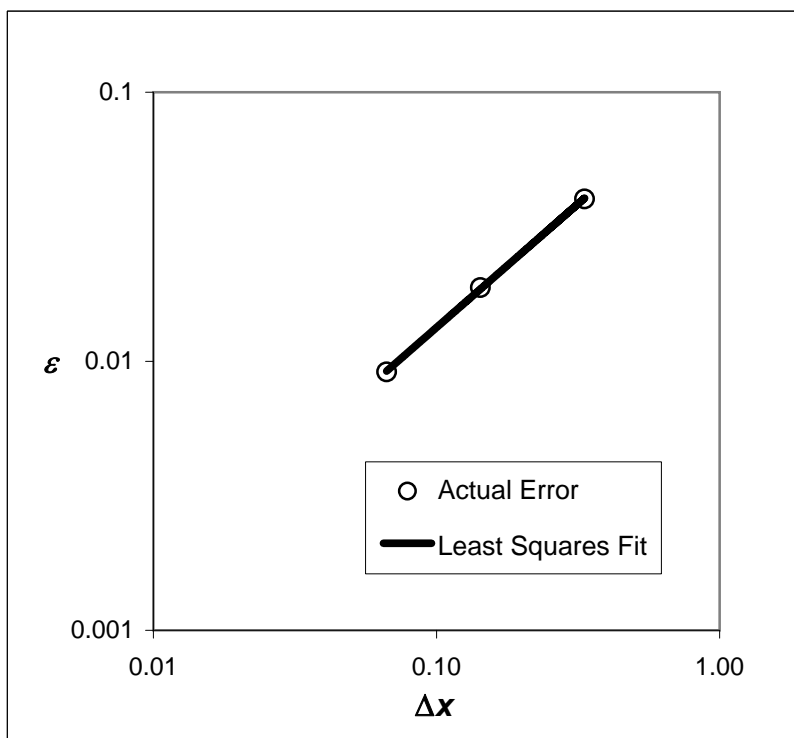
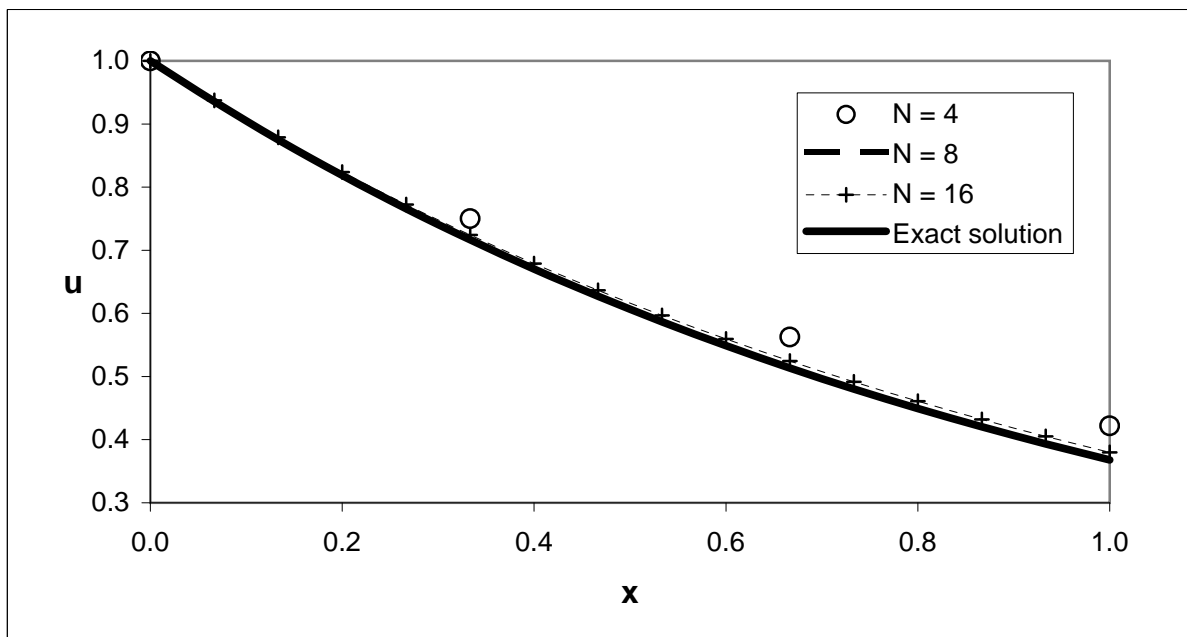
Δx = **0.067** **Eq. 5.34 (LHS)**

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	(RHS)
1	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1
2	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
3	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
4	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
5	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
6	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
7	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
8	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
9	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0
11	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0
12	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0
13	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0
14	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0
16	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0

x	Inverse Matrix																Result	Exact	Error
0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000	1.000	0.000
0.067	0.938	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.938	0.936	0.000
0.133	0.879	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.879	0.875	0.000
0.200	0.824	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.824	0.819	0.000
0.267	0.772	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.772	0.766	0.000
0.333	0.724	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.724	0.717	0.000
0.400	0.679	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.679	0.670	0.000
0.467	0.637	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.637	0.627	0.000
0.533	0.597	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.597	0.587	0.000
0.600	0.559	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.559	0.549	0.000
0.667	0.524	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.524	0.513	0.000
0.733	0.492	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.492	0.480	0.000
0.800	0.461	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.461	0.449	0.000
0.867	0.432	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.432	0.420	0.000
0.933	0.405	0.405	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.405	0.393	0.000
1.000	0.380	0.380	0.405	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.380	0.368	0.000
																			0.009

N	Δx	Error
4	0.333	0.040
8	0.143	0.019
16	0.067	0.009

1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
u
0.0
0.2
0.4
0.6
0.8
1.0
x
N = 4
N = 8
N = 16
Exact solution

0.1
0.01
0.001
ε
0.01
0.10
1.00
Δx
Actual Error
Least Squares Fit

Problem 5.88

5.88 Following the steps to convert the differential equation Eq. 5.28 (for $m = 1$) into a difference equation (for example, Eq. 5.34 for $N = 4$), solve

$$\frac{du}{dx} + u = 2\sin(x) \qquad 0 \le x \le 1 \qquad u(0) = 0$$

for $N = 4$, 8, and 16 and compare to the exact solution

$$u_{\text{exact}} = \sin(x) - \cos(x) + e^{-x}$$

Hints: Follow the rules for *Excel* array operations as described in Problem 5.87. Only the right side of the difference equations will change, compared to the solution method of Eq. 5.28 (for example, only the right side of Eq. 5.34 needs modifying).

New Eq. 5.34: $$-u_{i-1} + (1+\Delta x)u_i = 2\Delta x \cdot \sin(x_i)$$

N = **4**

Δx = **0.333**

	Eq. 5.34 (LHS)				**(RHS)**
	1.000	0.000	0.000	0.000	0
	-1.000	1.333	0.000	0.000	0.21813
	0.000	-1.000	1.333	0.000	0.41225
	0.000	0.000	-1.000	1.333	0.56098

x	**Inverse Matrix**				**Result**	**Exact**	**Error**
0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000
0.333	0.750	0.750	0.000	0.000	0.164	0.099	0.001
0.667	0.563	0.563	0.750	0.000	0.432	0.346	0.002
1.000	0.422	0.422	0.563	0.750	0.745	0.669	0.001
							0.066

N = **8**

Δx = **0.143**

	Eq. 5.34 (LHS)								**(RHS)**
	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
	-1.000	1.143	0.000	0.000	0.000	0.000	0.000	0.000	0.04068
	0.000	-1.000	1.143	0.000	0.000	0.000	0.000	0.000	0.08053
	0.000	0.000	-1.000	1.143	0.000	0.000	0.000	0.000	0.11873
	0.000	0.000	0.000	-1.000	1.143	0.000	0.000	0.000	0.15452
	0.000	0.000	0.000	0.000	-1.000	1.143	0.000	0.000	0.18717
	0.000	0.000	0.000	0.000	0.000	-1.000	1.143	0.000	0.21599
	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.143	0.24042

	Inverse Matrix										
x	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**Result**	**Exact**	**Error**
0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.143	0.875	0.875	0.000	0.000	0.000	0.000	0.000	0.000	0.036	0.019	0.000
0.286	0.766	0.766	0.875	0.000	0.000	0.000	0.000	0.000	0.102	0.074	0.000
0.429	0.670	0.670	0.766	0.875	0.000	0.000	0.000	0.000	0.193	0.157	0.000
0.571	0.586	0.586	0.670	0.766	0.875	0.000	0.000	0.000	0.304	0.264	0.000
0.714	0.513	0.513	0.586	0.670	0.766	0.875	0.000	0.000	0.430	0.389	0.000
0.857	0.449	0.449	0.513	0.586	0.670	0.766	0.875	0.000	0.565	0.526	0.000
1.000	0.393	0.393	0.449	0.513	0.586	0.670	0.766	0.875	0.705	0.669	0.000
											0.032

N = **16**

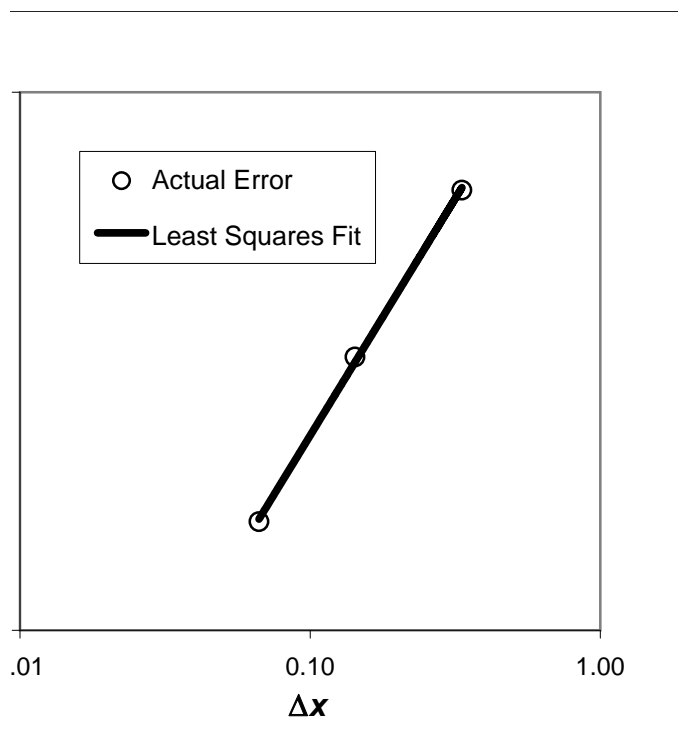
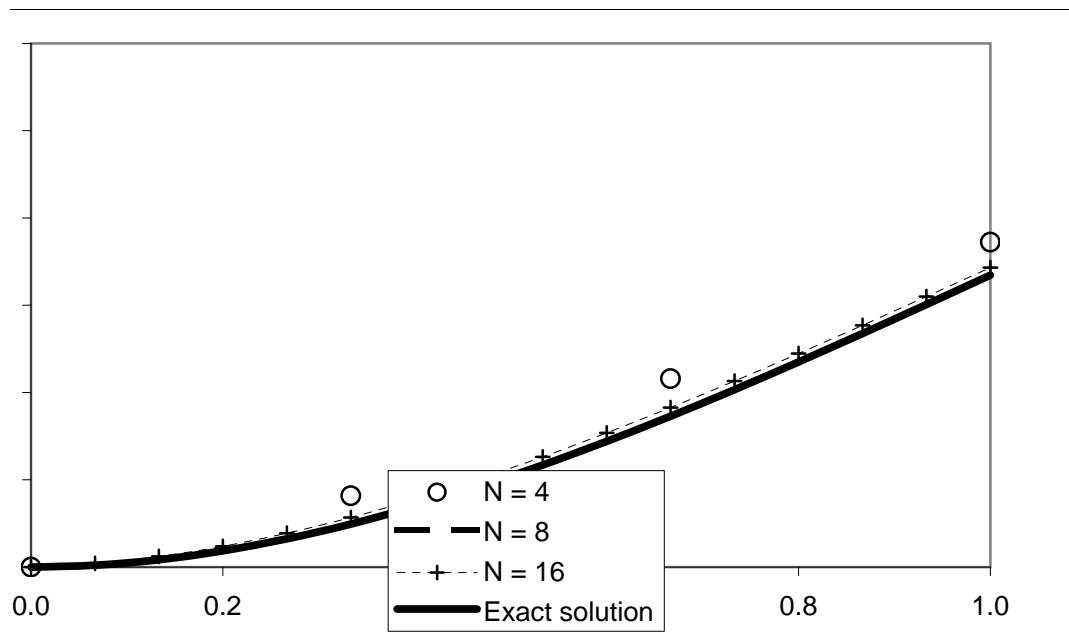
Δx = **0.067** **Eq. 5.34 (LHS)**

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**(RHS)**
1	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
2	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.00888
3	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.01773
4	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.02649
5	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.03514
6	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.04363
7	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.05192
8	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.05999
9	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.06779
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.07529
11	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.08245
12	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.08925
13	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.09565
14	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.10162
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.10715
16	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.1122

x	**Inverse Matrix**																**Result**	**Exact**	**Error**
0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.067	0.938	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.008	0.004	0.000
0.133	0.879	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.024	0.017	0.000
0.200	0.824	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.048	0.037	0.000
0.267	0.772	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.078	0.065	0.000
0.333	0.724	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.114	0.099	0.000
0.400	0.679	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.155	0.139	0.000
0.467	0.637	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.202	0.184	0.000
0.533	0.597	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.253	0.234	0.000
0.600	0.559	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.308	0.288	0.000
0.667	0.524	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.366	0.346	0.000
0.733	0.492	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.426	0.407	0.000
0.800	0.461	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.489	0.470	0.000
0.867	0.432	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.554	0.535	0.000
0.933	0.405	0.405	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.620	0.602	0.000
1.000	0.380	0.380	0.405	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.686	0.669	0.000
																			0.016

N	**Δx**	**Error**
4	0.333	0.066
8	0.143	0.032
16	0.067	0.016

N = 4
N = 8
N = 16
Exact solution
0.0
0.2
0.8
1.0

Actual Error
Least Squares Fit
.01
0.10
1.00
Δx

Problem 5.89

5.89 Following the steps to convert the differential equation Eq. 5.28 (for $m = 1$) into a difference equation (for example, Eq. 5.34 for $N = 4$), solve

$$\frac{du}{dx} + u = x^2 \qquad 0 \le x \le 1 \qquad u(0) = 2$$

For $N = 4$, 8, and 16 and compare to the extract solution

$$u_{\text{exact}} = x^2 - 2x + 2$$

Hint: Follow the hints provided in Problem 5.88.

New Eq. 5.34: $-u_{i-1} + (1+\Delta x)u_i = \Delta x \cdot x_i^2$

N = 4
Δx = 0.333

Eq. 5.34 (LHS)				(RHS)
1.000	0.000	0.000	0.000	2
-1.000	1.333	0.000	0.000	0.03704
0.000	-1.000	1.333	0.000	0.14815
0.000	0.000	-1.000	1.333	0.33333

x	**Inverse Matrix**				**Result**	**Exact**	**Error**
0.000	1.000	0.000	0.000	0.000	2.000	2.000	0.000
0.333	0.750	0.750	0.000	0.000	1.528	1.444	0.002
0.667	0.563	0.563	0.750	0.000	1.257	1.111	0.005
1.000	0.422	0.422	0.563	0.750	1.193	1.000	0.009
							0.128

N = 8
Δx = 0.143

Eq. 5.34 (LHS)								(RHS)
1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	2
-1.000	1.143	0.000	0.000	0.000	0.000	0.000	0.000	0.00292
0.000	-1.000	1.143	0.000	0.000	0.000	0.000	0.000	0.01166
0.000	0.000	-1.000	1.143	0.000	0.000	0.000	0.000	0.02624
0.000	0.000	0.000	-1.000	1.143	0.000	0.000	0.000	0.04665
0.000	0.000	0.000	0.000	-1.000	1.143	0.000	0.000	0.07289
0.000	0.000	0.000	0.000	0.000	-1.000	1.143	0.000	0.10496
0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.143	0.14286

Inverse Matrix

x	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**Result**	**Exact**	**Error**
0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	2.000	2.000	0.000
0.143	0.875	0.875	0.000	0.000	0.000	0.000	0.000	0.000	1.753	1.735	0.000
0.286	0.766	0.766	0.875	0.000	0.000	0.000	0.000	0.000	1.544	1.510	0.000
0.429	0.670	0.670	0.766	0.875	0.000	0.000	0.000	0.000	1.374	1.327	0.000
0.571	0.586	0.586	0.670	0.766	0.875	0.000	0.000	0.000	1.243	1.184	0.000
0.714	0.513	0.513	0.586	0.670	0.766	0.875	0.000	0.000	1.151	1.082	0.001
0.857	0.449	0.449	0.513	0.586	0.670	0.766	0.875	0.000	1.099	1.020	0.001
1.000	0.393	0.393	0.449	0.513	0.586	0.670	0.766	0.875	1.087	1.000	0.001
											0.057

N = **16**

Δx = **0.067** **Eq. 5.34 (LHS)**

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**(RHS)**
1	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	2
2	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.0003
3	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.00119
4	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.00267
5	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.00474
6	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.00741
7	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.01067
8	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.01452
9	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.01896
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.024
11	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.02963
12	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.03585
13	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.04267
14	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.05007
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.05807
16	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.06667

x	**Inverse Matrix**																**Result**	**Exact**	**Error**
0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	2.000	2.000	0.000
0.067	0.938	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.875	1.871	0.000
0.133	0.879	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.759	1.751	0.000
0.200	0.824	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.652	1.640	0.000
0.267	0.772	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.553	1.538	0.000
0.333	0.724	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.463	1.444	0.000
0.400	0.679	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.381	1.360	0.000
0.467	0.637	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.309	1.284	0.000
0.533	0.597	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.245	1.218	0.000
0.600	0.559	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	1.189	1.160	0.000
0.667	0.524	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	1.143	1.111	0.000
0.733	0.492	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	1.105	1.071	0.000
0.800	0.461	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	1.076	1.040	0.000
0.867	0.432	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	1.056	1.018	0.000
0.933	0.405	0.405	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	1.044	1.004	0.000
1.000	0.380	0.380	0.405	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	1.041	1.000	0.000
																			0.027

N	Δx	**Error**
4	0.333	0.128
8	0.143	0.057
16	0.067	0.027

N = 4
N = 8
N = 16
Exact solution
0.0
0.2
0.4
x
1.0

Actual Error
Least Squares Fit
.01
0.10
1.00
Δx

Problem 5.90

5.90 A 10-cm cube of mass $M = 5$ kg is sliding across an oiled surface. The oil viscosity is $\mu = 0.4\ \text{N} \cdot \text{s/m}^2$, and the thickness of the oil between the cube and surface is $\delta = 0.25$ mm. If the initial speed of the block is $u_0 = 1$ m/s, use the numerical method that was applied to the linear form of Eq. 5.28 to predict the cube motion for the first second of motion, Use $N = 4$, 8, and 16 and compare to the exact solution

$$u_{\text{exact}} = u_0 e^{-(A\mu/M\delta)t}$$

where A is the area of contact. Hint: Follow the hints provided in Problem 5.87.

Equation of motion:
$$M\frac{du}{dt} = -\mu\frac{du}{dy}A = -\mu A\frac{u}{\delta}$$

$$\frac{du}{dt} + \left(\frac{\mu A}{M\,\delta}\right)u = 0$$

$$\frac{du}{dt} + k \cdot u = 0$$

New Eq. 5.34:
$$-u_{i-1} + (1 + k \cdot \Delta x)u_i = 0$$

A =	0.01	m^2
δ =	0.25	mm
μ =	0.4	N.s/m^2
M =	5	kg
k =	3.2	s^{-1}

N = **4**

Δt = **0.333**

Eq. 5.34 (LHS)				(RHS)
1.000	0.000	0.000	0.000	1
-1.000	2.067	0.000	0.000	0
0.000	-1.000	2.067	0.000	0
0.000	0.000	-1.000	2.067	0

t	Inverse Matrix				Result	Exact	Error
0.000	1.000	0.000	0.000	0.000	1.000	1.000	0.000
0.333	0.484	0.484	0.000	0.000	0.484	0.344	0.005
0.667	0.234	0.234	0.484	0.000	0.234	0.118	0.003
1.000	0.113	0.113	0.234	0.484	0.113	0.041	0.001
							0.098

N = **8**

Δt = **0.143**

Eq. 5.34 (LHS)								(RHS)
1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1
-1.000	1.457	0.000	0.000	0.000	0.000	0.000	0.000	0
0.000	-1.000	1.457	0.000	0.000	0.000	0.000	0.000	0
0.000	0.000	-1.000	1.457	0.000	0.000	0.000	0.000	0
0.000	0.000	0.000	-1.000	1.457	0.000	0.000	0.000	0
0.000	0.000	0.000	0.000	-1.000	1.457	0.000	0.000	0
0.000	0.000	0.000	0.000	0.000	-1.000	1.457	0.000	0
0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.457	0

	Inverse Matrix										
t	1	2	3	4	5	6	7	8	Result	Exact	Error
0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000	1.000	0.000
0.143	0.686	0.686	0.000	0.000	0.000	0.000	0.000	0.000	0.686	0.633	0.000
0.286	0.471	0.471	0.686	0.000	0.000	0.000	0.000	0.000	0.471	0.401	0.001
0.429	0.323	0.323	0.471	0.686	0.000	0.000	0.000	0.000	0.323	0.254	0.001
0.571	0.222	0.222	0.323	0.471	0.686	0.000	0.000	0.000	0.222	0.161	0.000
0.714	0.152	0.152	0.222	0.323	0.471	0.686	0.000	0.000	0.152	0.102	0.000
0.857	0.104	0.104	0.152	0.222	0.323	0.471	0.686	0.000	0.104	0.064	0.000
1.000	0.072	0.072	0.104	0.152	0.222	0.323	0.471	0.686	0.072	0.041	0.000
											0.052

N = **16**

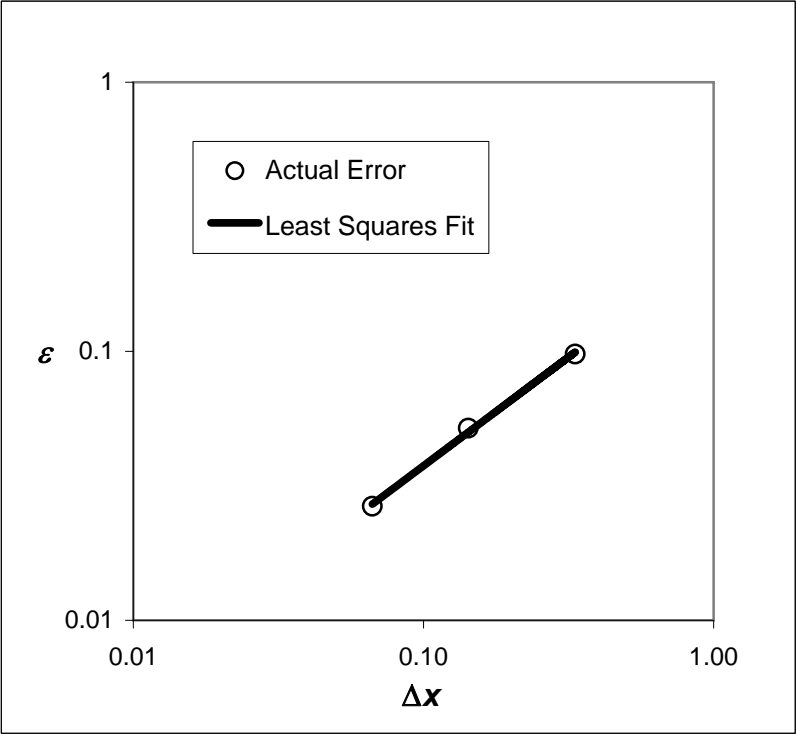
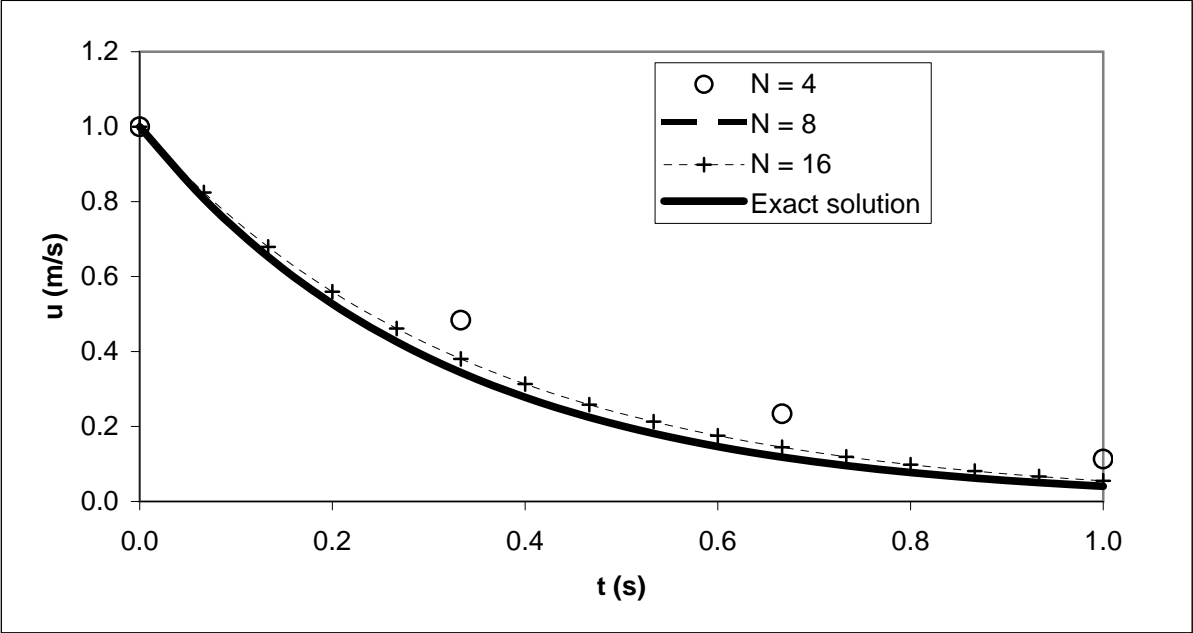
Δt = **0.067** **Eq. 5.34 (LHS)**

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**(RHS)**
1	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1
2	-1.000	1.213	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
3	0.000	-1.000	1.213	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
4	0.000	0.000	-1.000	1.213	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
5	0.000	0.000	0.000	-1.000	1.213	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
6	0.000	0.000	0.000	0.000	-1.000	1.213	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
7	0.000	0.000	0.000	0.000	0.000	-1.000	1.213	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
8	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.213	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
9	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.213	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.213	0.000	0.000	0.000	0.000	0.000	0.000	0
11	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.213	0.000	0.000	0.000	0.000	0.000	0
12	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.213	0.000	0.000	0.000	0.000	0
13	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.213	0.000	0.000	0.000	0
14	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.213	0.000	0.000	0
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.213	0.000	0
16	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.213	0

t	**Inverse Matrix**																**Result**	**Exact**	**Error**
0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000	1.000	0.000
0.067	0.824	0.824	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.824	0.808	0.000
0.133	0.679	0.679	0.824	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.679	0.653	0.000
0.200	0.560	0.560	0.679	0.824	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.560	0.527	0.000
0.267	0.461	0.461	0.560	0.679	0.824	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.461	0.426	0.000
0.333	0.380	0.380	0.461	0.560	0.679	0.824	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.380	0.344	0.000
0.400	0.313	0.313	0.380	0.461	0.560	0.679	0.824	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.313	0.278	0.000
0.467	0.258	0.258	0.313	0.380	0.461	0.560	0.679	0.824	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.258	0.225	0.000
0.533	0.213	0.213	0.258	0.313	0.380	0.461	0.560	0.679	0.824	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.213	0.181	0.000
0.600	0.175	0.175	0.213	0.258	0.313	0.380	0.461	0.560	0.679	0.824	0.000	0.000	0.000	0.000	0.000	0.000	0.175	0.147	0.000
0.667	0.145	0.145	0.175	0.213	0.258	0.313	0.380	0.461	0.560	0.679	0.824	0.000	0.000	0.000	0.000	0.000	0.145	0.118	0.000
0.733	0.119	0.119	0.145	0.175	0.213	0.258	0.313	0.380	0.461	0.560	0.679	0.824	0.000	0.000	0.000	0.000	0.119	0.096	0.000
0.800	0.098	0.098	0.119	0.145	0.175	0.213	0.258	0.313	0.380	0.461	0.560	0.679	0.824	0.000	0.000	0.000	0.098	0.077	0.000
0.867	0.081	0.081	0.098	0.119	0.145	0.175	0.213	0.258	0.313	0.380	0.461	0.560	0.679	0.824	0.000	0.000	0.081	0.062	0.000
0.933	0.067	0.067	0.081	0.098	0.119	0.145	0.175	0.213	0.258	0.313	0.380	0.461	0.560	0.679	0.824	0.000	0.067	0.050	0.000
1.000	0.055	0.055	0.067	0.081	0.098	0.119	0.145	0.175	0.213	0.258	0.313	0.380	0.461	0.560	0.679	0.824	0.055	0.041	0.000
																			0.027

N	Δt	**Error**
4	0.333	0.098
8	0.143	0.052
16	0.067	0.027

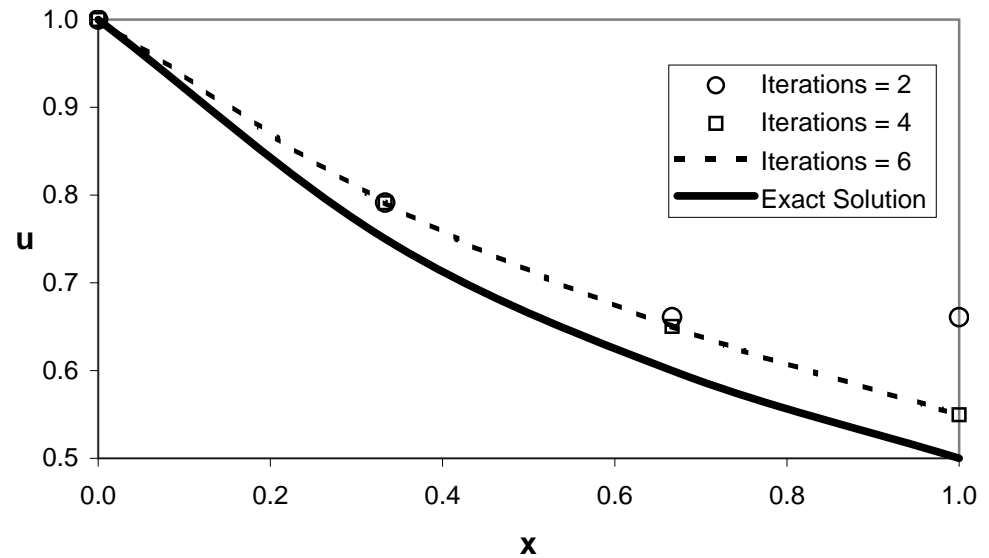
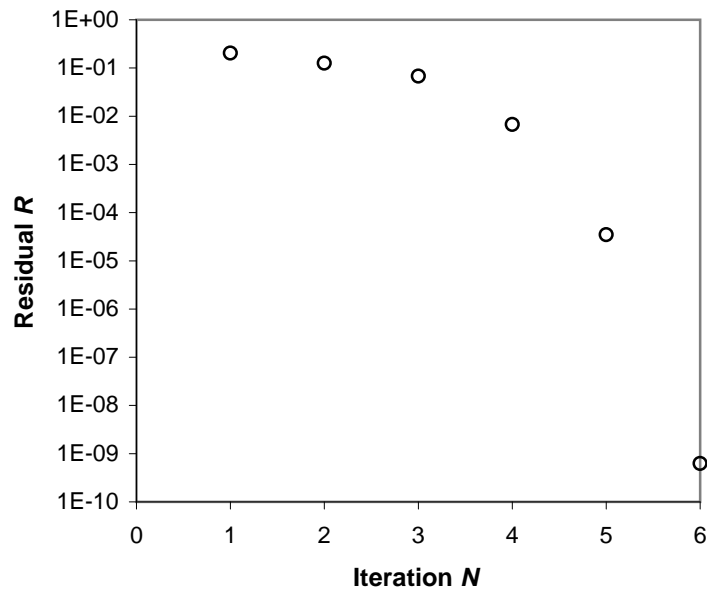
N = 4
N = 8
N = 16
Exact solution
u (m/s)
t (s)
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0

Actual Error
Least Squares Fit
ε
Δx
1
0.1
0.01
0.01
0.10
1.00

Problem 5.91

5.91 Use *Excel* to generate the solutions of Eq. 5.28 for $m = 2$ shown in Fig. 5.19.

$$u_i = \frac{u_{g_{i-1}} + \Delta x \, u_{g_i}^2}{1 + 2\Delta x \, u_{g_i}}$$

$\Delta x =$ 0.333

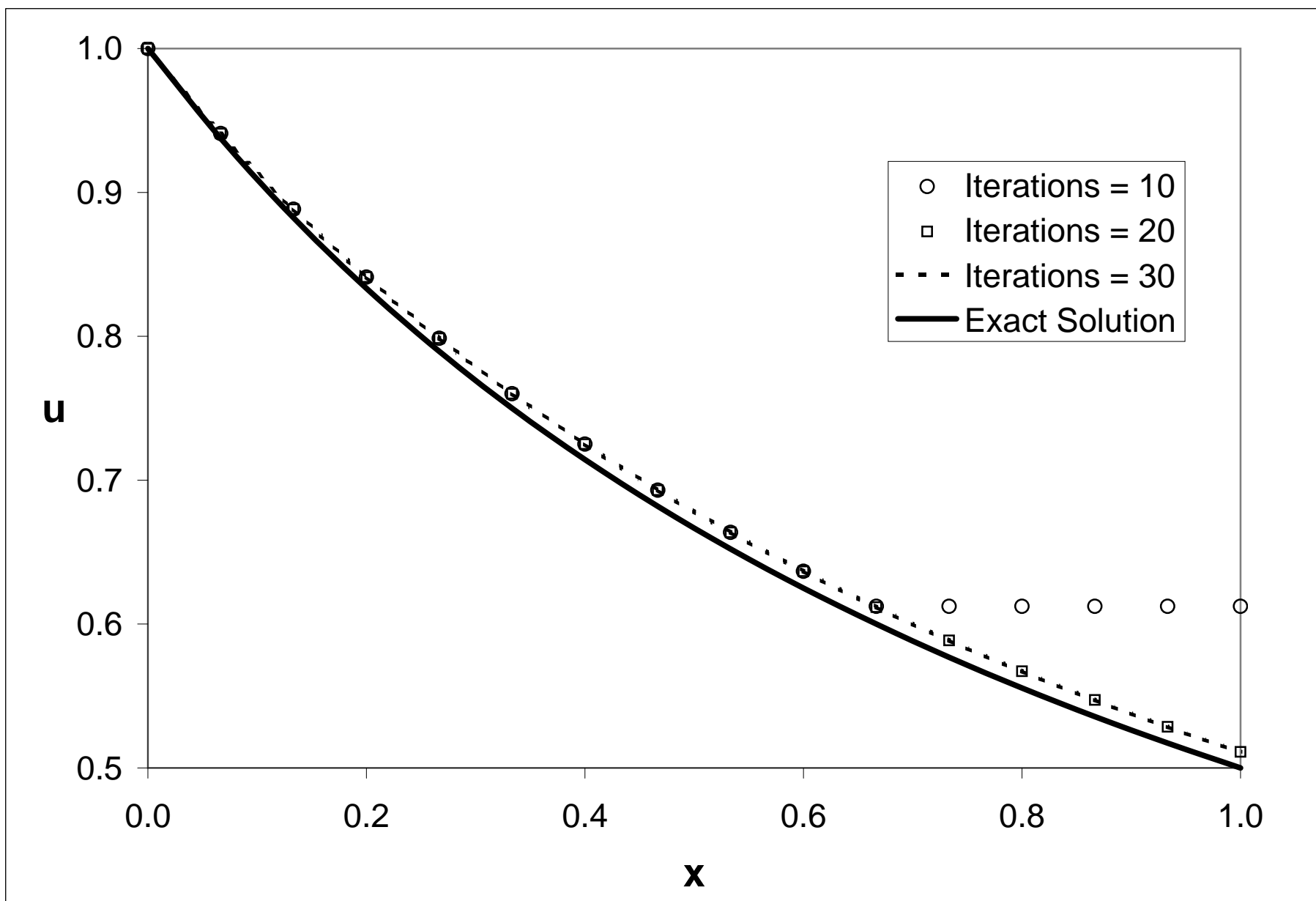
			x			
Iteration	0.000	0.333	0.667	1.000		**Residuals**
0	1.000	1.000	1.000	1.000		
1	1.000	0.800	0.800	0.800		0.204
2	1.000	0.791	0.661	0.661		0.127
3	1.000	0.791	0.650	0.560		0.068
4	1.000	0.791	0.650	0.550		0.007
5	1.000	0.791	0.650	0.550		0.000
6	1.000	0.791	0.650	0.550		0.000
Exact	**1.000**	**0.750**	**0.600**	**0.500**		

Problem 5.92

5.92 Use *Excel* to generate the solutions of Eq. 5.28 for $m = 2$, as shown in Fig. 5.19, except use 16 points and as many iterations as necessary to obtain reasonable convergence.

$$u_i = \frac{u_{g_{i-1}} + \Delta x\, u_{g_i}^2}{1 + 2\Delta x\, u_{g_i}}$$

$\Delta x =$ 0.0667

	x															
Iteration	0.000	0.067	0.133	0.200	0.267	0.333	0.400	0.467	0.533	0.600	0.667	0.733	0.800	0.867	0.933	1.000
0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1	1.000	0.941	0.941	0.941	0.941	0.941	0.941	0.941	0.941	0.941	0.941	0.941	0.941	0.941	0.941	0.941
2	1.000	0.941	0.889	0.889	0.889	0.889	0.889	0.889	0.889	0.889	0.889	0.889	0.889	0.889	0.889	0.889
3	1.000	0.941	0.888	0.842	0.842	0.842	0.842	0.842	0.842	0.842	0.842	0.842	0.842	0.842	0.842	0.842
4	1.000	0.941	0.888	0.841	0.799	0.799	0.799	0.799	0.799	0.799	0.799	0.799	0.799	0.799	0.799	0.799
5	1.000	0.941	0.888	0.841	0.799	0.761	0.761	0.761	0.761	0.761	0.761	0.761	0.761	0.761	0.761	0.761
6	1.000	0.941	0.888	0.841	0.799	0.760	0.726	0.726	0.726	0.726	0.726	0.726	0.726	0.726	0.726	0.726
7	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.694	0.694	0.694	0.694	0.694	0.694	0.694	0.694	0.694
8	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.664	0.664	0.664	0.664	0.664	0.664	0.664
9	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.637	0.637	0.637	0.637	0.637	0.637
10	**1.000**	**0.941**	**0.888**	**0.841**	**0.799**	**0.760**	**0.725**	**0.693**	**0.664**	**0.637**	**0.612**	**0.612**	**0.612**	**0.612**	**0.612**	**0.612**
11	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.589	0.589	0.589	0.589
12	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.568	0.568	0.568	0.568
13	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.548	0.548	0.548
14	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.529
15	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.512
16	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
17	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
18	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
19	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
20	**1.000**	**0.941**	**0.888**	**0.841**	**0.799**	**0.760**	**0.725**	**0.693**	**0.664**	**0.637**	**0.612**	**0.589**	**0.567**	**0.547**	**0.529**	**0.511**
21	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
22	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
23	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
24	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
25	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
26	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
27	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
28	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
29	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
30	**1.000**	**0.941**	**0.888**	**0.841**	**0.799**	**0.760**	**0.725**	**0.693**	**0.664**	**0.637**	**0.612**	**0.589**	**0.567**	**0.547**	**0.529**	**0.511**
Exact	**1.000**	**0.938**	**0.882**	**0.833**	**0.789**	**0.750**	**0.714**	**0.682**	**0.652**	**0.625**	**0.600**	**0.577**	**0.556**	**0.536**	**0.517**	**0.500**

1.0
0.9
0.8
0.7
0.6
0.5
u
0.0
0.2
0.4
0.6
0.8
1.0
x
Iterations = 10
Iterations = 20
Iterations = 30
Exact Solution

Problem 5.93

5.93 Use *Excel* to generate the solutions of Eq. 5.28 for $m = -1$, with $u(0) = 2$, using four and 16 points, with sufficient iterations, and compare to the exact solution

$$u_{\text{exact}} = \sqrt{4 - 2x}$$

To do so, follow the steps described in "Dealing with Nonlinearity" section.

$$\Delta u_i = u_i - u_{g_i}$$

$$\frac{1}{u_i} = \frac{1}{u_{g_i} + \Delta u_i} \approx \frac{1}{u_{g_i}}\left(1 - \frac{\Delta u_i}{u_{g_i}}\right)$$

$$\frac{u_i - u_{i-1}}{\Delta x} + \frac{1}{u_i} = 0$$

$$\frac{u_i - u_{i-1}}{\Delta x} + \frac{1}{u_{g_i}}\left(1 - \frac{u_i - u_{g_i}}{u_{g_i}}\right) = 0$$

$$\frac{u_i - u_{i-1}}{\Delta x} + \frac{1}{u_{g_i}}\left(2 - \frac{u_i}{u_{g_i}}\right) = 0$$

$$u_i\left(1 - \frac{\Delta x}{u_{g_i}^2}\right) = u_{i-1} - \frac{2\Delta x}{u_{g_i}}$$

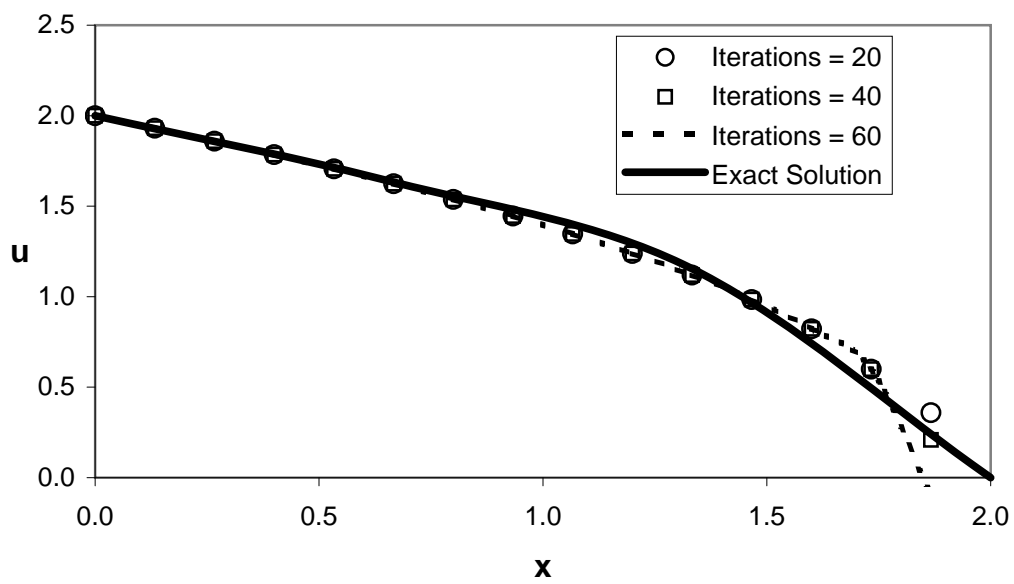
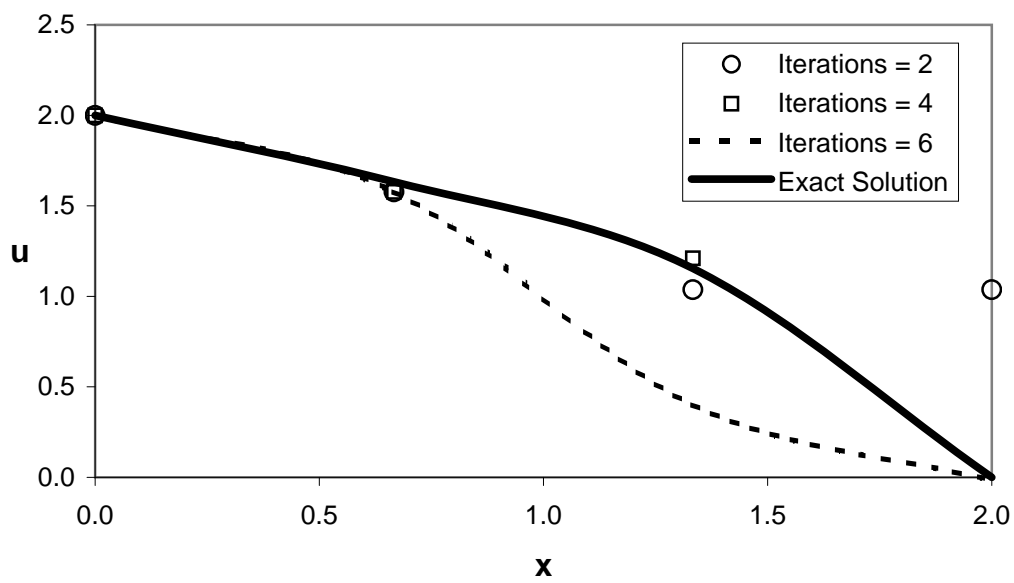
$$u_i = \frac{u_{i-1} - \dfrac{2\Delta x}{u_{g_i}}}{1 - \dfrac{\Delta x}{u_{g_i}^2}}$$

$\Delta x =$ 0.667

Iteration	x = 0.000	0.667	1.333	2.000
0	2.000	2.000	2.000	2.000
1	2.000	1.600	1.600	1.600
2	2.000	1.577	1.037	1.037
3	2.000	1.577	0.767	-0.658
4	2.000	1.577	1.211	-5.158
5	2.000	1.577	0.873	1.507
6	2.000	1.577	0.401	-0.017
Exact	**2.000**	**1.633**	**1.155**	**0.000**

$\Delta x =$ 0.133

Iteration	x = 0.000	0.133	0.267	0.400	0.533	0.667	0.800	0.933	1.067	1.200	1.333	1.467	1.600	1.733	1.867	2.000
0	2.000	2.000	2.000	2.000	2.000	2.000	2.000	2.000	2.000	2.000	2.000	2.000	2.000	2.000	2.000	2.000
1	2.000	1.931	1.931	1.931	1.931	1.931	1.931	1.931	1.931	1.931	1.931	1.931	1.931	1.931	1.931	1.931
2	2.000	1.931	1.859	1.859	1.859	1.859	1.859	1.859	1.859	1.859	1.859	1.859	1.859	1.859	1.859	1.859
3	2.000	1.931	1.859	1.785	1.785	1.785	1.785	1.785	1.785	1.785	1.785	1.785	1.785	1.785	1.785	1.785
4	2.000	1.931	1.859	1.785	1.707	1.707	1.707	1.707	1.707	1.707	1.707	1.707	1.707	1.707	1.707	1.707
5	2.000	1.931	1.859	1.785	1.706	1.625	1.625	1.625	1.625	1.625	1.625	1.625	1.625	1.625	1.625	1.625
6	2.000	1.931	1.859	1.785	1.706	1.624	1.539	1.539	1.539	1.539	1.539	1.539	1.539	1.539	1.539	1.539
7	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.447	1.447	1.447	1.447	1.447	1.447	1.447	1.447	1.447
8	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.348	1.348	1.348	1.348	1.348	1.348	1.348	1.348
9	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.242	1.242	1.242	1.242	1.242	1.242	1.242
10	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.124	1.124	1.124	1.124	1.124	1.124
11	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.991	0.991	0.991	0.991	0.991
12	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.836	0.836	0.836	0.836
13	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.639	0.639	0.639
14	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.601	0.329	0.329
15	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.899	2.061
16	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.363	0.795
17	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	9.602	0.034
18	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.572	-0.016
19	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.225	-0.034
20	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.359	-0.070
21	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	3.969	-0.160
22	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.537	-1.332
23	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.191	0.797
24	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.300	-0.182
25	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.600	-0.584
26	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.246	1.734
27	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.403	0.097
28	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	-0.345	0.178
29	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	-11.373	0.572
30	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.623	-19.981
31	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.261	0.637
32	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.442	-0.234
33	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	-0.013	-1.108
34	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	-0.027	0.255
35	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	-0.059	1.023
36	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	-0.136	-0.366
37	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	-0.414	132.420
38	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	5.624	-0.416
39	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.554	27.391
40	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.209	0.545
41	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.329	-0.510
42	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.919	1.749
43	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.367	0.802
44	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	-11.148	0.044
45	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.624	0.252
46	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.262	0.394
47	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.443	-2.929
48	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	-0.010	0.542
49	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	-0.019	-0.918
50	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	-0.041	0.322
51	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	-0.090	3.048
52	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	-0.231	-0.180
53	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	-1.171	-0.402
54	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.916	-2.886
55	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.366	1.025
56	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	-18.029	0.122
57	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.614	2.526
58	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.256	0.520
59	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	0.426	-0.509
60	2.000	1.931	1.859	1.785	1.706	1.624	1.538	1.445	1.346	1.239	1.120	0.984	0.822	0.599	-0.097	1.962
Exact	**2.000**	**1.932**	**1.862**	**1.789**	**1.713**	**1.633**	**1.549**	**1.461**	**1.366**	**1.265**	**1.155**	**1.033**	**0.894**	**0.730**	**0.516**	**0.000**

Iterations = 2
Iterations = 4
Iterations = 6
Exact Solution
u
x
2.5
2.0
1.5
1.0
0.5
0.0
0.0
0.5
1.0
1.5
2.0
Iterations = 20
Iterations = 40
Iterations = 60
Exact Solution
u
x

Problem 5.94

5.94 You (someone whose mass is M = 70 kg) fall into a fast moving river (the speed of the water is U = 7.5 m/s). The equation of motion for your speed u is

$$M\frac{du}{dt} = k(U-u)^2$$

where k = 10 N • s²/m² is a constant indicating the drag of the water. Use *Excel* to generate and plot your speed versus time (for the first 10 s) using the same approach as the solutions of Eq. 5.28 for m = 2, as shown in Fig. 5.19, except use 16 points and as many iterations as necessary to obtain reasonable convergence. Compare your results to the exact solution.

$$u_{\text{exact}} = \frac{kU^2t}{M+kUt}$$

Hint: Use a substitution for $(U-u)$ so the equation of motion looks similar to Eq. 5.28.

$$M\frac{du}{dt} = k(U-u)^2$$

$$v = U - u$$

$$dv = -du$$

$$-M\frac{dv}{dt} = kv^2$$

$$\frac{dv}{dt} + \frac{k}{M}v^2 = 0$$

$$v_i^2 \approx 2v_{g_i}v_i - v_{g_i}^2$$

$$\frac{v_i - v_{i-1}}{\Delta t} + \frac{k}{M}\left(2v_{g_i}v_i - v_{g_i}^2\right) = 0$$

$$v_i = \frac{v_{g_{i-1}} + \frac{k}{M}\Delta t\, v_{g_i}^2}{1 + 2\frac{k}{M}\Delta t\, v_{g_i}}$$

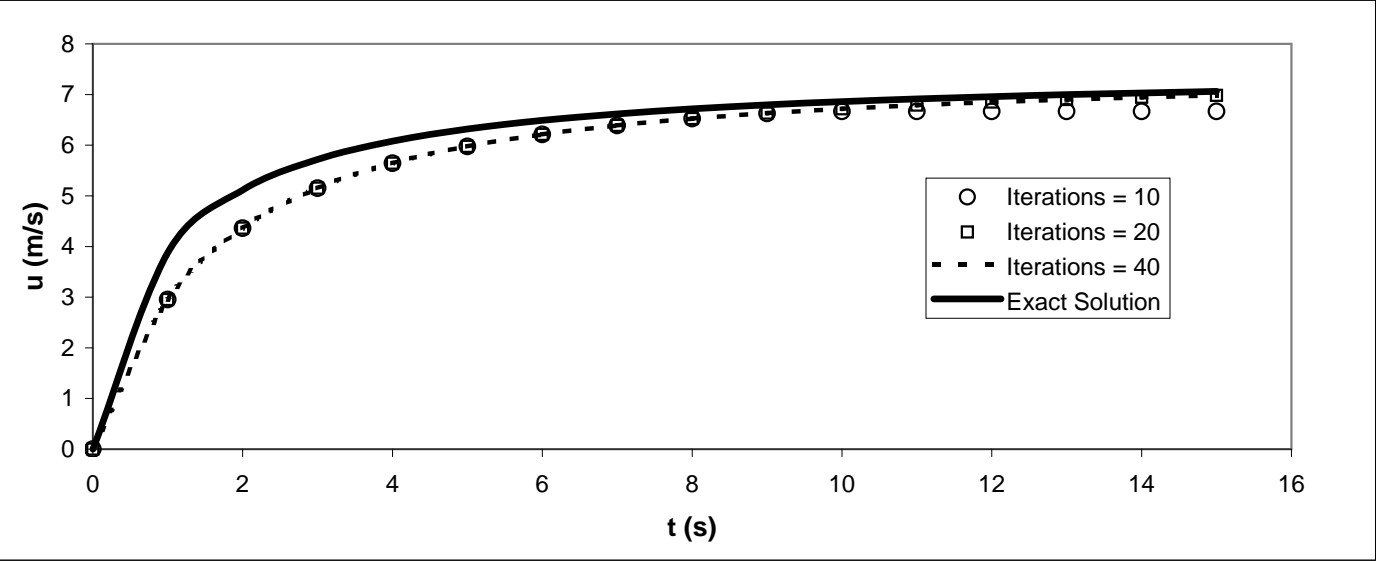
Δt = 1.000

k = 10 N.s²/m²

M = 70 kg

	t															
Iteration	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	7.500	7.500	7.500	7.500	7.500	7.500	7.500	7.500	7.500	7.500	7.500	7.500	7.500	7.500	7.500	7.500
1	7.500	4.943	4.943	4.943	4.943	4.943	4.943	4.943	4.943	4.943	4.943	4.943	4.943	4.943	4.943	4.943
2	7.500	4.556	3.496	3.496	3.496	3.496	3.496	3.496	3.496	3.496	3.496	3.496	3.496	3.496	3.496	3.496
3	7.500	4.547	3.153	2.623	2.623	2.623	2.623	2.623	2.623	2.623	2.623	2.623	2.623	2.623	2.623	2.623
4	7.500	4.547	3.139	2.364	2.061	2.061	2.061	2.061	2.061	2.061	2.061	2.061	2.061	2.061	2.061	2.061
5	7.500	4.547	3.139	2.350	1.870	1.679	1.679	1.679	1.679	1.679	1.679	1.679	1.679	1.679	1.679	1.679
6	7.500	4.547	3.139	2.350	1.857	1.536	1.407	1.407	1.407	1.407	1.407	1.407	1.407	1.407	1.407	1.407
7	7.500	4.547	3.139	2.350	1.857	1.525	1.297	1.205	1.205	1.205	1.205	1.205	1.205	1.205	1.205	1.205
8	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.119	1.051	1.051	1.051	1.051	1.051	1.051	1.051	1.051
9	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.982	0.930	0.930	0.930	0.930	0.930	0.930	0.930
10	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.874	0.832	0.832	0.832	0.832	0.832	0.832
11	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.786	0.752	0.752	0.752	0.752	0.752
12	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.713	0.686	0.686	0.686	0.686
13	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.653	0.629	0.629	0.629
14	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.601	0.581	0.581
15	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.557	0.540
16	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.519
17	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
18	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
19	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
20	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
21	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
22	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
23	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
24	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
25	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
26	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
27	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
28	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
29	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
30	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
31	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
32	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
33	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
34	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
35	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
36	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
37	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
38	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
39	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516
40	7.500	4.547	3.139	2.350	1.857	1.525	1.288	1.112	0.976	0.868	0.781	0.709	0.649	0.598	0.554	0.516

Above values are for v! To get u we compute u = U - v

Iteration																
10	0.000	2.953	4.361	5.150	5.643	5.975	6.212	6.388	6.524	6.626	6.668	6.668	6.668	6.668	6.668	6.668
20	0.000	2.953	4.361	5.150	5.643	5.975	6.212	6.388	6.524	6.632	6.719	6.791	6.851	6.902	6.946	6.984
40	0.000	2.953	4.361	5.150	5.643	5.975	6.212	6.388	6.524	6.632	6.719	6.791	6.851	6.902	6.946	6.984
Exact	**0.000**	**3.879**	**5.114**	**5.720**	**6.081**	**6.320**	**6.490**	**6.618**	**6.716**	**6.795**	**6.860**	**6.913**	**6.959**	**6.998**	**7.031**	**7.061**

Problem 6.1

6.1 Consider the flow field with velocity given by $\vec{V} = [A(y^2 - x^2) - Bx]\hat{i} + [2Axy + By]\hat{j}$; $A = 1\ \text{ft}^{-1}\cdot\text{s}^{-1}$, $B = 1\ \text{ft}^{-1}\cdot\text{s}^{-1}$; the coordinates are measured in feet. The density is 2 slug/ft^3, and gravity acts in the negative y direction. Calculate the acceleration of a fluid particle and the pressure gradient at point $(x, y) = (1, 1)$.

Given: Velocity field

Find: Acceleration of particle and pressure gradient at (1,1)

Solution:

NOTE: Units of B are s^{-1} not $ft^{-1}s^{-1}$

Basic equations

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = \underbrace{u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}}_{\text{convective acceleration}} + \frac{\partial\vec{V}}{\partial t}$$

total acceleration of a particle; convective acceleration; local acceleration

$$\rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

For this flow

$$u(x,y) = A\cdot\left(y^2 - x^2\right) - B\cdot x \qquad v(x,y) = 2\cdot A\cdot x\cdot y + B\cdot y$$

$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = \left[A\cdot\left(y^2 - x^2\right) - B\cdot x\right]\cdot\frac{\partial}{\partial x}\left[A\cdot\left(y^2 - x^2\right) - B\cdot x\right] + (2\cdot A\cdot x\cdot y + B\cdot y)\cdot\frac{\partial}{\partial y}\left[A\cdot\left(y^2 - x^2\right) - B\cdot x\right]$$

$$a_x = (B + 2\cdot A\cdot x)\cdot\left(A\cdot x^2 + B\cdot x + A\cdot y^2\right)$$

$$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v = \left[A\cdot\left(y^2 - x^2\right) - B\cdot x\right]\cdot\frac{\partial}{\partial x}(2\cdot A\cdot x\cdot y + B\cdot y) + (2\cdot A\cdot x\cdot y + B\cdot y)\cdot\frac{\partial}{\partial y}(2\cdot A\cdot x\cdot y + B\cdot y)$$

$$a_y = (B + 2\cdot A\cdot x)\cdot(B\cdot y + 2\cdot A\cdot x\cdot y) - 2\cdot A\cdot y\cdot\left[B\cdot x + A\cdot\left(x^2 - y^2\right)\right]$$

Hence at (1,1)

$$a_x = (1 + 2\cdot 1\cdot 1)\cdot\frac{1}{s}\times\left(1\cdot 1^2 + 1\cdot 1 + 1\cdot 1^2\right)\cdot\frac{ft}{s} \qquad a_x = 9\cdot\frac{ft}{s^2}$$

$$a_y = (1 + 2\cdot 1\cdot 1)\cdot\frac{1}{s}\times(1\cdot 1 + 2\cdot 1\cdot 1\cdot 1)\cdot\frac{ft}{s} - 2\cdot 1\cdot 1\cdot\frac{1}{s}\times\left[1\cdot 1 + 1\cdot\left(1^2 - 1^2\right)\right]\cdot\frac{ft}{s} \qquad a_y = 7\cdot\frac{ft}{s^2}$$

$$a = \sqrt{a_x^2 + a_y^2} \qquad \theta = \text{atan}\left(\frac{a_y}{a_x}\right) \qquad a = 11.4\cdot\frac{ft}{s^2} \qquad \theta = 37.9\cdot\text{deg}$$

For the pressure gradient

$$\frac{\partial}{\partial x}p = \rho\cdot g_x - \rho\cdot a_x = -2\cdot\frac{slug}{ft^3}\times 9\cdot\frac{ft}{s^2}\times\frac{lbf\cdot s^2}{slug\cdot ft} \qquad \frac{\partial}{\partial x}p = -18\cdot\frac{\frac{lbf}{ft^2}}{ft} = -0.125\cdot\frac{psi}{ft}$$

$$\frac{\partial}{\partial y}p = \rho\cdot g_y - \rho\cdot a_y = 2\cdot\frac{slug}{ft^3}\times(-32.2 - 7)\cdot\frac{ft}{s^2}\times\frac{lbf\cdot s^2}{slug\cdot ft} \qquad \frac{\partial}{\partial y}p = -78.4\cdot\frac{\frac{lbf}{ft^2}}{ft} = -0.544\cdot\frac{psi}{ft}$$

Problem 6.2

[2]

6.2 An incompressible frictionless flow field is given by $\vec{V} = (Ax - By)\hat{i} - Ay\hat{j}$, where $A = 1\ \text{s}^{-1}$, $B = 3\ \text{s}^{-1}$, and the coordinates are measured in meters. Find the magnitude and direction of the acceleration of a fluid particle at point $(x, y) = (0.7, 2)$. Find the pressure gradient at the same point, if $\vec{g} = -g\hat{j}$ and the fluid is water.

Given:

Velocity field

Find:

Acceleration of particle and pressure gradient at (0.7,2)

Solution:

Basic equations

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\ \text{acceleration}\\ \text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\ \text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\ \text{acceleration}}}$$

$$\rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

For this flow

$$u(x,y) = A\cdot x - B\cdot y \qquad v(x,y) = -A\cdot y$$

$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = (A\cdot x - B\cdot y)\cdot\frac{\partial}{\partial x}(A\cdot x - B\cdot y) + (-A\cdot y)\cdot\frac{\partial}{\partial y}(A\cdot x - B\cdot y) \qquad a_x = A^2\cdot x$$

$$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v = (A\cdot x - B\cdot y)\cdot\frac{\partial}{\partial x}(-A\cdot y) + (-A\cdot y)\cdot\frac{\partial}{\partial y}(-A\cdot y) \qquad a_y = A^2\cdot y$$

Hence at (0.7,2)

$$a_x = \left(\frac{1}{s}\right)^2 \times 0.7\cdot m \qquad a_x = 0.7\frac{m}{s^2}$$

$$a_y = \left(\frac{1}{s}\right)^2 \times 2\cdot m \qquad a_y = 2\frac{m}{s^2}$$

$$a = \sqrt{a_x^2 + a_y^2} \qquad \theta = \operatorname{atan}\left(\frac{a_y}{a_x}\right) \qquad a = 2.12\frac{m}{s^2} \qquad \theta = 70.7\cdot\text{deg}$$

For the pressure gradient

$$\frac{\partial}{\partial x}p = \rho\cdot g_x - \rho\cdot a_x = -1000\cdot\frac{kg}{m^3} \times 0.7\cdot\frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} \qquad \frac{\partial}{\partial x}p = -700\cdot\frac{Pa}{m} = -0.7\cdot\frac{kPa}{m}$$

$$\frac{\partial}{\partial y}p = \rho\cdot g_y - \rho\cdot a_y = 1000\cdot\frac{kg}{m^3} \times (-9.81 - 2)\cdot\frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} \qquad \frac{\partial}{\partial y}p = -11800\cdot\frac{Pa}{m} = -11.8\cdot\frac{kPa}{m}$$

Problem 6.3

Given: Horizontal flow of water described by the velocity field

$$\vec{V} = (Ax + Bt)\hat{i} + (-Ay + Bt)\hat{j}$$

where: $A = 5\ s^{-1}$, $B = 10\ ft \cdot s^{-2}$, coordinates x, y in ft, t in s.

Find: (a) Expressions for (i) local, (ii) convective, (iii) total, acceleration
(b) Evaluate at point (2,2) for $t = 5\ s$
(c) Evaluate ∇p at same point and time

Solution:

Basic equations: $\frac{D\vec{V}}{Dt} = \vec{a}_p = \underbrace{\frac{\partial \vec{V}}{\partial t}}_{local} + \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{convective}$ (last term $= 0$) ; $\rho\vec{g} - \nabla p = \rho\frac{D\vec{V}}{Dt}$

Assumptions: (1) frictionless flow
(2) ρ = constant = 1.94 slug/ft^3

$$\frac{\partial \vec{V}}{\partial t} = \frac{\partial}{\partial t}\left[(Ax+Bt)\hat{i} + (-Ay+Bt)\hat{j}\right] = B\hat{i} + B\hat{j} = 10(\hat{i}+\hat{j})\ ft/s^2 \quad \longleftarrow \vec{a}_{local}$$

$$u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = (Ax+Bt)\frac{\partial}{\partial x}\left[(Ax+Bt)\hat{i} + (-Ay+Bt)\hat{j}\right] + (-Ay+Bt)\frac{\partial}{\partial y}\left[(Ax+Bt)\hat{i} + (-Ay+Bt)\hat{j}\right]$$

$$= (Ax+Bt)\left[A\hat{i}\right] + (-Ay+Bt)\left[-A\hat{j}\right]$$

$$u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = A(Ax+Bt)\hat{i} - A(-Ay+Bt)\hat{j} \quad \longleftarrow$$

$$= \frac{5}{s}\left(\frac{5}{s}\times 2ft + \frac{10ft}{s^2}\times 5s\right)\hat{i} - \frac{5}{s}\left(-\frac{5}{s}\times 2ft + \frac{10ft}{s^2}\times 5s\right)\hat{j} = 300\hat{i} - 200\hat{j}\ \frac{ft}{s^2} \quad \longleftarrow \vec{a}_{conv}$$

$$\vec{a} = \vec{a}_{local} + \vec{a}_{conv} = \left[B + A(Ax+Bt)\right]\hat{i} + \left[B - A(-Ay+Bt)\right]\hat{j} = 310\hat{i} - 190\hat{j}\ \frac{ft}{s^2} \quad \longleftarrow \vec{a}$$

From Euler's equation,

$$\nabla p = \rho\vec{g} - \rho\frac{D\vec{V}}{Dt} = 1.94\ \frac{slug}{ft^3}\left[-32.2\hat{k} - (310\hat{i} - 190\hat{j})\right]\frac{ft}{s^2}\times\frac{lbf\cdot s^2}{ft\cdot slug}$$

$$\nabla p = -601\hat{i} + 367\hat{j} - 62\hat{k}\ \frac{lbf/ft^2}{ft} = -4.17\hat{i} + 2.56\hat{j} - 0.43\hat{k}\ psi/ft$$

Note: $\nabla \cdot \vec{V} = 0$ as required for incompressible flow

Given: Velocity field, $\vec{V} = (Ax - By)t\hat{i} - (Ay + Bx)t\hat{j}$

where $A = 1\ s^{-2}$

$B = 2\ s^{-2}$

coordinates x, y are in meters

Fluid density is $\rho = 1500\ kg/m^3$. Body forces are negligible

Find: ∇P at location (1,2) at $t = 1$ s.

Solution:

Basic equations: $\rho\vec{g} - \nabla P = \rho \frac{D\vec{V}}{Dt}$ (with $\rho\vec{g}$ crossed out $\to 0$)

$$\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

Assumptions: (1) frictionless flow

Substituting for the velocity field in the equation for $\frac{D\vec{V}}{Dt}$,

$$\frac{D\vec{V}}{Dt} = \frac{\partial}{\partial t}\left[(Ax-By)t\hat{i} - (Ay+Bx)t\hat{j}\right] + (Ax-By)t\frac{\partial}{\partial x}\left[(Ax-By)t\hat{i} - (Ay+Bx)t\hat{j}\right]$$
$$-(Ay+Bx)t\frac{\partial}{\partial y}\left[(Ax-By)t\hat{i} - (Ay+Bx)t\hat{j}\right]$$

$$= \left[(Ax-By)\hat{i} - (Ay+Bx)\hat{j}\right] + (Ax-By)t\left[At\hat{i} - Bt\hat{j}\right] - (Ay+Bx)t\left[-Bt\hat{i} - At\hat{j}\right]$$

$$= \hat{i}\left\{Ax - By + A^2xt^2 - ABwritten yt^2 + ABy t^2 + B^2xt^2\right\} + \hat{j}\left\{-Ay - Bx - ABxt^2 + B^2yt^2 + A^2yt^2 + ABxt^2\right\}$$

$$\frac{D\vec{V}}{Dt} = \hat{i}\left\{Ax - By + xt^2(A^2+B^2)\right\} + \hat{j}\left\{-Ay - Bx + yt^2(A^2+B^2)\right\}$$

Then,

$$\nabla P = -\rho\frac{D\vec{V}}{Dt} = -\rho\left[\hat{i}\left\{Ax - By + xt^2(A^2+B^2)\right\} + \hat{j}\left\{-Ay - Bx + yt^2(A^2+B^2)\right\}\right]$$

At location (1,2) at $t = 1$ s

$$\nabla P = -1500\frac{kg}{m^3}\left[\hat{i}\left\{\frac{1}{s^2}\times 1m - \frac{2}{s^2}\times 2 + 1m\times 1s^2\left(\frac{(1)^2+(2)^2}{s^4}\right)\right\}\right.$$
$$\left. + \hat{j}\left\{-\frac{1}{s^2}\times 2m - \frac{2}{s^2}\times 1m + 2m\times 1s^2\times\left(\frac{(1)^2+(2)^2}{s^4}\right)\right\}\right]\frac{N\cdot s^2}{kg\cdot m}$$

$$\nabla P = -(3.0\hat{i} + 9.0\hat{j})\ \frac{kN/m^2}{m} \quad \leftarrow \nabla P$$

Note: $\nabla\cdot\vec{V} = 0$ as required for incompressible flow

Problem 6.5

6.5 Consider the flow field with velocity given by $\vec{V} = [A(x^2 - y^2) - 3Bx]\hat{i} - [2Axy - 3By]\hat{j}$, where $A = 1\ \text{ft}^{-1}\cdot\text{s}^{-1}$, $B = 1\ \text{s}^{-1}$, and the coordinates are measured in feet. The density is 2 slug/ft^3 and gravity acts in the negative y direction. Determine the acceleration of a fluid particle and the pressure gradient at point $(x, y) = (1, 1)$.

Given: Velocity field

Find: Acceleration of particle and pressure gradient at (1,1)

Solution:

Basic equations

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective acceleration}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local acceleration}}$$

(total acceleration of a particle)

$$\rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

For this flow

$$u(x,y) = A\cdot\left(x^2 - y^2\right) - 3\cdot B\cdot x \qquad v(x,y) = -2\cdot A\cdot x\cdot y + 3\cdot B\cdot y$$

$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = \left[A\cdot\left(x^2 - y^2\right) - 3\cdot B\cdot x\right]\cdot\frac{\partial}{\partial x}\left[A\cdot\left(x^2 - y^2\right) - 3\cdot B\cdot x\right] \ldots$$

$$+ (-2\cdot A\cdot x\cdot y + 3\cdot B\cdot y)\cdot\frac{\partial}{\partial y}\left[A\cdot\left(x^2 - y^2\right) - 3\cdot B\cdot x\right]$$

$$a_x = (2\cdot A\cdot x - 3\cdot B)\cdot\left(A\cdot x^2 - 3\cdot B\cdot x + A\cdot y^2\right)$$

$$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v = \left[A\cdot\left(x^2 - y^2\right) - 3\cdot B\cdot x\right]\cdot\frac{\partial}{\partial x}(-2\cdot A\cdot x\cdot y + 3\cdot B\cdot y) + (-2\cdot A\cdot x\cdot y + 3\cdot B\cdot y)\cdot\frac{\partial}{\partial y}(-2\cdot A\cdot x\cdot y + 3\cdot B\cdot y)$$

$$a_y = (3\cdot B\cdot y - 2\cdot A\cdot x\cdot y)\cdot(3\cdot B - 2\cdot A\cdot x) - 2\cdot A\cdot y\cdot\left[A\cdot\left(x^2 - y^2\right) - 3\cdot B\cdot x\right]$$

Hence at (1,1)

$$a_x = (2\cdot 1\cdot 1 - 3\cdot 1)\cdot\frac{1}{s}\times\left(1\cdot 1^2 - 3\cdot 1\cdot 1 + 1\cdot 1^2\right)\cdot\frac{\text{ft}}{s} \qquad a_x = 1\cdot\frac{\text{ft}}{s^2}$$

$$a_y = (3\cdot 1\cdot 1 - 2\cdot 1\cdot 1\cdot 1)\cdot\frac{1}{s}\times(3\cdot 1 - 2\cdot 1\cdot 1)\cdot\frac{\text{ft}}{s} - 2\cdot 1\cdot 1\cdot\frac{1}{s}\times\left[1\cdot\left(1^2 - 1^2\right) - 3\cdot 1\cdot 1\right]\cdot\frac{\text{ft}}{s} \qquad a_y = 7\cdot\frac{\text{ft}}{s^2}$$

$$a = \sqrt{a_x^{\,2} + a_y^{\,2}} \qquad \theta = \text{atan}\left(\frac{a_y}{a_x}\right) \qquad a = 7.1\cdot\frac{\text{ft}}{s^2} \qquad \theta = 81.9\cdot\text{deg}$$

For the pressure gradient

$$\frac{\partial}{\partial x}p = \rho\cdot g_x - \rho\cdot a_x = -2\cdot\frac{\text{slug}}{\text{ft}^3}\times 1\cdot\frac{\text{ft}}{s^2}\times\frac{\text{lbf}\cdot s^2}{\text{slug}\cdot\text{ft}} \qquad \frac{\partial}{\partial x}p = -2\cdot\frac{\frac{\text{lbf}}{\text{ft}^2}}{\text{ft}} = -0.0139\cdot\frac{\text{psi}}{\text{ft}}$$

$$\frac{\partial}{\partial y}p = \rho\cdot g_y - \rho\cdot a_y = 2\cdot\frac{\text{slug}}{\text{ft}^3}\times(-32.2 - 7)\cdot\frac{\text{ft}}{s^2}\times\frac{\text{lbf}\cdot s^2}{\text{slug}\cdot\text{ft}} \qquad \frac{\partial}{\partial y}p = -78.4\cdot\frac{\frac{\text{lbf}}{\text{ft}^2}}{\text{ft}} = -0.544\cdot\frac{\text{psi}}{\text{ft}}$$

Problem 6.6 [3]

6.6 Consider the flow field with velocity given by $\vec{V} = Ax\sin(2\pi\omega t)\hat{i} - Ay\sin(2\pi\omega t)\hat{j}$, where $A = 2\ \mathrm{s}^{-1}$ and $\omega = 1\ \mathrm{s}^{-1}$. The fluid density is 2 kg/m^3. Find expressions for the local acceleration, the convective acceleration, and the total acceleration. Evaluate these at point (1, 1) at t = 0, 0.5 and 1 seconds. Evaluate ∇p at the same point and times.

Given: Velocity field

Find: Expressions for local, convective and total acceleration; evaluate at several points; evaluate pressure gradient

Solution:

The given data is $A = 2\cdot\frac{1}{s}$ $\quad \omega = 1\cdot\frac{1}{s}$ $\quad \rho = 2\cdot\frac{kg}{m^3}$ $\quad u = A\cdot x\cdot\sin(2\cdot\pi\cdot\omega\cdot t)$ $\quad v = -A\cdot y\cdot\sin(2\cdot\pi\cdot\omega\cdot t)$

Check for incompressible flow $\quad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$

Hence $\quad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = A\cdot\sin(2\cdot\pi\cdot\omega\cdot t) - A\cdot\sin(2\cdot\pi\cdot\omega\cdot t) = 0 \quad$ Incompressible flow

The governing equation for acceleration is

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\text{total acceleration of a particle}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective acceleration}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local acceleration}}$$

The local acceleration is then $\quad$ *x* - component $\quad \frac{\partial}{\partial t}u = 2\cdot\pi\cdot A\cdot\omega\cdot x\cdot\cos(2\cdot\pi\cdot\omega\cdot t)$

y - component $\quad \frac{\partial}{\partial t}v = -2\cdot\pi\cdot A\cdot\omega\cdot y\cdot\cos(2\cdot\pi\cdot\omega\cdot t)$

For the present steady, 2D flow, the convective acceleration is

x - component $\quad u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = A\cdot x\cdot\sin(2\cdot\pi\cdot\omega\cdot t)\cdot(A\cdot\sin(2\cdot\pi\cdot\omega\cdot t)) + (-A\cdot y\cdot\sin(2\cdot\pi\cdot\omega\cdot t))\cdot 0 = A^2\cdot x\cdot\sin(2\cdot\pi\cdot\omega\cdot t)^2$

y - component $\quad u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v = A\cdot x\cdot\sin(2\cdot\pi\cdot\omega\cdot t)\cdot 0 + (-A\cdot y\cdot\sin(2\cdot\pi\cdot\omega\cdot t))\cdot(-A\cdot\sin(2\cdot\pi\cdot\omega\cdot t)) = A^2\cdot y\cdot\sin(2\cdot\pi\cdot\omega\cdot t)^2$

The total acceleration is then $\quad$ *x* - component $\quad \frac{\partial}{\partial t}u + u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = 2\cdot\pi\cdot A\cdot\omega\cdot x\cdot\cos(2\cdot\pi\cdot\omega\cdot t) + A^2\cdot x\cdot\sin(2\cdot\pi\cdot\omega\cdot t)^2$

y - component $\quad \frac{\partial}{\partial t}v + u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v = -2\cdot\pi\cdot A\cdot\omega\cdot y\cdot\cos(2\cdot\pi\cdot\omega\cdot t) + A^2\cdot y\cdot\sin(2\cdot\pi\cdot\omega\cdot t)^2$

Evaluating at point (1,1) at

$t = 0 \cdot s$	Local	$12.6 \cdot \frac{m}{s^2}$	and	$-12.6 \cdot \frac{m}{s^2}$	Convective	$0 \cdot \frac{m}{s^2}$	and	$0 \cdot \frac{m}{s^2}$
	Total	$12.6 \cdot \frac{m}{s^2}$	and	$-12.6 \cdot \frac{m}{s^2}$				
$t = 0.5 \cdot s$	Local	$-12.6 \cdot \frac{m}{s^2}$	and	$12.6 \cdot \frac{m}{s^2}$	Convective	$0 \cdot \frac{m}{s^2}$	and	$0 \cdot \frac{m}{s^2}$
	Total	$-12.6 \cdot \frac{m}{s^2}$	and	$12.6 \cdot \frac{m}{s^2}$				
$t = 1 \cdot s$	Local	$12.6 \cdot \frac{m}{s^2}$	and	$-12.6 \cdot \frac{m}{s^2}$	Convective	$0 \cdot \frac{m}{s^2}$	and	$0 \cdot \frac{m}{s^2}$
	Total	$12.6 \cdot \frac{m}{s^2}$	and	$-12.6 \cdot \frac{m}{s^2}$				

The governing equation (assuming inviscid flow) for computing the pressure gradient is $\rho \frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$ (6.1)

Hence, the components of pressure gradient (neglecting gravity) are

$$\frac{\partial}{\partial x} p = -\rho \cdot \frac{Du}{Dt} \qquad \frac{\partial}{\partial x} p = -\rho \cdot \left(2 \cdot \pi \cdot A \cdot \omega \cdot x \cdot \cos(2 \cdot \pi \cdot \omega \cdot t) + A^2 \cdot x \cdot \sin(2 \cdot \pi \cdot \omega \cdot t)^2\right)$$

$$\frac{\partial}{\partial y} p = -\rho \cdot \frac{Dv}{Dt} \qquad \frac{\partial}{\partial x} p = -\rho \cdot \left(-2 \cdot \pi \cdot A \cdot \omega \cdot y \cdot \cos(2 \cdot \pi \cdot \omega \cdot t) + A^2 \cdot y \cdot \sin(2 \cdot \pi \cdot \omega \cdot t)^2\right)$$

Evaluated at (1,1) and time					
	$t = 0 \cdot s$	*x* comp.	$-25.1 \cdot \frac{Pa}{m}$	*y* comp.	$25.1 \cdot \frac{Pa}{m}$
	$t = 0.5 \cdot s$	*x* comp.	$25.1 \cdot \frac{Pa}{m}$	*y* comp.	$-25.1 \cdot \frac{Pa}{m}$
	$t = 1 \cdot s$	*x* comp.	$-25.1 \cdot \frac{Pa}{m}$	*y* comp.	$25.1 \cdot \frac{Pa}{m}$

Problem 6.7

6.7 The x component of velocity in an incompressible flow field is given by $u = Ax$, where $A = 2\ s^{-1}$ and the coordinates are measured in meters. The pressure at point $(x, y) = (0, 0)$ is $p_0 = 190$ kPa (gage). The density is $\rho = 1.50\ kg/m^3$ and the z axis is vertical. Evaluate the simplest possible y component of velocity. Calculate the fluid acceleration and determine the pressure gradient at point $(x, y) = (2, 1)$. Find the pressure distribution along the positive x axis.

Given: Velocity field

Find: Simplest y component of velocity; Acceleration of particle and pressure gradient at (2,1); pressure on x axis

Solution:

Basic equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\\text{acceleration}}} \qquad \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

For this flow $u(x,y) = A\cdot x$ $\qquad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ $\qquad$ so $\qquad v(x,y) = -\int \frac{\partial}{\partial x}u\,dy = -\int A\,dy = -A\cdot y + c$

Hence $v(x,y) = -A\cdot y$ $\qquad$ is the simplest y component of velocity

For acceleration

$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = A\cdot x\cdot\frac{\partial}{\partial x}(A\cdot x) + (-A\cdot y)\cdot\frac{\partial}{\partial y}(A\cdot x) = A^2\cdot x \qquad a_x = A^2\cdot x$$

$$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v = A\cdot x\cdot\frac{\partial}{\partial x}(-A\cdot y) + (-A\cdot y)\cdot\frac{\partial}{\partial y}(-A\cdot y) \qquad a_y = A^2\cdot y$$

Hence at (2,1) $\qquad a_x = \left(\frac{2}{s}\right)^2 \times 2\cdot m \qquad a_y = \left(\frac{2}{s}\right)^2 \times 1\cdot m \qquad a_x = 8\frac{m}{s^2} \qquad a_y = 4\frac{m}{s^2}$

$$a = \sqrt{a_x^{\,2} + a_y^{\,2}} \qquad \theta = \operatorname{atan}\left(\frac{a_y}{a_x}\right) \qquad a = 8.94\frac{m}{s^2} \qquad \theta = 26.6\cdot deg$$

For the pressure gradient

$$\frac{\partial}{\partial x}p = \rho\cdot g_x - \rho\cdot a_x = -1.50\cdot\frac{kg}{m^3} \times 8\cdot\frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} \qquad \frac{\partial}{\partial x}p = -12\cdot\frac{Pa}{m}$$

$$\frac{\partial}{\partial y}p = \rho\cdot g_y - \rho\cdot a_y = -1.50\cdot\frac{kg}{m^3} \times 4\cdot\frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} \qquad \frac{\partial}{\partial y}p = -6\cdot\frac{Pa}{m}$$

$$\frac{\partial}{\partial z}p = \rho\cdot g_z - \rho\cdot a_z = 1.50 \times \frac{kg}{m^3} \times (-9.81)\cdot\frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} \qquad \frac{\partial}{\partial y}p = -14.7\cdot\frac{Pa}{m}$$

For the pressure on the x axis $\qquad dp = \frac{\partial}{\partial x}p \qquad p - p_0 = \int_0^x (\rho\cdot g_x - \rho\cdot a_x)\,dx = \int_0^x (-\rho\cdot A^2\cdot x)\,dx = -\frac{1}{2}\cdot\rho\cdot A^2\cdot x^2$

$$p(x) = p_0 - \frac{1}{2}\cdot\rho\cdot A^2\cdot x^2 \qquad p(x) = 190\cdot kPa - \frac{1}{2}\cdot 1.5\cdot\frac{kg}{m^3} \times \left(\frac{2}{s}\right)^2 \times \frac{N\cdot s^2}{kg\cdot m} \times x^2 \qquad p(x) = 190 - \frac{3}{1000}\cdot x^2$$

(p in kPa, x in m)

Problem 6.8

6.8 The velocity field for a plane source located distance $h = 1$ m above an infinite wall aligned along the x axis is given by

$$\vec{V} = \frac{q}{2\pi[x^2 + (y-h)^2]}[x\hat{i} + (y-h)\hat{j}] + \frac{q}{2\pi[x^2 + (y+h)^2]}[x\hat{i} + (y+h)\hat{j}]$$

where $q = 2$ m^3/s/m. The fluid density is 1000 kg/m^3 and body forces are negligible. Derive expressions for the velocity and acceleration of a fluid particle that moves along the wall, and plot from $x = 0$ to $x = +10h$. Verify that the velocity and acceleration normal to the wall are zero. Plot the pressure gradient $\partial p/\partial x$ along the wall. Is the pressure gradient along the wall adverse (does it oppose fluid motion) or not?

Given: Velocity field

Find: Expressions for velocity and acceleration along wall; plot; verify vertical components are zero; plot pressure gradient

Solution:

The given data is $q = 2 \cdot \frac{\frac{m^3}{s}}{m}$ $\quad h = 1 \cdot m \quad \rho = 1000 \cdot \frac{kg}{m^3}$

$$u = \frac{q \cdot x}{2 \cdot \pi \left[x^2 + (y-h)^2\right]} + \frac{q \cdot x}{2 \cdot \pi \left[x^2 + (y+h)^2\right]} \qquad v = \frac{q \cdot (y-h)}{2 \cdot \pi \left[x^2 + (y-h)^2\right]} + \frac{q \cdot (y+h)}{2 \cdot \pi \left[x^2 + (y+h)^2\right]}$$

The governing equation for acceleration is

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\text{total acceleration of a particle}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective acceleration}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local acceleration}}$$

For steady, 2D flow this reduces to (after considerable math!)

x - component
$$a_x = u \cdot \frac{\partial}{\partial x} u + v \cdot \frac{\partial}{\partial y} u = -\frac{q^2 \cdot x \cdot \left[\left(x^2 + y^2\right)^2 - h^2 \cdot \left(h^2 - 4 \cdot y^2\right)\right]}{\left[x^2 + (y+h)^2\right]^2 \cdot \left[x^2 + (y-h)^2\right]^2 \cdot \pi^2}$$

y - component
$$a_y = u \cdot \frac{\partial}{\partial x} v + v \cdot \frac{\partial}{\partial y} v = -\frac{q^2 \cdot y \cdot \left[\left(x^2 + y^2\right)^2 - h^2 \cdot \left(h^2 + 4 \cdot x^2\right)\right]}{\pi^2 \cdot \left[x^2 + (y+h)^2\right]^2 \cdot \left[x^2 + (y-h)^2\right]^2}$$

For motion along the wall $\quad y = 0 \cdot m$

$$u = \frac{q \cdot x}{\pi \cdot \left(x^2 + h^2\right)} \qquad v = 0 \quad \text{(No normal velocity)} \qquad a_x = -\frac{q^2 \cdot x \cdot \left(x^2 - h^2\right)}{\pi^2 \cdot \left(x^2 + h^2\right)^3} \qquad a_y = 0 \quad \text{(No normal acceleration)}$$

The governing equation (assuming inviscid flow) for computing the pressure gradient is $\rho \frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$ (6.1)

Hence, the component of pressure gradient (neglecting gravity) along the wall is

$$\frac{\partial}{\partial x} p = -\rho \cdot \frac{Du}{Dt} \qquad \frac{\partial}{\partial x} p = \frac{\rho \cdot q^2 \cdot x \cdot \left(x^2 - h^2\right)}{\pi^2 \cdot \left(x^2 + h^2\right)^3}$$

The plots of velocity, acceleration, and pressure gradient are shown in the associated *Excel* workbook. From the plots it is clear that the fluid experiences an adverse pressure gradient from the origin to x = 1 m, then a negative one promoting fluid acceleration. If flow separates, it will likely be in the region x = 0 to x = h.

6.8 The velocity field for a plane source located distance $h = 1$ m above an infinite wall aligned along the x axis is given by

$$\vec{V} = \frac{q}{2\pi[x^2 + (y-h)^2]}[x\hat{i} + (y-h)\hat{j}] + \frac{q}{2\pi[x^2 + (y+h)^2]}[x\hat{i} + (y+h)\hat{j}]$$

where $q = 2$ m^3/s/m. The fluid density is 1000 kg/m^3 and body forces are negligible. Derive expressions for the velocity and acceleration of a fluid particle that moves along the wall, and plot from $x = 0$ to $x = +10h$. Verify that the velocity and acceleration normal to the wall are zero. Plot the pressure gradient $\partial p/\partial x$ along the wall. Is the pressure gradient along the wall adverse (does it oppose fluid motion) or not?

Given: Velocity field

Find: Expressions for velocity and acceleration along wall; plot; verify vertical components are zero; plot pressure gradient

Solution:

The velocity, acceleration and pressure gradient are given by

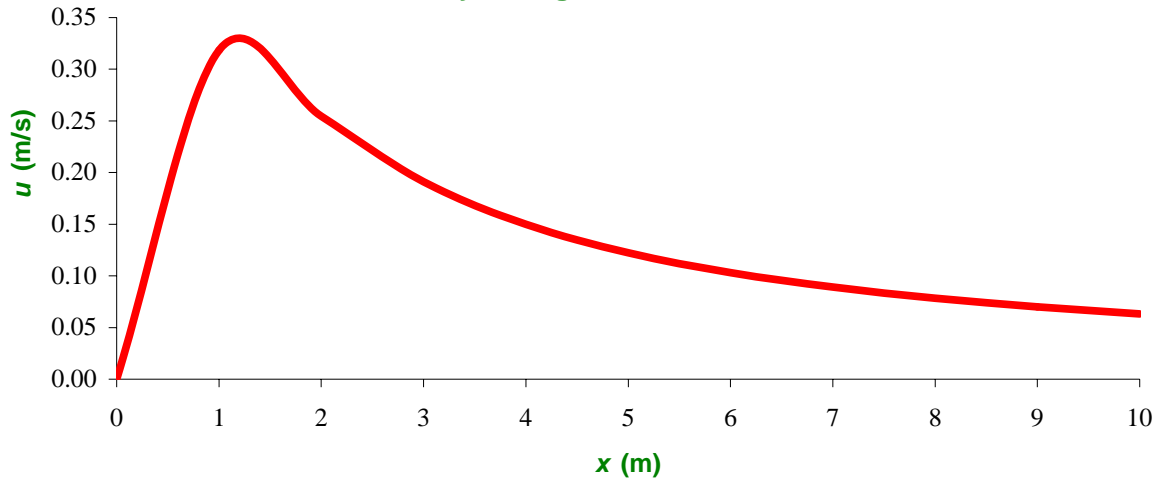
$$u = \frac{q \cdot x}{\pi \cdot (x^2 + h^2)}$$

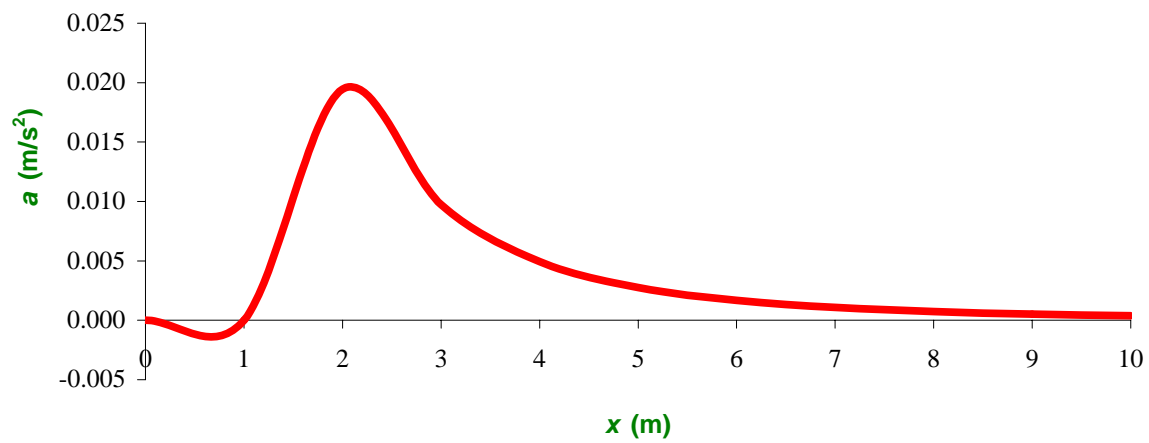
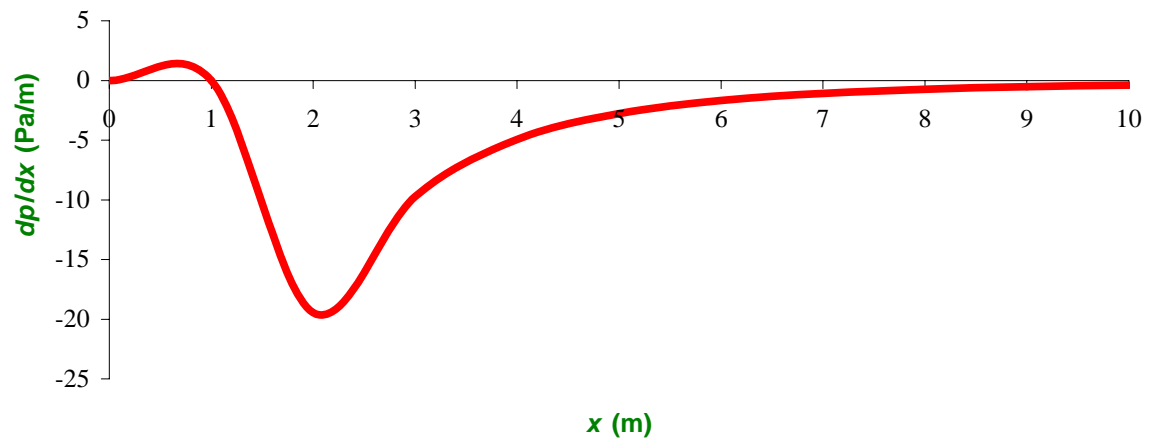
$$a_x = -\frac{q^2 \cdot x \cdot (x^2 - h^2)}{\pi^2 \cdot (x^2 + h^2)^3}$$

$$\frac{\partial}{\partial x} p = \frac{\rho \cdot q^2 \cdot x \cdot (x^2 - h^2)}{\pi^2 \cdot (x^2 + h^2)^3}$$

$q =$ 2 m^3/s/m
$h =$ 1 m
$\rho =$ 1000 kg/m^3

x (m)	*u* (m/s)	*a* (m/s^2)	*dp*/*dx* (Pa/m)
0.0	0.00	0.00000	0.00
1.0	0.32	0.00000	0.00
2.0	0.25	0.01945	-19.45
3.0	0.19	0.00973	-9.73
4.0	0.15	0.00495	-4.95
5.0	0.12	0.00277	-2.77
6.0	0.10	0.00168	-1.68
7.0	0.09	0.00109	-1.09
8.0	0.08	0.00074	-0.74
9.0	0.07	0.00053	-0.53
10.0	0.06	0.00039	-0.39

Velocity Along Wall Near A Source
u (m/s)
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
0
1
2
3
4
5
6
7
8
9
10
x (m)

Acceleration Along Wall Near A Source
a (m/s2)
0.025
0.020
0.015
0.010
0.005
0.000
-0.005
0
1
2
3
4
5
6
7
8
9
10
x (m)

Pressure Gradient Along Wall
dp/dx (Pa/m)
5
0
-5
-10
-15
-20
-25
0
1
2
3
4
5
6
7
8
9
10
x (m)

Given: The velocity distribution in a steady, 2-D flow field in the xy plane is given by $\vec{V} = (Ax - B)\hat{i} + (C - Ay)\hat{j}$, where $A = 2\ s^{-1}$, $B = 5\ m \cdot s^{-1}$, $C = 3\ m \cdot s^{-1}$, and the body force distribution is $\vec{g} = -g\hat{k}$

Find: (a) Does the velocity field represent the flow of an incompressible fluid?
(b) Find the stagnation point of the flow field.
(c) Obtain an expression for the pressure gradient.
(d) Evaluate Δp between origin and point (1,3) if $\rho = 1.2\ kg/m^3$

Solution:

(a) Apply the continuity equation, $\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} = 0$, for the given conditions. If ρ = constant, then

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 = \frac{\partial}{\partial x}(2x - 5) + \frac{\partial}{\partial y}(3 - 2y) = 2 - 2 = 0 \checkmark$$

$\therefore$ velocity field represents an incompressible flow

(b) At the stagnation point, $\vec{V} = 0$. For $\vec{V} = 0$, then

$$u = 2x - 5 = 0 \quad \text{and} \quad v = (3 - 2y) = 0$$

Thus stagnation point is at $(x, y) = \left(\frac{5}{2}, \frac{3}{2}\right)$

(c) Euler's equation, $\rho\vec{g} - \nabla p = \rho \frac{D\vec{V}}{Dt}$, can be used to obtain an expression for the pressure gradient

$$\nabla p = \rho\vec{g} - \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \rho\left[\frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}\right]$$

(steady: $\frac{\partial \vec{V}}{\partial t} = 0$; $w = 0$)

$$\nabla p = \rho\left[\vec{g} - u\frac{\partial \vec{V}}{\partial x} - v\frac{\partial \vec{V}}{\partial y}\right] = \rho\left[-g\hat{k} - (2x - 5)\,2\hat{i} - (3 - 2y)(-2\hat{j})\right]$$

$$\nabla p = -\rho\left[(4x - 10)\hat{i} + (4y - 6)\hat{j} + g\hat{k}\right] \qquad \nabla p$$

(d) Since $p = p(x, y, z)$ we can write

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz = -\rho(4x - 10)dx - \rho(4y - 6)dy - \rho g\,dz$$

We can integrate to obtain Δp between any two points in the field if, and only if, the integral of the right hand side is independent of the path of integration. This is true for the present case.

$$\therefore p_{1,3} - p_{0,0} = -\rho\left\{\int_0^1 (4x - 10)dx + \int_0^3 (4y - 6)dy\right\} = -\rho\left\{\left[2x^2 - 10x\right]_0^1 + \left[2y^2 - 6y\right]_0^3\right\}$$

$$= -\rho\{-8 - 0\} = 8\rho$$

$$p_{1,3} - p_{0,0} = 8\ \frac{m^2}{s^2} \times 1.2\ \frac{kg}{m^3} \cdot \frac{N \cdot s^2}{kg \cdot m} = 9.6\ N/m^2 \qquad \Delta p$$

Problem 6.10

Given: Frictionless, incompressible flow field with

$$\vec{V} = Ax\hat{i} - Ay\hat{j}$$
$$\vec{g} = -g\hat{k}$$

At $(0,0,0)$ $p = p_0$

Find: Expression for the pressure field $p(x,y,z)$

Solution:

Basic equations: $\rho\vec{g} - \nabla p = \rho\frac{D\vec{V}}{Dt}$

$$\frac{D\vec{V}}{Dt} = \frac{\partial\vec{V}}{\partial t} + u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}$$

$$\nabla p = \rho\left(\vec{g} - \frac{D\vec{V}}{Dt}\right) = \rho\left(-g\hat{k} - u\frac{\partial\vec{V}}{\partial x} - v\frac{\partial\vec{V}}{\partial y}\right)$$
$$= -\rho\left[g\hat{k} + Ax(A\hat{i}) - Ay(-A\hat{j})\right]$$

$$\nabla p = -\rho\left[A^2x\hat{i} + A^2y\hat{j} + g\hat{k}\right]$$

$$\hat{i}\frac{\partial p}{\partial x} + \hat{j}\frac{\partial p}{\partial y} + \hat{k}\frac{\partial p}{\partial z} = -\rho\left[A^2x\hat{i} + A^2y\hat{j} + g\hat{k}\right]$$

$$\frac{\partial p}{\partial x} = -\rho A^2 x \qquad \frac{\partial p}{\partial y} = -\rho A^2 y \qquad \frac{\partial p}{\partial z} = -\rho g$$

$$p = p(x,y,z)$$
$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz = -\rho A^2 x\,dx - \rho A^2 y\,dy - \rho g\,dz$$

* $$p - p_0 = \int_{p_0}^{p} dp = -\int_0^x \rho A^2 x\,dx - \int_0^y \rho A^2 y\,dy - \int_0^z \rho g\,dz$$

$$p - p_0 = -\rho\left[\frac{A^2x^2}{2} + \frac{A^2y^2}{2} + gz\right]$$

$$p = p_0 - \rho\left[\frac{A^2x^2}{2} + \frac{A^2y^2}{2} + gz\right] \qquad p$$

* We can integrate to obtain Δp between any two points in the flow field if, and only if, the integral of the right hand side is independent of the path of integration. This is true for the present case

Problem 6.11

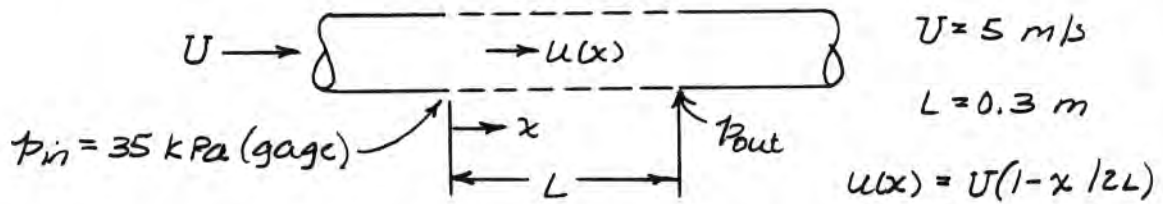
Given: Porous pipe with liquid ($\mu = 0$, $\rho = 900\ kg/m^3$)

$U \rightarrow$ $\rightarrow u(x)$

$p_{in} = 35\ kPa$ (gage) $\rightarrow x$ p_{out} L

$U = 5\ m/s$

$L = 0.3\ m$

$u(x) = U(1 - x/2L)$

Find: (a) Expression for acceleration along ℄.

(b) Expression for pressure gradient along ℄.

(c) Evaluate p_{out}

Solution: Computing equations (acceleration and Euler in x-direction)

$$a_{px} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}; \quad \rho g_x - \frac{\partial p}{\partial x} = \rho a_{px}$$

($v\frac{\partial u}{\partial y} = 0(1)$, $w\frac{\partial u}{\partial z} = 0(1)$, $\frac{\partial u}{\partial t} = 0(2)$, $\rho g_x = 0(3)$)

Assumptions: (1) $v = w = 0$ along ℄

(2) steady flow

(3) $g_x = 0$

Then

$$a_{px} = u\frac{\partial u}{\partial x} = U\left(1 - \frac{x}{2L}\right)U\left(-\frac{1}{2L}\right) = -\frac{U^2}{2L}\left(1 - \frac{x}{2L}\right)$$

a_{px}

From Euler

$$\frac{\partial p}{\partial x} = \frac{dp}{dx} = -\rho a_{px} = \rho\frac{U^2}{2L}\left(1 - \frac{x}{2L}\right)$$

$\frac{dp}{dx}$

Integrating,

$$p_{out} - p_{in} = \int_0^L \frac{dp}{dx}\,dx = \rho\frac{U^2}{2L}\int_0^L\left(1 - \frac{x}{2L}\right)dx = \rho\frac{U^2}{2L}\left(x - \frac{x^2}{4L}\right)\Big]_0^L$$

or

$$p_{out} = p_{in} + \frac{\rho U^2}{2L}\left(\frac{3}{4}L\right) = p_{in} + \frac{3}{8}\rho U^2$$

$$= 35\ kPa + \frac{3}{8} \times 900\ \frac{kg}{m^3} \times (5)^2\ \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$p_{out} = 43.4\ kPa\ (gage)$$

p_{out}

Problem 6.12

Given: Liquid, ρ = constant and negligible viscosity, is pumped at total volume flow rate, Q, through two small holes into the narrow gap between closely space parallel plates. The liquid flowing away from the holes has only radial motion. Flow may be assumed uniform at any section.

(a) Show that $V_r = Q/2\pi r h$, where h is the spacing between the plates.

(b) Obtain an expression for a_r and $\partial p/\partial r$

Solution:

Apply the conservation of mass to a CV with outer edge at r.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$ (the first term ≈ 0 (1))

Assumptions: (1) steady flow
(2) incompressible flow
(3) uniform flow at each section

Then

$$0 = \int_{CS} \vec{V} \cdot d\vec{A} = -2 \times \frac{Q}{2} + V_r 2\pi r h$$

and $$V_r = \frac{Q}{2\pi r h}$$ ← V_r

From Eq. 6.4a

$$g_r - \frac{1}{\rho}\frac{\partial p}{\partial r} = a_r = \frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z \frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}$$

Since $V_r = V_r(r)$ and $V_\theta = 0$, then

$$a_r = V_r \frac{\partial V_r}{\partial r} = \frac{Q}{2\pi r h}\left[\frac{Q}{2\pi h}\left(-\frac{1}{r^2}\right)\right] = -\left(\frac{Q}{2\pi r h}\right)^2 \frac{1}{r}$$

$$a_r = -\frac{V_r^2}{r}$$ ← a_r

Since $g_r = 0$, then

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} = a_r$$

$$\frac{\partial p}{\partial r} = -\rho a_r = \rho \frac{V_r^2}{r}$$ ← $\frac{\partial p}{\partial r}$

Problem 6.13

6.13 The velocity field for a plane vortex sink is given by $\vec{V} = (-q/2\pi r)\hat{e}_r + (K/2\pi r)\hat{e}_\theta$, where $q = 2$ m^3/s/m and $K = 1$ m^3/s/m. The fluid density is 1000 kg/m^3. Find the acceleration at $(1, 0)$, $(1, \pi/2)$, and $(2, 0)$. Evaluate ∇p under the same conditions.

Given: Velocity field

Find: The acceleration at several points; evaluate pressure gradient

Solution:

The given data is $q = 2 \cdot \frac{\frac{m^3}{s}}{m}$ $\quad K = 1 \cdot \frac{\frac{m^3}{s}}{m}$ $\quad \rho = 1000 \cdot \frac{kg}{m^3}$ $\quad V_r = -\frac{q}{2 \cdot \pi \cdot r}$ $\quad V_\theta = \frac{K}{2 \cdot \pi \cdot r}$

The governing equations for this 2D flow are

$$\rho a_r = \rho\left(\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z \frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}\right) = \rho g_r - \frac{\partial p}{\partial r} \quad (6.3a)$$

$$\rho a_\theta = \rho\left(\frac{\partial V_\theta}{\partial t} + V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z \frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r}\right) = \rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} \quad (6.3b)$$

The total acceleration for this steady flow is then

r - component $\quad a_r = V_r \cdot \frac{\partial}{\partial r} V_r + \frac{V_\theta}{r} \cdot \frac{\partial}{\partial \theta} V_r \qquad a_r = -\frac{q^2}{4 \cdot \pi^2 \cdot r^3}$

θ - component $\quad a_\theta = V_r \cdot \frac{\partial}{\partial r} V_\theta + \frac{V_\theta}{r} \cdot \frac{\partial}{\partial \theta} V_\theta \qquad a_\theta = \frac{q \cdot K}{4 \cdot \pi^2 \cdot r^3}$

Evaluating at point (1,0) $\quad a_r = -0.101 \frac{m}{s^2} \qquad a_\theta = 0.0507 \frac{m}{s^2}$

Evaluating at point (1,π/2) $\quad a_r = -0.101 \frac{m}{s^2} \qquad a_\theta = 0.0507 \frac{m}{s^2}$

Evaluating at point (2,0) $\quad a_r = -0.0127 \frac{m}{s^2} \qquad a_\theta = 0.00633 \frac{m}{s^2}$

From Eq. 6.3, pressure gradient is $\quad \frac{\partial}{\partial r} p = -\rho \cdot a_r \qquad \frac{\partial}{\partial r} p = \frac{\rho \cdot q^2}{4 \cdot \pi^2 \cdot r^3}$

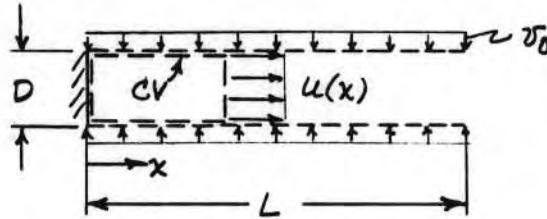
$$\frac{1}{r} \cdot \frac{\partial}{\partial \theta} p = -\rho \cdot a_\theta \qquad \frac{1}{r} \cdot \frac{\partial}{\partial \theta} p = -\frac{\rho \cdot q \cdot K}{4 \cdot \pi^2 \cdot r^3}$$

Evaluating at point (1,0) $\quad \frac{\partial}{\partial r} p = 101 \cdot \frac{Pa}{m} \qquad \frac{1}{r} \cdot \frac{\partial}{\partial \theta} p = -50.5 \cdot \frac{Pa}{m}$

Evaluating at point (1,π/2) $\quad \frac{\partial}{\partial r} p = 101 \cdot \frac{Pa}{m} \qquad \frac{1}{r} \cdot \frac{\partial}{\partial \theta} p = -50.5 \cdot \frac{Pa}{m}$

Evaluating at point (2,0) $\quad \frac{\partial}{\partial r} p = 12.7 \cdot \frac{Pa}{m} \qquad \frac{1}{r} \cdot \frac{\partial}{\partial \theta} p = -6.33 \cdot \frac{Pa}{m}$

Problem 6.14

Given: Circular tube with porous wall; incompressible flow, uniform in x direction.

Find: (a) Algebraic expression for a_{p_x} at x.
(b) Pressure gradient at x.
(c) Integrate to obtain p at $x = 0$.

Solution: Apply conservation of mass using the CV shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$ (first term $= 0(1)$)

$$a_{p_x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \;;\; -\frac{\partial p}{\partial x} + \rho g_x = \rho a_{p_x}$$

($v\frac{\partial u}{\partial y} \approx 0(5)$, $w\frac{\partial u}{\partial z} \approx 0(5)$, $\frac{\partial u}{\partial t} = 0(1)$)

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each cross-section
(4) Horizontal; $g_x = 0$
(5) $v \approx 0$ in channel ($w \approx 0$ too)
(6) Inviscid flow

Then
$$\int \vec{V} \cdot d\vec{A} = \left\{-|v_0 \pi D x|\right\} + \left\{+\left|u \frac{\pi D^2}{4}\right|\right\} = 0 \quad \text{or} \quad u(x) = 4 v_0 \frac{x}{D}$$

and
$$a_{p_x} = 4 v_0 \frac{x}{D}\left(4 v_0 \frac{1}{D}\right) = 16 v_0^2 \frac{x}{D^2}$$

a_{p_x}

From the Euler equation,
$$-\frac{\partial p}{\partial x} = \rho a_{p_x} \quad \text{so} \quad \frac{\partial p}{\partial x} = -\rho a_{p_x} = -16 \rho v_0^2 \frac{x}{D^2}$$

$\frac{\partial p}{\partial x}$

Since $v \approx w \approx 0$, then $p(x)$ and $dp = \frac{\partial p}{\partial x} dx$. Integrating
$$\int_0^L dp = p_L - p(0) = \int_0^L -16 \rho v_0^2 \frac{x}{D^2} \, dx = -\frac{16 \rho v_0^2}{D^2} \left.\frac{x^2}{2}\right]_0^L = -\frac{8 \rho v_0^2 L^2}{D^2}$$

Thus, since $p_L = p_{atm}$, the gage pressure at $x = 0$ is
$$p(0) = 8 \rho v_0^2 \left(\frac{L}{D}\right)^2$$

$p(0)$

Problem 6.15

6.15 An incompressible liquid with negligible viscosity and density $\rho = 850\,\text{kg/m}^3$ flows steadily through a horizontal pipe. The pipe cross-section area linearly varies from 100 cm^2 to 25 cm^2 over a length of 2 m. Develop an expression for and plot the pressure gradient and pressure versus position along the pipe, if the inlet centerline velocity is 1 m/s and inlet pressure is 250 kPa.

Given: Flow in a pipe with variable area

Find: Expression for pressure gradient and pressure; Plot them

Solution:

Assumptions: 1) Incompressible flow 2) Flow profile remains unchanged so centerline velocity can represent average velocity

Basic equations

$$Q = V \cdot A \qquad \vec{a}_p = \underset{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}}{\frac{D\vec{V}}{Dt}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underset{\substack{\text{local}\\\text{acceleration}}}{\frac{\partial \vec{V}}{\partial t}} \qquad \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

For this 1D flow

$$Q = u_i \cdot A_i = u \cdot A \qquad A = A_i - \frac{(A_i - A_e)}{L} \cdot x \qquad \text{so} \qquad u(x) = u_i \cdot \frac{A_i}{A} = u_i \cdot \frac{A_i}{A_i - \left[\frac{(A_i - A_e)}{L} \cdot x\right]}$$

$$a_x = u \cdot \frac{\partial}{\partial x}u + v \cdot \frac{\partial}{\partial y}u = u_i \cdot \frac{A_i}{A_i - \left[\frac{(A_i - A_e)}{L} \cdot x\right]} \cdot \frac{\partial}{\partial x}\left[u_i \cdot \frac{A_i}{A_i - \left[\frac{(A_i - A_e)}{L} \cdot x\right]}\right] = \frac{A_i^2 \cdot L^2 \cdot u_i^2 \cdot (A_e - A_i)}{(A_i \cdot L + A_e \cdot x - A_i \cdot x)^3}$$

For the pressure

$$\frac{\partial}{\partial x}p = -\rho \cdot a_x - \rho \cdot g_x = -\frac{\rho \cdot A_i^2 \cdot L^2 \cdot u_i^2 \cdot (A_e - A_i)}{(A_i \cdot L + A_e \cdot x - A_i \cdot x)^3}$$

and

$$dp = \frac{\partial}{\partial x}p \cdot dx \qquad p - p_i = \int_0^x \frac{\partial}{\partial x}p\, dx = \int_0^x -\frac{\rho \cdot A_i^2 \cdot L^2 \cdot u_i^2 \cdot (A_e - A_i)}{(A_i \cdot L + A_e \cdot x - A_i \cdot x)^3}\, dx$$

This is a tricky integral, so instead consider the following:

$$\frac{\partial}{\partial x}p = -\rho \cdot a_x = -\rho \cdot u \cdot \frac{\partial}{\partial x}u = -\frac{1}{2} \cdot \rho \cdot \frac{\partial}{\partial x}\left(u^2\right)$$

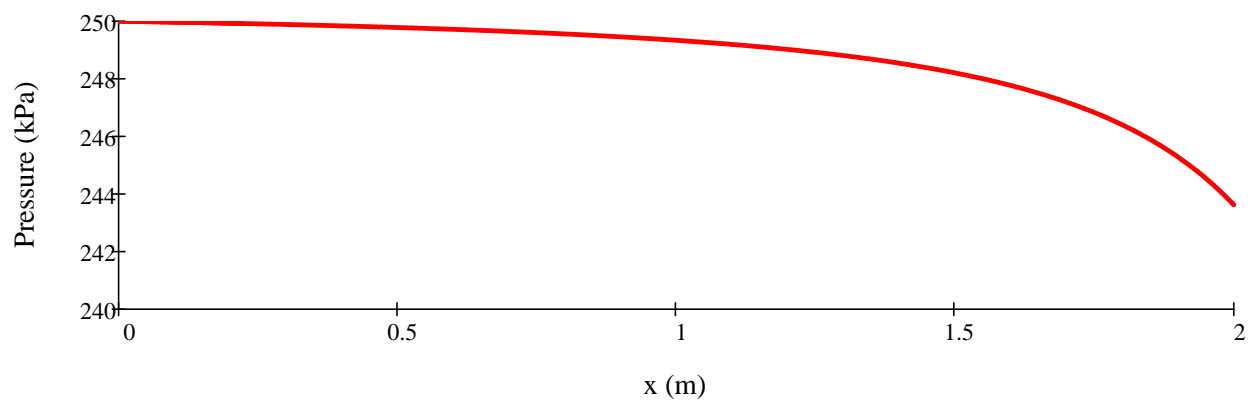
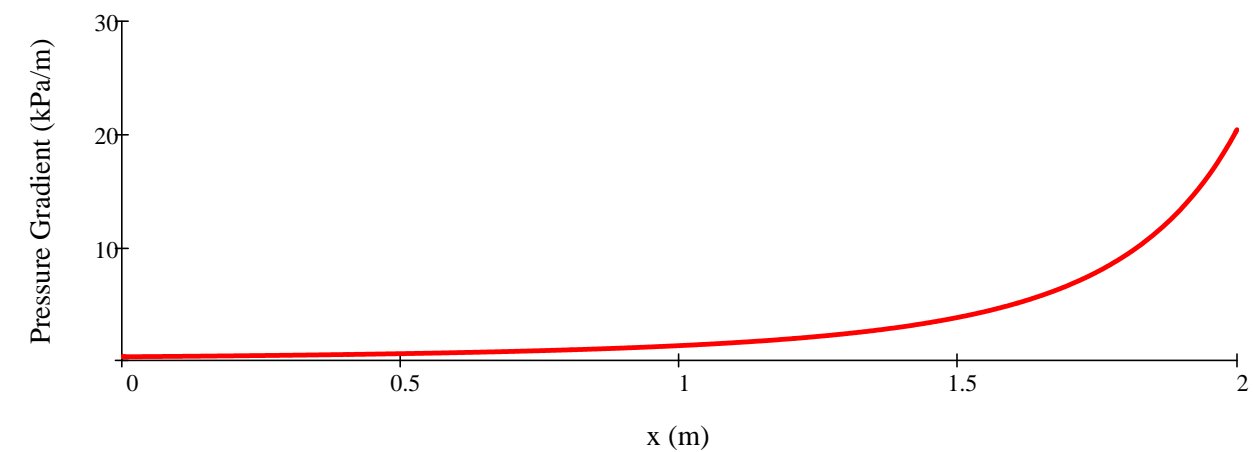
Hence

$$p - p_i = \int_0^x \frac{\partial}{\partial x}p\, dx = -\frac{\rho}{2} \cdot \int_0^x \frac{\partial}{\partial x}\left(u^2\right) dx = \frac{\rho}{2} \cdot \left(u(x=0)^2 - u(x)^2\right)$$

$$p(x) = p_i + \frac{\rho}{2} \cdot \left(u_i^2 - u(x)^2\right)$$

which we recognise as the Bernoulli equation!

$$p(x) = p_i + \frac{\rho \cdot u_i^2}{2} \cdot \left[1 - \left[\frac{A_i}{A_i - \left[\frac{(A_i - A_e)}{L} \cdot x\right]}\right]^2\right]$$

The following plots can be done in *Excel*

Problem 6.16

6.16 An incompressible liquid with negligible viscosity and density $\rho = 750\ \text{kg/m}^3$ flows steadily through a 10-m-long convergent-divergent section of pipe for which the area varies as

$$A(x) = A_0(1 + e^{-x/a} - e^{-x/2a})$$

where $A_0 = 0.1\ \text{m}^2$ and $a = 1$ m. Develop an expression for and plot the pressure gradient and pressure versus position along the pipe, if the inlet centerline velocity is 1 m/s and inlet pressure is 200 kPa.

Given: Flow in a pipe with variable area

Find: Expression for pressure gradient and pressure; Plot them

Solution:

Assumptions: 1) Incompressible flow 2) Flow profile remains unchanged so centerline velocity can represent average velocity

Basic equations $\quad Q = V \cdot A \qquad \vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total acceleration}\\ \text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective acceleration}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local acceleration}} \qquad \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$

For this 1D flow $\quad Q = u_0 \cdot A_0 = u \cdot A \qquad A(x) = A_0 \cdot \left(1 + e^{-\frac{x}{a}} - e^{-\frac{x}{2 \cdot a}}\right)$

so $\quad u(x) = u_0 \cdot \frac{A_0}{A} = \frac{u_0}{\left(1 + e^{-\frac{x}{a}} - e^{-\frac{x}{2 \cdot a}}\right)}$

$$a_x = u \cdot \frac{\partial}{\partial x}u + v \cdot \frac{\partial}{\partial y}u = \frac{u_0}{\left(1 + e^{-\frac{x}{a}} - e^{-\frac{x}{2 \cdot a}}\right)} \cdot \frac{\partial}{\partial x}\left[\frac{u_0}{\left(1 + e^{-\frac{x}{a}} - e^{-\frac{x}{2 \cdot a}}\right)}\right] = \frac{u_0^2 \cdot e^{-\frac{x}{2 \cdot a}} \cdot \left(2 \cdot e^{-\frac{x}{2 \cdot a}} - 1\right)}{2 \cdot a \cdot \left(e^{-\frac{x}{a}} - e^{-\frac{x}{2 \cdot a}} + 1\right)^3}$$

For the pressure $\quad \frac{\partial}{\partial x}p = -\rho \cdot a_x - \rho \cdot g_x = -\frac{\rho \cdot u_0^2 \cdot e^{-\frac{x}{2 \cdot a}} \cdot \left(2 \cdot e^{-\frac{x}{2 \cdot a}} - 1\right)}{2 \cdot a \cdot \left(e^{-\frac{x}{a}} - e^{-\frac{x}{2 \cdot a}} + 1\right)^3}$

and $\quad dp = \frac{\partial}{\partial x}p \cdot dx \qquad p - p_i = \int_0^x \frac{\partial}{\partial x}p\, dx = \int_0^x -\frac{\rho \cdot u_0^2 \cdot e^{-\frac{x}{2 \cdot a}} \cdot \left(2 \cdot e^{-\frac{x}{2 \cdot a}} - 1\right)}{2 \cdot a \cdot \left(e^{-\frac{x}{a}} - e^{-\frac{x}{2 \cdot a}} + 1\right)^3}\, dx$

This is a tricky integral, so instead consider the following: $\quad \frac{\partial}{\partial x}p = -\rho \cdot a_x = -\rho \cdot u \cdot \frac{\partial}{\partial x}u = -\frac{1}{2} \cdot \rho \cdot \frac{\partial}{\partial x}\left(u^2\right)$

Hence

$$p - p_i = \int_0^x \frac{\partial}{\partial x} p \, dx = -\frac{\rho}{2} \cdot \int_0^x \frac{\partial}{\partial x}\left(u^2\right) dx = \frac{\rho}{2} \cdot \left(u(x=0)^2 - u(x)^2\right)$$

$$p(x) = p_0 + \frac{\rho}{2} \cdot \left(u_0{}^2 - u(x)^2\right)$$

which we recognise as the Bernoulli equation!

$$p(x) = p_0 + \frac{\rho \cdot u_0{}^2}{2} \cdot \left[1 - \left[\frac{1}{\left(1 + e^{-\frac{x}{a}} - e^{-\frac{x}{2 \cdot a}}\right)}\right]^2\right]$$

The following plots can be done in *Excel*

Problem 6.17

6.17 A nozzle for an incompressible, inviscid fluid of density $\rho = 1000\ \text{kg/m}^3$ consists of a converging section of pipe. At the inlet the diameter is $D_i = 100$ mm, and at the outlet the diameter is $D_o = 20$ mm. The nozzle length is $L = 500$ mm, and the diameter decreases linearly with distance x along the nozzle. Derive and plot the acceleration of a fluid particle, assuming uniform flow at each section, if the speed at the inlet is $V_i = 1$ m/s. Plot the pressure gradient through the nozzle, and find its maximum absolute value. If the pressure gradient must be no greater than 5 MPa/m in absolute value, how long would the nozzle have to be?

Given: Nozzle geometry

Find: Acceleration of fluid particle; Plot; Plot pressure gradient; find L such that pressure gradient < 5 MPa/m in absolute value

Solution:

The given data is $D_i = 0.1\cdot\text{m}$ $\quad D_o = 0.02\cdot\text{m}$ $\quad L = 0.5\cdot\text{m}$ $\quad V_i = 1\cdot\frac{\text{m}}{\text{s}}$ $\quad \rho = 1000\cdot\frac{\text{kg}}{\text{m}^3}$

For a linear decrease in diameter
$$D(x) = D_i + \frac{D_o - D_i}{L}\cdot x$$

From continuity
$$Q = V\cdot A = V\cdot\frac{\pi}{4}\cdot D^2 = V_i\cdot\frac{\pi}{4}\cdot D_i^2 \qquad Q = 0.00785\frac{\text{m}^3}{\text{s}}$$

Hence
$$V(x)\cdot\frac{\pi}{4}\cdot D(x)^2 = Q \qquad V(x) = \frac{4\cdot Q}{\pi\cdot\left(D_i + \frac{D_o - D_i}{L}\cdot x\right)^2}$$

or
$$V(x) = \frac{V_i}{\left(1 + \frac{D_o - D_i}{L\cdot D_i}\cdot x\right)^2}$$

The governing equation for this flow is
$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} \qquad (6.2a)$$

or, for steady 1D flow, in the notation of the problem

$$a_x = V\cdot\frac{d}{dx}V = \frac{V_i}{\left(1 + \frac{D_o - D_i}{L\cdot D_i}\cdot x\right)^2}\cdot\frac{d}{dx}\frac{V_i}{\left(1 + \frac{D_o - D_i}{L\cdot D_i}\cdot x\right)^2} \qquad a_x(x) = -\frac{2\cdot V_i^2\cdot(D_o - D_i)}{D_i\cdot L\cdot\left[1 + \frac{(D_o - D_i)}{D_i\cdot L}\cdot x\right]^5}$$

This is plotted in the associated *Excel* workbook

From Eq. 6.2a, pressure gradient is

$$\frac{\partial}{\partial x}p = -\rho\cdot a_x \qquad \frac{\partial}{\partial x}p = \frac{2\cdot\rho\cdot V_i^2\cdot(D_o - D_i)}{D_i\cdot L\cdot\left[1 + \frac{(D_o - D_i)}{D_i\cdot L}\cdot x\right]^5}$$

This is also plotted in the associated *Excel* workbook. Note that the pressure gradient is always negative: separation is unlikely to occur in the nozzle

At the inlet $\frac{\partial}{\partial x}p = -3.2 \cdot \frac{kPa}{m}$ At the exit $\frac{\partial}{\partial x}p = -10 \cdot \frac{MPa}{m}$

To find the length *L* for which the absolute pressure gradient is no more than 5 MPa/m, we need to solve

$$\left|\frac{\partial}{\partial x}p\right| \le 5 \cdot \frac{MPa}{m} = \frac{2 \cdot \rho \cdot V_i^2 \cdot \left(D_o - D_i\right)}{D_i \cdot L \cdot \left[1 + \frac{\left(D_o - D_i\right)}{D_i \cdot L} \cdot x\right]^5}$$

with *x* = *L* m (the largest pressure gradient is at the outlet)

Hence $$L \ge \frac{2 \cdot \rho \cdot V_i^2 \cdot \left(D_o - D_i\right)}{D_i \cdot \left(\frac{D_o}{D_i}\right)^5 \cdot \left|\frac{\partial}{\partial x}p\right|}$$ $L \ge 1 \cdot m$

This result is also obtained using *Goal Seek* in the *Excel* workbook

Problem 6.17

6.17 A nozzle for an incompressible, inviscid fluid of density $\rho =$ 1000 kg/m^3 consists of a converging section of pipe. At the inlet the diameter is $D_i = 100$ mm, and at the outlet the diameter is $D_o =$ 20 mm. The nozzle length is $L = 500$ mm, and the diameter decreases linearly with distance x along the nozzle. Derive and plot the acceleration of a fluid particle, assuming uniform flow at each section, if the speed at the inlet is $V_i = 1$ m/s. Plot the pressure gradient through the nozzle, and find its maximum absolute value. If the pressure gradient must be no greater than 5 MPa/m in absolute value, how long would the nozzle have to be?

Given: Nozzle geometry

Find: Acceleration of fluid particle; Plot; Plot pressure gradient; find L such that pressure gradient < 5 MPa/m in absolute value

Solution:

The acceleration and pressure gradient are given by

$$a_x(x) = -\frac{2 \cdot V_i^2 \cdot (D_o - D_i)}{D_i \cdot L \cdot \left[1 + \frac{(D_o - D_i)}{D_i \cdot L} \cdot x\right]^5}$$

$$\frac{\partial}{\partial x} p = \frac{2 \cdot \rho \cdot V_i^2 \cdot (D_o - D_i)}{D_i \cdot L \cdot \left[1 + \frac{(D_o - D_i)}{D_i \cdot L} \cdot x\right]^5}$$

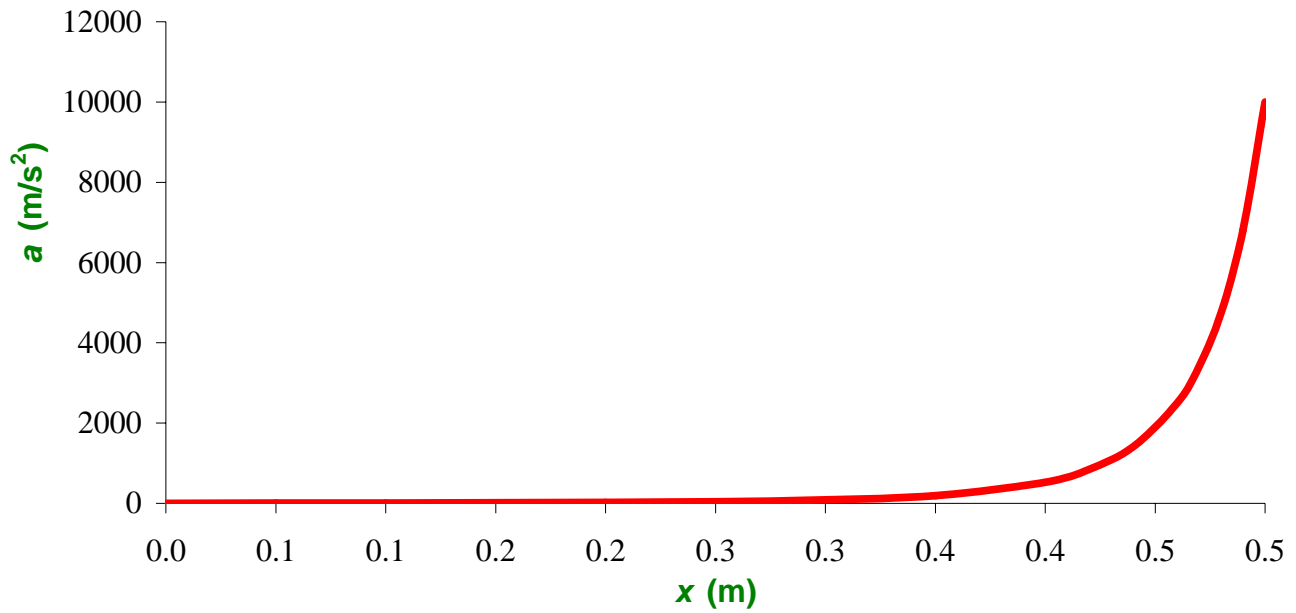
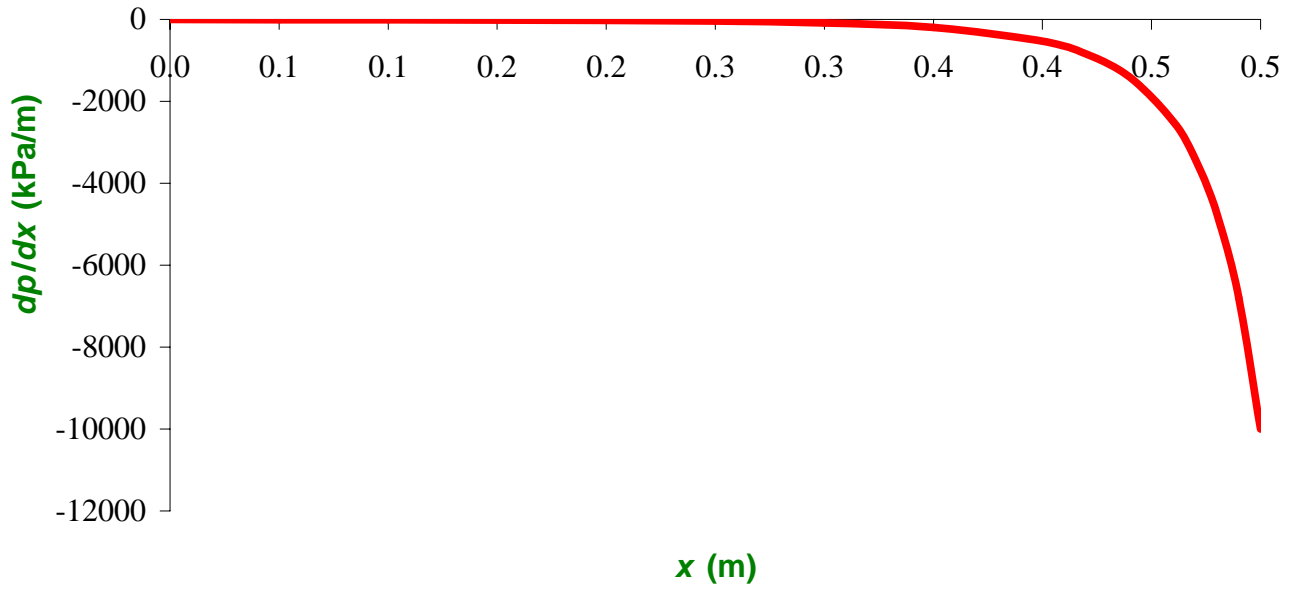
$D_i =$	0.1	m
$D_o =$	0.02	m
$L =$	0.5	m
$V_i =$	1	m/s
$\rho =$	1000	kg/m^3

x (m)	a (m/s^2)	dp/dx (kPa/m)
0.000	3.20	-3.20
0.050	4.86	-4.86
0.100	7.65	-7.65
0.150	12.6	-12.6
0.200	22.0	-22.0
0.250	41.2	-41.2
0.300	84.2	-84.2
0.350	194	-194
0.400	529	-529
0.420	843	-843
0.440	1408	-1408
0.460	2495	-2495
0.470	3411	-3411
0.480	4761	-4761
0.490	6806	-6806
0.500	10000	-10000

For the length L required for the pressure gradient to be less than 5 MPa/m (abs) use *Goal Seek*

$L =$ 1.00 m

x (m)	dp/dx (kPa/m)
1.00	-5000

Acceleration Through A Nozzle
12000
10000
8000
6000
4000
2000
0
a (m/s2)
0.0
0.1
0.1
0.2
0.2
0.3
0.3
0.4
0.4
0.5
0.5
x (m)

Pressure Gradient Along A Nozzle
0
-2000
-4000
-6000
-8000
-10000
-12000
dp/dx (kPa/m)
0.0
0.1
0.1
0.2
0.2
0.3
0.3
0.4
0.4
0.5
0.5
x (m)

Problem 6.18

[3]

6.18 A diffuser for an incompressible, inviscid fluid of density $\rho = 1000\ \text{kg/m}^3$ consists of a diverging section of pipe. At the inlet the diameter is $D_i = 0.25$ m, and at the outlet the diameter is $D_o = 0.75$ m. The diffuser length is $L = 1$ m, and the diameter increases linearly with distance x along the diffuser. Derive and plot the acceleration of a fluid particle, assuming uniform flow at each section, if the speed at the inlet is $V_i = 5$ m/s. Plot the pressure gradient through the diffuser, and find its maximum value. If the pressure gradient must be no greater than 25 kPa/m, how long would the diffuser have to be?

Given: Diffuser geometry

Find: Acceleration of a fluid particle; plot it; plot pressure gradient; find *L* such that pressure gradient is less than 25 kPa/m

Solution:

The given data is $D_i = 0.25\cdot m$ $\quad D_o = 0.75\cdot m$ $\quad L = 1\cdot m$ $\quad V_i = 5\cdot\frac{m}{s}$ $\quad \rho = 1000\cdot\frac{kg}{m^3}$

For a linear increase in diameter $\quad D(x) = D_i + \frac{D_o - D_i}{L}\cdot x$

From continuity $\quad Q = V\cdot A = V\cdot\frac{\pi}{4}\cdot D^2 = V_i\cdot\frac{\pi}{4}\cdot D_i^{\,2}$ $\quad Q = 0.245\frac{m^3}{s}$

Hence $\quad V(x)\cdot\frac{\pi}{4}\cdot D(x)^2 = Q$ $\quad V(x) = \frac{4\cdot Q}{\pi\cdot\left(D_i + \frac{D_o - D_i}{L}\cdot x\right)^2}$ $\quad$ or $\quad V(x) = \frac{V_i}{\left(1 + \frac{D_o - D_i}{L\cdot D_i}\cdot x\right)^2}$

The governing equation for this flow is

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} \qquad (6.2a)$$

or, for steady 1D flow, in the notation of the problem $\quad a_x = V\cdot\frac{d}{dx}V = \frac{V_i}{\left(1 + \frac{D_o - D_i}{L\cdot D_i}\cdot x\right)^2}\cdot\frac{d}{dx}\frac{V_i}{\left(1 + \frac{D_o - D_i}{L\cdot D_i}\cdot x\right)^2}$

Hence $\quad a_x(x) = -\frac{2\cdot V_i^{\,2}\cdot\left(D_o - D_i\right)}{D_i\cdot L\cdot\left[1 + \frac{\left(D_o - D_i\right)}{D_i\cdot L}\cdot x\right]^5}$

This is plotted in the associated *Excel* workbook

From Eq. 6.2a, pressure gradient is $\quad \frac{\partial}{\partial x}p = -\rho\cdot a_x$ $\quad \frac{\partial}{\partial x}p = \frac{2\cdot\rho\cdot V_i^{\,2}\cdot\left(D_o - D_i\right)}{D_i\cdot L\cdot\left[1 + \frac{\left(D_o - D_i\right)}{D_i\cdot L}\cdot x\right]^5}$

This is also plotted in the associated *Excel* workbook. Note that the pressure gradient is adverse: separation is likely to occur in the diffuser, and occur near the entrance

At the inlet $\frac{\partial}{\partial x}p = 100 \cdot \frac{kPa}{m}$ At the exit $\frac{\partial}{\partial x}p = 412 \cdot \frac{Pa}{m}$

To find the length *L* for which the pressure gradient is no more than 25 kPa/m, we need to solve

$$\frac{\partial}{\partial x}p \le 25 \cdot \frac{kPa}{m} = \frac{2 \cdot \rho \cdot V_i^2 \cdot (D_o - D_i)}{D_i \cdot L \cdot \left[1 + \frac{(D_o - D_i)}{D_i \cdot L} \cdot x\right]^5}$$

with *x* = 0 m (the largest pressure gradient is at the inlet)

Hence $$L \ge \frac{2 \cdot \rho \cdot V_i^2 \cdot (D_o - D_i)}{D_i \cdot \frac{\partial}{\partial x}p}$$ $L \ge 4 \cdot m$

This result is also obtained using *Goal Seek* in the *Excel* workbook

Problem 6.18

6.18 A diffuser for an incompressible, inviscid fluid of density $\rho = 1000\ \text{kg/m}^3$ consists of a diverging section of pipe. At the inlet the diameter is $D_i = 0.25$ m, and at the outlet the diameter is $D_o = 0.75$ m. The diffuser length is $L = 1$ m, and the diameter increases linearly with distance x along the diffuser. Derive and plot the acceleration of a fluid particle, assuming uniform flow at each section, if the speed at the inlet is $V_i = 5$ m/s. Plot the pressure gradient through the diffuser, and find its maximum value. If the pressure gradient must be no greater than 25 kPa/m, how long would the diffuser have to be?

Given: Diffuser geometry

Find: Acceleration of a fluid particle; plot it; plot pressure gradient; find L such that pressure gradient is less than 25 kPa/m

Solution:

The acceleration and pressure gradient are given by

$D_i = 0.25$ m
$D_o = 0.75$ m
$L = 1$ m
$V_i = 5$ m/s
$\rho = 1000\ \text{kg/m}^3$

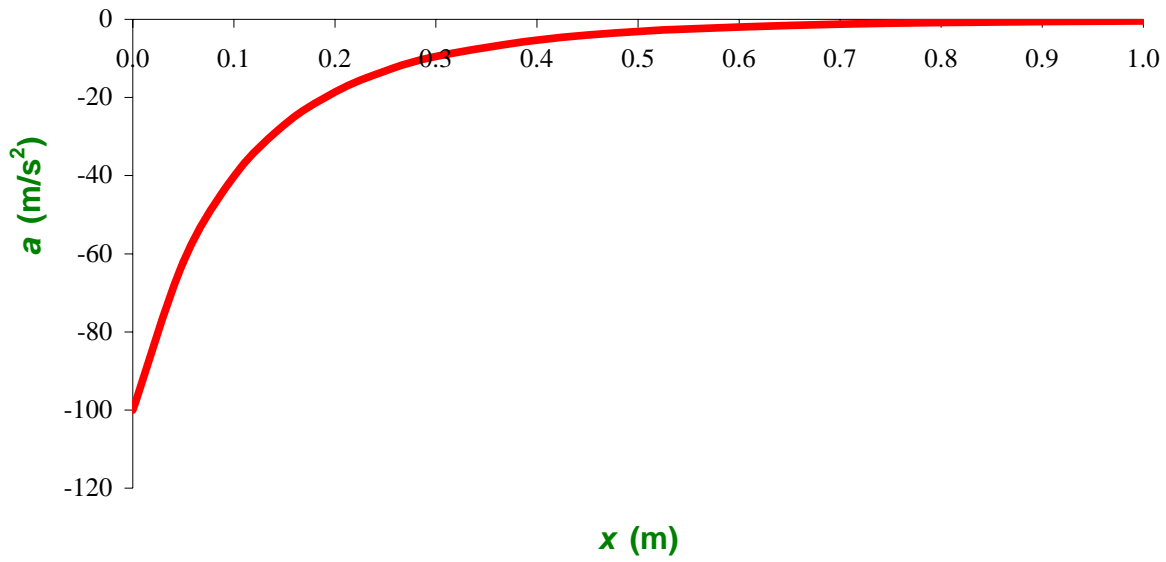
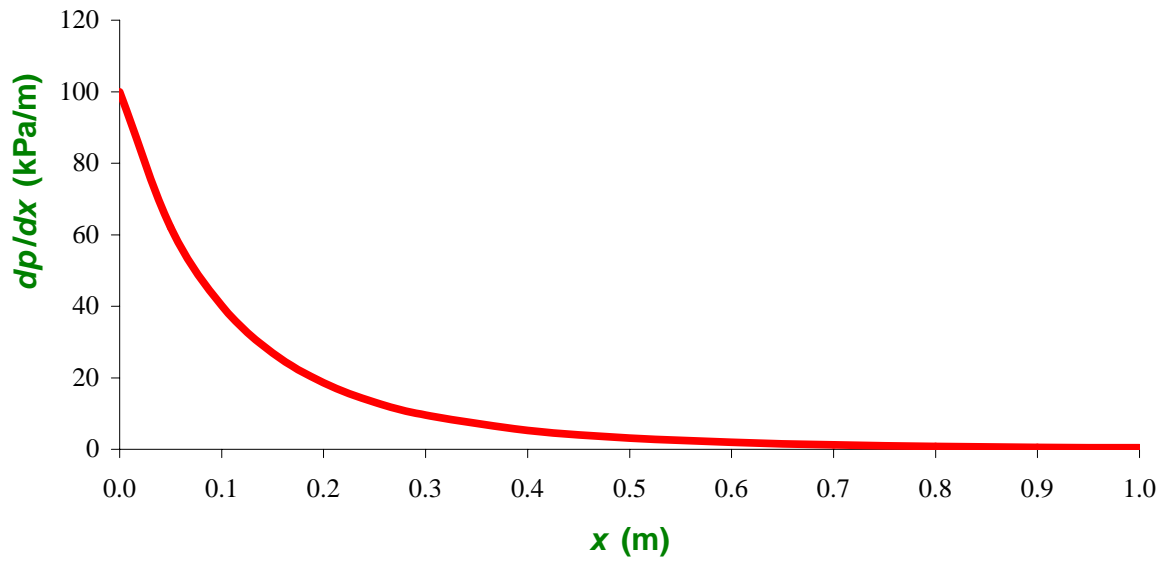
$$a_x(x) = -\frac{2 \cdot V_i^2 \cdot (D_o - D_i)}{D_i \cdot L \cdot \left[1 + \frac{(D_o - D_i)}{D_i \cdot L} \cdot x\right]^5}$$

$$\frac{\partial}{\partial x} p = \frac{2 \cdot \rho \cdot V_i^2 \cdot (D_o - D_i)}{D_i \cdot L \cdot \left[1 + \frac{(D_o - D_i)}{D_i \cdot L} \cdot x\right]^5}$$

x (m)	*a* (m/s^2)	*dp*/*dx* (kPa/m)
0.00	-100	100
0.05	-62.1	62.1
0.10	-40.2	40.2
0.15	-26.9	26.93
0.20	-18.59	18.59
0.25	-13.17	13.17
0.30	-9.54	9.54
0.40	-5.29	5.29
0.50	-3.125	3.125
0.60	-1.940	1.940
0.70	-1.256	1.256
0.80	-0.842	0.842
0.90	-0.581	0.581
1.00	-0.412	0.412

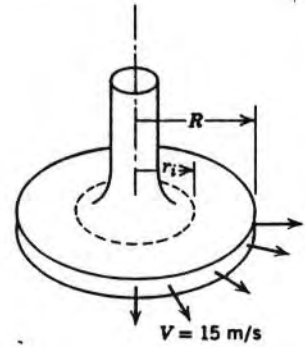
For the length L required for the pressure gradient to be less than 25 kPa/m use *Goal Seek*

$L =$ 4.00 m

x (m)	*dp*/*dx* (kPa/m)
0.0	25.0

Given: Steady, incompressible flow of air between parallel discs as shown

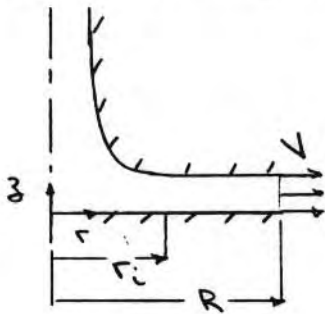
$$\vec{V} = V\frac{R}{r}\hat{e}_r \quad \text{for } r_i \le r \le R$$

where $V = 15$ m/s, $r_i = R/2$, $R = 75$ mm

Find: magnitude and direction of the net pressure force that acts on the upper plate between r_i and R.

Solution:

Basic equations: $\rho\vec{g} - \nabla p = \rho\frac{D\vec{V}}{Dt}$ $\qquad \vec{F} = -\int p\,d\vec{A}$

Assumptions: (1) incompressible flow
(2) steady flow
(3) frictionless flow
(4) uniform flow at each section.

To determine the pressure distribution $p(r)$, apply Eulers equation in the r direction

$$-\frac{\partial p}{\partial r} + \rho g_r = \rho a_r = \rho V_r \frac{\partial V_r}{\partial r}$$

$$\frac{\partial p}{\partial r} = -\rho V_r \frac{\partial V_r}{\partial r} = -\rho V\frac{R}{r}\frac{\partial}{\partial r}\left(V\frac{R}{r}\right) = \rho V\frac{R}{r}\frac{VR}{r^2}$$

$$\frac{dp}{dr} = \rho V^2\frac{R^2}{r^3}$$

$$dp = \rho V^2\frac{R^2}{r^3}\,dr$$

Integrating we obtain

$$p - p_{atm} = \int_{p_{atm}}^{p} dp = \rho V^2R^2\int_R^r r^{-3}\,dr = \rho V^2R^2\left[-\frac{1}{2r^2}\right]_R^r = \frac{1}{2}\rho V^2R^2\left[\frac{1}{R^2} - \frac{1}{r^2}\right]$$

Then

$$F_z = \int (p - p_{atm})\,dA = \int_{R/2}^{R}\frac{1}{2}\rho V^2R^2\left[\frac{1}{R^2} - \frac{1}{r^2}\right]2\pi r\,dr = \rho V^2R^2\pi\left[\frac{r^2}{2R^2} - \ln r\right]_{R/2}^{R}$$

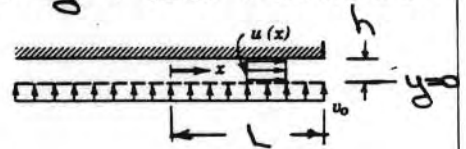
$$= \rho V^2R^2\pi\left[\frac{1}{2R^2}\left(R^2 - \frac{R^2}{4}\right) - \ln\frac{R}{R/2}\right] = \rho V^2R^2\pi\left[0.375 - \ln 2\right] = -0.318\pi\rho V^2R^2$$

$$= -0.318\pi \times 1.23\,\frac{kg}{m^3} \times (15)^2\,\frac{m^2}{s^2} \times (0.075)^2\,m^2 \times \frac{N\cdot s^2}{kg\cdot m}$$

$F_z = -1.56$ N ($F_z < 0$, so force acts down) $\qquad F_z$

Problem 6.20

Given: Air flows into the narrow gap between closely spaced parallel plates through a porous surface as shown. The uniform velocity in the x direction is $u = v_0 x/h$. Assume the flow is incompressible with $\rho = 1.23\ kg/m^3$ and that friction is negligible.

$v_0 = 15 mm/s$, $L = 22 mm$, $h = 1.2 mm$

Find: (a) the pressure gradient at the point (L, h)
(b) an equation for the flow streamlines in the cavity

Solution:
Eulers equation, $\rho\vec{g} - \nabla p = \rho \frac{D\vec{V}}{Dt}$, can be used to determine the pressure gradient for incompressible frictionless flow.

We need first to determine the velocity field. With $u = v_0 x/h$, for 2-D, incompressible flow we can use the continuity equation to determine v.

Since $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, then $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{v_0 x}{h}\right) = -\frac{v_0}{h}$

Then $v = \int \frac{\partial v}{\partial y} dy + f(x) = -\frac{v_0}{h} y + f(x)$

But $v = v_0$ at $y = 0$ and hence $f(x) = v_0$ and $v = v_0\left(1 - \frac{y}{h}\right)$

Then
$$\nabla p = \rho\vec{g} - \rho\frac{D\vec{V}}{Dt} = \rho\left[\vec{g} - u\frac{\partial \vec{V}}{\partial x} - v\frac{\partial \vec{V}}{\partial y}\right] = \rho\left[-g\hat{j} - \frac{v_0 x}{h}\left(\frac{v_0}{h}\hat{i}\right) - v_0\left(1 - \frac{y}{h}\right)\left(-\frac{v_0}{h}\hat{j}\right)\right]$$

$$\nabla p = \rho\left[-g\hat{j} - \frac{v_0^2 x}{h^2}\hat{i} - \frac{v_0^2}{h}\left(1 - \frac{y}{h}\right)\hat{j}\right]$$

At the point $(x, y) = (L, h)$
$$\nabla p = \rho\left[-\frac{v_0^2 L}{h^2}\hat{i} - g\hat{j}\right]$$

$$= 1.23\frac{kg}{m^3}\left[-(15)^2\frac{mm^2}{s^2} \times 0.022 m \times \frac{1}{(1.2)^2 mm^2}\hat{i} - 9.81\frac{m}{s^2}\hat{j}\right]\cdot\frac{N \cdot s^2}{kg \cdot m}$$

$$\nabla p\big]_{L,h} = -4.23\hat{i} - 12.1\hat{j}\ N/m^3 \qquad \nabla p$$

(b) The slope of the streamlines is given by $\frac{dy}{dx} = \frac{v}{u}$

$\therefore \frac{dy}{dx} = \frac{v_0(1 - y/h)}{\frac{v_0 x}{h}}$ and separating variables, we can write

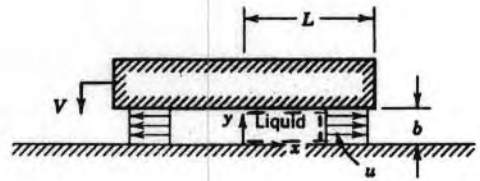
$\frac{d\left(\frac{y}{h}\right)}{(1 - y/h)} = \frac{d\left(\frac{x}{h}\right)}{x/h}$ Then integrating we obtain

$-\ln(1 - y/h) = \ln\frac{x}{h} - \ln c$

or
$$\frac{x}{h}\left(1 - \frac{y}{h}\right) = \text{constant}$$

Problem 6.21

Given: Upper plane surface moving downward at constant speed V causes incompressible liquid layer to be squeezed between surfaces as shown. Depth w in z direction and $w \gg L$.

Find: (a) Show that $u = Vx/b$ within the gap ($b = b_0 - Vt$)
(b) expression for a_x
(c) $\partial p/\partial x$
(d) $p(x)$
(e) net pressure force on upper surface

Solution:

Basic equations: $0 = \frac{\partial}{\partial t}\int_{cv} \rho\, d\forall + \int_{cs} \rho \vec{V}\cdot d\vec{A}$

$-\nabla p + \rho\vec{g} = \rho \frac{D\vec{V}}{Dt}$ $\qquad \vec{F} = -\int p\, d\vec{A}$

(a) For the deformable CV shown

$$0 = \frac{\partial}{\partial t}\int_0^y \rho w x\, dy + \rho u w y = \rho w x \frac{dy}{dt} + \rho u w y$$

But $dy/dt = -V$ and hence $u = \frac{Vx}{y}$

If $y = b_0$ at $t = 0$, then $y = b = b_0 - Vt$ at any time t

$$\therefore\ u = \frac{Vx}{b}$$ $\qquad u(t)$

(b) $a_x = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$ (terms 2 and 3 $\to 0$ (1))

Assumptions: (1) $u \neq u(y)$, $w = 0$

$$a_x = \frac{Vx}{b}\left(\frac{V}{b}\right) + \frac{\partial u}{\partial b}\frac{\partial b}{\partial t} = \frac{V^2 x}{b^2} + \left(-\frac{Vx}{b^2}\right)(-V) = \frac{2V^2 x}{b^2}$$ $\qquad a_x$

(c) From Euler's equation in the x direction with $g_x = 0$

$$\frac{\partial p}{\partial x} = -\rho a_x = -\frac{2\rho V^2 x}{b^2}$$ $\qquad \frac{\partial p}{\partial x}$

(d) $$p - p_{atm} = \int_L^x \frac{\partial p}{\partial x}dx = \int_L^x -\frac{2\rho V^2}{b^2}x\, dx = -\frac{\rho V^2 x^2}{b^2}\Big]_L^x = \frac{\rho V^2 L^2}{b^2}\left[1 - \left(\frac{x}{L}\right)^2\right]$$ $\qquad p(x)$

(e) $$F_y = \int_A (p - p_{atm})\, dA = 2\int_0^L \frac{\rho V^2 L^2}{b^2}\left[1 - \left(\frac{x}{L}\right)^2\right] w\, dx$$

$$= 2\int_0^1 \frac{\rho V^2 L^3}{b^2}\left[1 - \left(\frac{x}{L}\right)^2\right] w\, d\left(\frac{x}{L}\right) = \frac{2\rho V^2 L^3 w}{b^2}\left[\left(\frac{x}{L}\right) - \frac{1}{3}\left(\frac{x}{L}\right)^3\right]_0^1$$

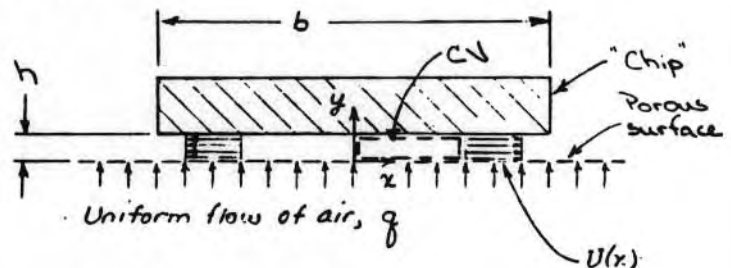
$$F_y = \frac{4\rho V^2 L^3 w}{3b^2}$$ (upward, since $F_y > 0$) $\qquad F_y$

Problem 6.22

Given: Rectangular "chip" floats on thin layer of air of thickness, $h = 0.5$ mm above a porous surface as shown. Chip width $b = 20$ mm; length L (perpendicular to diagram) $\gg b$; no flow in z direction. Flow in x direction under chip may be assumed uniform; $\rho =$ constant; neglect frictional effects

Find: (a) Use a suitably chosen CV to show $U(x) = qx/h$ in the gap
(b) Find an expression for $\vec{a}_p$ in the gap
(c) Estimate the maximum value of $\vec{a}_p$
(d) Obtain an expression for $\partial p/\partial x$
(e) Sketch the pressure distribution under the chip
(f) Is the net pressure force on the chip directed up or down?
(g) Estimate the mass per unit length of the chip if $q = 0.06$ m³/sec/m²

Solution:

Assumptions:
(1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) uniform flow at porous surface and in the gap at x.

(a) Apply continuity equation to CV, $0 = \frac{\partial}{\partial t}\int \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$ (unsteady term $= 0$ (1))
Then

$$0 = \{-|\rho q x L|\} + \{+|\rho U h L|\} \quad \text{or} \quad U = q\frac{x}{h}$$ ← $U(x)$

(b) Apply the substantial derivative definition

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$$

($w = 0$ given; $\frac{\partial \vec{V}}{\partial t} = 0$ (1))

Obtain v from differential continuity $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$$\therefore \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{q}{h} \quad \text{and} \quad v - v_0 = \int_0^y -\frac{q}{h}\,dy + f(x) = -\frac{q}{h}y + f(x)$$

$$\text{or} \quad v = q\left(1 - \frac{y}{h}\right) \qquad [f(x) = 0 \text{ since } v = v_0 = q = \text{const. along } y = 0]$$

$$a_{p_x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = q\frac{x}{h}\left(\frac{q}{h}\right) = \frac{q^2 x}{h^2}$$

$$a_{p_y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = q\left(1 - \frac{y}{h}\right)\left(-\frac{q}{h}\right) = \frac{q^2}{h}\left(\frac{y}{h} - 1\right)$$

$$\vec{a}_p = \frac{q^2 x}{h^2}\hat{i} + \frac{q^2}{h}\left(\frac{y}{h} - 1\right)\hat{j} = \frac{q^2}{h}\left[\frac{x}{h}\hat{i} + \left(\frac{y}{h} - 1\right)\hat{j}\right]$$

(c) The magnitude of $|\vec{a}_p| = \frac{q^2}{h}\left[\left(\frac{x}{h}\right)^2 + \left(\frac{y}{h} - 1\right)^2\right]^{1/2}$ is a maximum at $x = \frac{b}{2}$, $y = 0$

$$|\vec{a}_p|_{max} = \frac{q^2}{h}\left[\left(\frac{b}{2h}\right)^2 + 1\right]^{1/2} = 144 \text{ m/s}^2$$ ← $|\vec{a}_p|_{max}$

(d) To obtain $\partial p/\partial x$ write the x component of the Euler equation

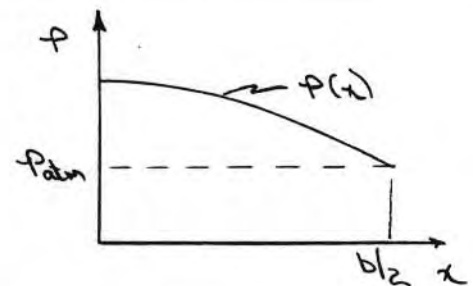
$$-\frac{\partial p}{\partial x} + \rho g_x = \rho a_{p_x} \quad \therefore \quad \frac{\partial p}{\partial x} = -\rho a_{p_x} = -\frac{\rho q^2 x}{h^2}$$ ← $\frac{\partial p}{\partial x}$

(e) To obtain an expression for the pressure distribution, $p(x)$ we need to separate variables and integrate noting that $p = p_{atm}$ at $x = b/2$. Thus

$$p - p_{atm} = \int_{b/2}^{x} \frac{\partial p}{\partial x}\,dx = -\int_{b/2}^{x} \frac{\rho q^2}{h^2} x = -\frac{\rho q^2 x^2}{2h^2}\Big]_{b/2}^{x}$$

$$p - p_{atm} = \frac{\rho q^2}{2h^2}\left[\left(\frac{b}{2}\right)^2 - x^2\right] = \frac{\rho q^2 b^2}{8h^2}\left[1 - \left(\frac{2x}{b}\right)^2\right]$$

$$p = p_{atm} + \frac{\rho q^2 b^2}{8h^2}\left[1 - \left(\frac{2x}{b}\right)^2\right]$$ ← $p(x)$

(f) The net pressure force on the chip is up. Note that the pressure on the chip is greater than p_{atm} over the entire chip surface

(g) To estimate the mass per unit weight of the chip we must determine the net pressure force on the chip

$$F_{net} = \int_A (p - p_{atm})\,dA = 2\int_0^{b/2} \frac{\rho q^2 b^2}{8h^2}\left[1 - \left(\frac{2x}{b}\right)^2\right] L\,dx$$

$$= \frac{\rho q^2 b^2 L}{4h^2}\left[x - \frac{4}{3}\frac{x^3}{b^2}\right]_0^{b/2} = \frac{\rho q^2 b^2 L}{4h^2}\left[\frac{b}{2} - \frac{1}{3}\frac{b}{2}\right]$$

$$F_{net} = \frac{\rho q^2 b^3 L}{12h^2}$$

The weight of the chip, $W = Mg$, must be balanced by the net pressure force. Hence

$$Mg = F_{net} = \frac{\rho q^2 b^3 L}{12h^2}$$

$$\frac{M}{L} = \frac{\rho q^2 b^3}{12h^2 g}$$

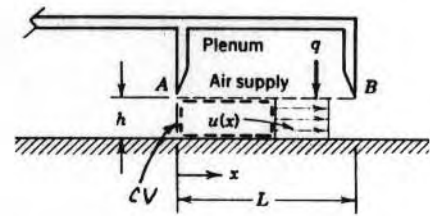
$$= \frac{1}{12} \times 1.23\,\frac{kg}{m^3} \times (0.06)^2 \frac{m^6}{s^2 \cdot m^4} \times (0.02)^3\,m^3 \times \frac{1}{(0.0005)^2\,m^2} \times \frac{s^2}{9.81\,m}$$

$$\frac{M}{L} = 1.20 \times 10^{-3}\ kg/m$$ ← mass/length

Problem 6.23

Given: Load pallet supported by air:

Flow is incompressible, uniform, and frictionless; $h \ll L$.

No flow across plane at $x = 0$.

Find: (a) Use a suitable CV to show $u(x) = qx/h$ in the gap.
(b) Calculate the acceleration of a fluid particle in the gap.
(c) Evaluate the pressure gradient, $\partial p/\partial x$.
(d) Sketch the pressure distribution; indicate pressure at $x = L$.

Solution: Choose a CV in the gap, from 0 to x, as shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$ (first term $= 0(1)$)

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \qquad -\frac{\partial p}{\partial x} + \rho g_x = \rho a_{p_x}$$

($v\,\partial u/\partial y = 0(3)$, $w\,\partial u/\partial z = 0(4)$, $\partial u/\partial t = 0(1)$, $\rho g_x = 0(5)$)

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) No variation with z
(5) Horizontal, so $g_x = 0$

From continuity,

$$0 = \{-|\rho q w x|\} + \{+|\rho u(x) w h|\} \quad \text{so} \quad u(x) = q\frac{x}{h}$$

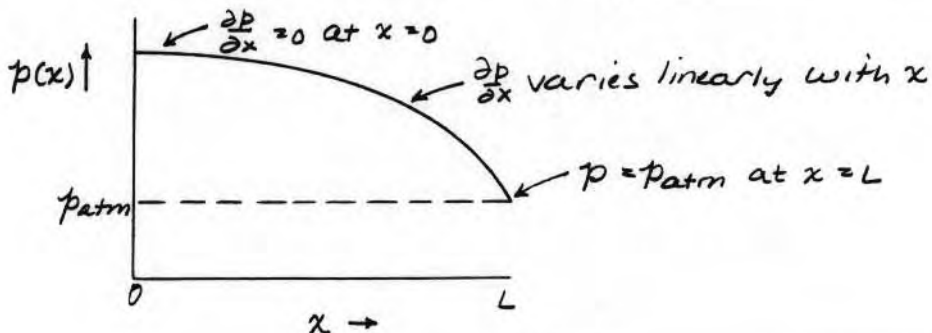
$u(x)$

The acceleration is $a_{p_x} = \left(q\frac{x}{h}\right)\left(q\frac{1}{h}\right) = \frac{q^2 x}{h^2}$

a_{p_x}

The pressure gradient is $\frac{\partial p}{\partial x} = -\rho a_{p_x} = -\frac{\rho q^2 x}{h^2}$

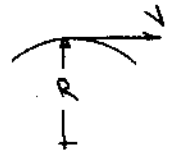
$\frac{\partial p}{\partial x}$

Sketching:

Sketch

Problem 6.24

Given: Air at 20 psia, 100°F flows around a smooth corner

Velocity = 150 ft/s

Radius of curvature of streamline is 3 in.

Find: (a) magnitude of centripetal acceleration in G's

(b) pressure gradient, $\frac{\partial p}{\partial r}$

Solution:

Basic equations: $\rho\vec{g} - \nabla p = \rho\frac{D\vec{V}}{Dt}$ ......(1)

$\frac{D\vec{V}}{Dt} = \vec{a}_p$ ....(2) $\qquad p = \rho RT$ .....(3)

Assumptions: (1) ρ = constant

(2) frictionless flow

(3) $\vec{g} = -g\hat{k}$

Writing the r component of equation (1)

$$g_r^{\,0} - \frac{1}{\rho}\frac{\partial p}{\partial r} = a_r = \frac{\partial V_r}{\partial t}^{0} + V_r^{\,0}\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta}^{0} + V_z^{\,0}\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}$$

$$a_r = -\frac{V_\theta^2}{r} \qquad \frac{a_r}{g} = -\frac{V_\theta^2}{rg} = -(150)^2\frac{ft^2}{s^2} \times \frac{1}{3\,in} \times \frac{12\,in}{ft} \times \frac{s^2}{32.2\,ft}$$

$$\frac{a_r}{g} = -2800 \text{ G's}$$

$\frac{a_r}{g}$

Also

$$\frac{\partial p}{\partial r} = \rho\frac{V_\theta^2}{r}$$

where $\rho = \frac{p}{RT} = 20\frac{lbf}{in^2} \times \frac{lbm\,°R}{53.3\,ft\text{-}lbf} \times \frac{1}{560°R} \times 144\frac{in^2}{ft^2} \times \frac{slug}{32.2\,lbm}$

$$\rho = 0.003\ slug/ft^3$$

$$\frac{\partial p}{\partial r} = \rho\frac{V_\theta^2}{r} = 0.003\frac{slug}{ft^3} \times (150)^2\frac{ft^2}{s^2} \times \frac{1}{3\,in} \times \frac{12\,in}{ft} \times \frac{lbf\text{-}s^2}{ft\text{-}slug}$$

$$\frac{\partial p}{\partial r} = 270\ \frac{lbf/ft^2}{ft}$$

$\frac{\partial p}{\partial r}$

Problem 6.25

[2]

6.25 The velocity field for a plane doublet is given in Table 6.2. Find an expression for the pressure gradient at any point (r, θ).

Given: Velocity field for doublet

Find: Expression for pressure gradient

Solution:

Basic equations

$$\rho a_r = \rho\left(\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}\right) = \rho g_r - \frac{\partial p}{\partial r}$$

$$\rho a_\theta = \rho\left(\frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r}\right) = \rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta}$$

For this flow $\quad V_r(r,\theta) = -\frac{\Lambda}{r^2}\cdot\cos(\theta) \qquad V_\theta(r,\theta) = -\frac{\Lambda}{r^2}\cdot\sin(\theta) \qquad V_z = 0$

Hence for r momentum

$$\rho\cdot g_r - \frac{\partial}{\partial r}p = \rho\cdot\left(V_r\cdot\frac{\partial}{\partial r}V_r + \frac{V_\theta}{r}\cdot\frac{\partial}{\partial \theta}V_r - \frac{V_\theta^{\,2}}{r}\right)$$

Ignoring gravity

$$\frac{\partial}{\partial r}p = -\rho\cdot\left[\left(-\frac{\Lambda}{r^2}\cdot\cos(\theta)\right)\cdot\frac{\partial}{\partial r}\left(-\frac{\Lambda}{r^2}\cdot\cos(\theta)\right) + \frac{\left(-\frac{\Lambda}{r^2}\cdot\sin(\theta)\right)}{r}\cdot\frac{\partial}{\partial \theta}\left(-\frac{\Lambda}{r^2}\cdot\cos(\theta)\right) - \frac{\left(-\frac{\Lambda}{r^2}\cdot\sin(\theta)\right)^2}{r}\right] \qquad \frac{\partial}{\partial r}p = \frac{2\cdot\Lambda^2\cdot\rho}{r^5}$$

For θ momentum

$$\rho\cdot g_\theta - \frac{1}{r}\cdot\frac{\partial}{\partial \theta}p = \rho\cdot\left(V_r\cdot\frac{\partial}{\partial r}V_\theta + \frac{V_\theta}{r}\cdot\frac{\partial}{\partial \theta}V_\theta + \frac{V_r\cdot V_\theta}{r}\right)$$

Ignoring gravity

$$\frac{\partial}{\partial \theta}p = -r\cdot\rho\cdot\left[\left(-\frac{\Lambda}{r^2}\cdot\cos(\theta)\right)\cdot\frac{\partial}{\partial r}\left(-\frac{\Lambda}{r^2}\cdot\sin(\theta)\right) + \frac{\left(-\frac{\Lambda}{r^2}\cdot\sin(\theta)\right)}{r}\cdot\frac{\partial}{\partial \theta}\left(-\frac{\Lambda}{r^2}\cdot\sin(\theta)\right) + \frac{\left(-\frac{\Lambda}{r^2}\cdot\sin(\theta)\right)\cdot\left(-\frac{\Lambda}{r^2}\cdot\cos(\theta)\right)}{r}\right] \qquad \frac{\partial}{\partial \theta}p = 0$$

The pressure gradient is purely radial

Problem 6.26

Given: The velocity field for steady, frictionless, incompressible flow (from right to left) over a stationary circular cylinder of radius, a, is given by

$$\vec{V} = U\left[\left(\frac{a}{r}\right)^2 - 1\right]\cos\theta\,\hat{e}_r + U\left[\left(\frac{a}{r}\right)^2 + 1\right]\sin\theta\,\hat{e}_\theta$$

Consider flow along the streamline forming the cylinder surface, i.e. $r = a$

Find: The pressure gradient along cylinder surface
Plot $V(r)$ along $\theta = \pi/2$ for $r > a$

Solution:

Basic equation: $\rho\vec{g} - \nabla p = \rho\frac{D\vec{V}}{Dt}$

Assumptions: (1) neglect body force

Along the surface, $r = a$, $\vec{V} = 2U\sin\theta\,\hat{i}_\theta$.

Computing equations:

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}$$

$$-\frac{1}{\rho r}\frac{\partial p}{\partial \theta} = \frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r}$$

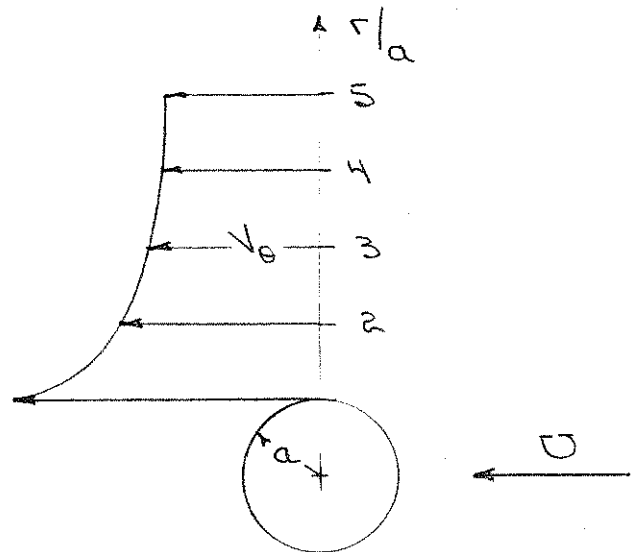
$$\frac{\partial p}{\partial r} = \rho\frac{V_\theta^2}{r} = \rho\frac{[2U\sin\theta]^2}{a} = \frac{4U^2\rho}{a}\sin^2\theta$$

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = -\rho\frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} = -\rho\left[\frac{2U\sin\theta}{a}\right](2U\cos\theta) = -\frac{4U^2\rho}{a}\sin\theta\cos\theta$$

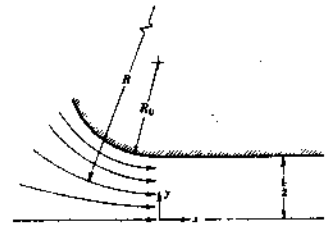
$$\nabla p = \hat{i}_r\frac{\partial p}{\partial r} + \hat{i}_\theta\frac{1}{r}\frac{\partial p}{\partial \theta} = \frac{4\rho U^2}{a}\sin\theta\left(\hat{i}_r\sin\theta - \hat{i}_\theta\cos\theta\right) \longleftarrow \nabla p$$

Along $\theta = \frac{\pi}{2}$, $\vec{V} = U\left[\left(\frac{a}{r}\right)^2 + 1\right]\hat{e}_\theta$

$\frac{r}{a}$	V_θ
1	$2U$
2	$1.25U$
3	$1.111U$
4	$1.063U$
5	$1.04U$

Problem 6.27

Given: Radius of curvature of streamlines at wind tunnel inlet is modeled as

$$R = \frac{h/2}{y} R_0$$

Speed along each streamline assumed constant at $V = 20\,m/s$; $h = 0.15\,m$, $R = 0.6\,m$

Find: Δp between $y = 0$ and tunnel wall ($y = h/2$)

Solution:

Basic equation: $\frac{\partial p}{\partial n} = \rho \frac{V^2}{R}$

Assumptions: (1) steady flow (2) frictionless flow
(3) neglect body forces
(4) constant speed along each streamline

At the inlet section, $p = p(y)$

$$\therefore \quad \frac{dp}{dn} = -\frac{dp}{dy} = \rho \frac{V^2}{R} = \rho V^2 \frac{2y}{R_0 h}$$

$$\therefore \; dp = -\frac{\rho V^2}{R_0 h} 2y \, dy$$

$$p_{h/2} - p_0 = \int_0^{h/2} dp = -\frac{2\rho V^2}{R_0 h} \int_0^{h/2} y \, dy = -\frac{2\rho V^2}{R_0 h} \left. \frac{y^2}{2} \right]_0^{h/2}$$

$$p_{h/2} - p_0 = -\frac{\rho V^2}{R_0 h} \frac{h^2}{4} = -\frac{\rho V h}{4 R_0}$$

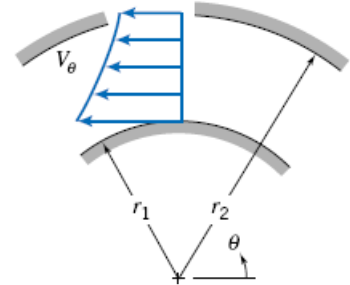
$$p_{h/2} - p_0 = -1.225 \frac{kg}{m^3} \times \left(20 \frac{m}{s}\right)^2 \times 0.15\,m \times \frac{1}{4} \times \frac{1}{0.6\,m} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$p_{h/2} - p_0 = -30.6 \; N/m^2 \longleftarrow \quad p_{h/2} - p_0$$

Problem 6.28

6.28 Repeat Example 6.1, but with the somewhat more realistic assumption that the flow is similar to a free vortex (irrotational) profile, $V_\theta = c/r$ (where c is a constant), as shown in Fig. P6.28. In doing so, prove that the flow rate is given by $Q = k\sqrt{\Delta p}$, where k is

$$k = w \ln\left(\frac{r_2}{r_1}\right)\sqrt{\frac{2r_2^2 r_1^2}{\rho(r_2^2 - r_1^2)}}$$

and w is the depth of the bend.

Given: Velocity field for free vortex flow in elbow

Find: Similar solution to Example 6.1; find k (above)

Solution:

Basic equation $\quad \frac{\partial}{\partial r}p = \frac{\rho \cdot V^2}{r} \quad$ with $\quad V = V_\theta = \frac{c}{r}$

Assumptions: 1) Frictionless 2) Incompressible 3) free vortex

For this flow $\quad p \neq p(\theta) \quad$ so $\quad \frac{\partial}{\partial r}p = \frac{d}{dr}p = \frac{\rho \cdot V^2}{r} = \frac{\rho \cdot c^2}{r^3}$

Hence
$$\Delta p = p_2 - p_1 = \int_{r_1}^{r_2} \frac{\rho \cdot c^2}{r^3}\,dr = \frac{\rho \cdot c^2}{2}\cdot\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) = \frac{\rho \cdot c^2 \cdot \left(r_2^2 - r_1^2\right)}{2 \cdot r_1^2 \cdot r_2^2} \qquad (1)$$

Next we obtain c in terms of Q

$$Q = \int \vec{V}\,d\vec{A} = \int_{r_1}^{r_2} V \cdot w\,dr = \int_{r_1}^{r_2} \frac{w \cdot c}{r}\,dr = w \cdot c \cdot \ln\left(\frac{r_2}{r_1}\right)$$

Hence
$$c = \frac{Q}{w \cdot \ln\left(\frac{r_2}{r_1}\right)}$$

Using this in Eq 1
$$\Delta p = p_2 - p_1 = \frac{\rho \cdot c^2 \cdot \left(r_2^2 - r_1^2\right)}{2 \cdot r_1^2 \cdot r_2^2} = \frac{\rho \cdot Q^2 \cdot \left(r_2^2 - r_1^2\right)}{2 \cdot w^2 \cdot \ln\left(\frac{r_2}{r_1}\right)^2 \cdot r_1^2 \cdot r_2^2}$$

Solving for Q
$$Q = w \cdot \ln\left(\frac{r_2}{r_1}\right) \cdot \sqrt{\frac{2 \cdot r_1^2 \cdot r_2^2}{\rho \cdot \left(r_2^2 - r_1^2\right)}} \cdot \sqrt{\Delta p} \qquad k = w \cdot \ln\left(\frac{r_2}{r_1}\right) \cdot \sqrt{\frac{2 \cdot r_1^2 \cdot r_2^2}{\rho \cdot \left(r_2^2 - r_1^2\right)}}$$

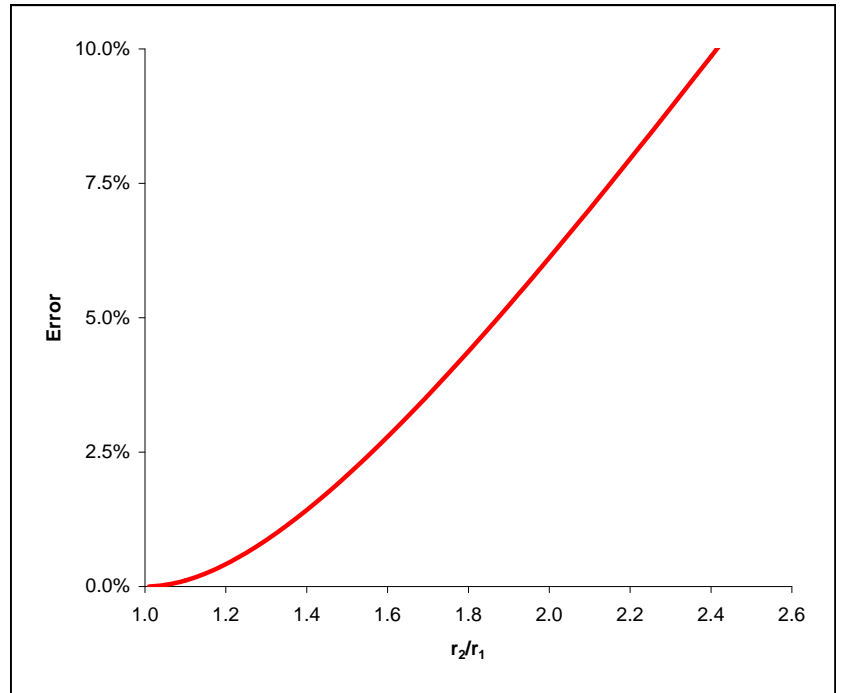
Problem 6.29

6.29 Using the analyses of Example 6.1 and Problem 6.28, plot the discrepancy (percent) between the flow rates obtained from assuming uniform flow and the free vortex (irrotational) profile as a function of inner radius r_1.

From Example 6.1:
$$Q_{Uniform} = V \cdot A = w \cdot (r_2 - r_1) \cdot \sqrt{\frac{1}{\rho \cdot \ln\left(\frac{r_2}{r_1}\right)}} \cdot \sqrt{\Delta p}$$
or
$$\frac{Q_{Uniform} \cdot \sqrt{\rho}}{w \cdot r_1 \cdot \sqrt{\Delta p}} = \frac{\left(\frac{r_2}{r_1} - 1\right)}{\sqrt{\ln\left(\frac{r_2}{r_1}\right)}} \qquad \text{Eq. 1}$$

From Problem 6.28:
$$\frac{Q \cdot \sqrt{\rho}}{w \cdot r_1 \cdot \sqrt{\Delta p}} = \left(\frac{r_2}{r_1}\right) \cdot \ln\left(\frac{r_2}{r_1}\right) \cdot \sqrt{\frac{2}{\left[\left(\frac{r_2}{r_1}\right)^2 - 1\right]}} \qquad \text{Eq. 2}$$

Instead of plotting as a function of inner radius we plot as a function of r_2/r_1

r_2/r_1	Eq. 1	Eq. 2	Error
1.01	0.100	0.100	0.0%
1.05	0.226	0.226	0.0%
1.10	0.324	0.324	0.1%
1.15	0.401	0.400	0.2%
1.20	0.468	0.466	0.4%
1.25	0.529	0.526	0.6%
1.30	0.586	0.581	0.9%
1.35	0.639	0.632	1.1%
1.40	0.690	0.680	1.4%
1.45	0.738	0.726	1.7%
1.50	0.785	0.769	2.1%
1.55	0.831	0.811	2.4%
1.60	0.875	0.851	2.8%
1.65	0.919	0.890	3.2%
1.70	0.961	0.928	3.6%
1.75	1.003	0.964	4.0%
1.80	1.043	1.000	4.4%
1.85	1.084	1.034	4.8%
1.90	1.123	1.068	5.2%
1.95	1.162	1.100	5.7%
2.00	1.201	1.132	6.1%
2.05	1.239	1.163	6.6%
2.10	1.277	1.193	7.0%
2.15	1.314	1.223	7.5%
2.20	1.351	1.252	8.0%
2.25	1.388	1.280	8.4%
2.30	1.424	1.308	8.9%
2.35	1.460	1.335	9.4%
2.40	1.496	1.362	9.9%
2.45	1.532	1.388	10.3%
2.50	1.567	1.414	10.8%

Given: Velocity field $\vec{V} = (Ax + B)\hat{i} - Ay\hat{j}$ where $A = 1\,s^{-1}$, $B = 2\,m/s$ and coordinates are measured in meters.

Show: that streamlines are given by $(x + B/A)y = \text{constant}$.

Plot: streamlines through points $(x, y) = (1,1), (1,2), (2,2)$.

Find:
(a) velocity vector & acceleration vector at (1,2); show these on streamline plot
(b) component of $\vec{a}_p$ along the streamline at (1,2); express as a vector.
(c) pressure gradient along streamline at (1,2) for air
(d) relative value of pressure at points (1,1) & (2,2)

Solution:

The slope of a streamline is $\left.\frac{dy}{dx}\right)_{s.l} = \frac{v}{u} = \frac{-Ay}{Ax+B} = \frac{-y}{x+B/A}$

Then $\frac{dy}{y} + \frac{dx}{x+B/A} = 0$ and $\ln y + \ln(x + B/A) = \ln c$.

and

$$(x + B/A)y = \text{constant} \quad \text{Streamlines}$$

For
(1,1) $(x+2)y = 3$
(1,2) $(x+2)y = 6$
(2,2) $(x+2)y = 8$

These streamlines are shown in the plot at the end of the problem solution

The particle acceleration $\vec{a}_p = \frac{D\vec{V}}{Dt} = \cancel{\frac{\partial \vec{V}}{\partial t}}^{\,0(1)} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + \cancel{w\frac{\partial \vec{V}}{\partial z}}^{\,0(2)}$.

Assumptions: (1) steady flow (given)
(2) 2-D (given) $\vec{V} \neq \vec{V}(z)$.

$\vec{a}_p = (Ax+B)\frac{\partial}{\partial x}\left[(Ax+B)\hat{i} - Ay\hat{j}\right] - Ay\frac{\partial}{\partial y}\left[(Ax+B)\hat{i} - Ay\hat{j}\right]$

$\vec{a}_p = (Ax+B)A\hat{i} - Ay(-A\hat{j}) = A(Ax+B)\hat{i} + A^2y\hat{j}$

At point (1,2).

$\vec{a}_p = \frac{1}{s}\left(\frac{1}{s}\times 1m + 2\frac{m}{s}\right)\hat{i} + \frac{1}{s^2}\times 2m\hat{j} = 3\hat{i} + 2\hat{j}\ m/s^2 \quad \vec{a}_{(1,2)}$

$\vec{V} = \left(\frac{1}{s}\times 1m + 2\frac{m}{s}\right)\hat{i} - \frac{1}{s}2m\hat{j} = 3\hat{i} - 2\hat{j}\ m/s \quad \vec{V}_{(1,2)}$

$\vec{V}$ and $\vec{a}$ are shown on the streamline plot.

(b) The component of $\vec{a}_p$ along (tangent to) the streamline is given by $a_t = \vec{a}_p \cdot \hat{e}_t$ where $\hat{e}_t = \frac{\vec{V}}{|\vec{V}|}$

Thus $\hat{e}_t = \frac{3\hat{i} - 2\hat{j}}{\left[3^2 + (-2)^2\right]^{1/2}} = 0.832\hat{i} - 0.555\hat{j}$

and

$a_t = \vec{a}_p \cdot \hat{e}_t = (3\hat{i}+2\hat{j})\,m/s^2 \cdot (0.832\hat{i} - 0.555\hat{j}) = 1.39\ m/s^2$

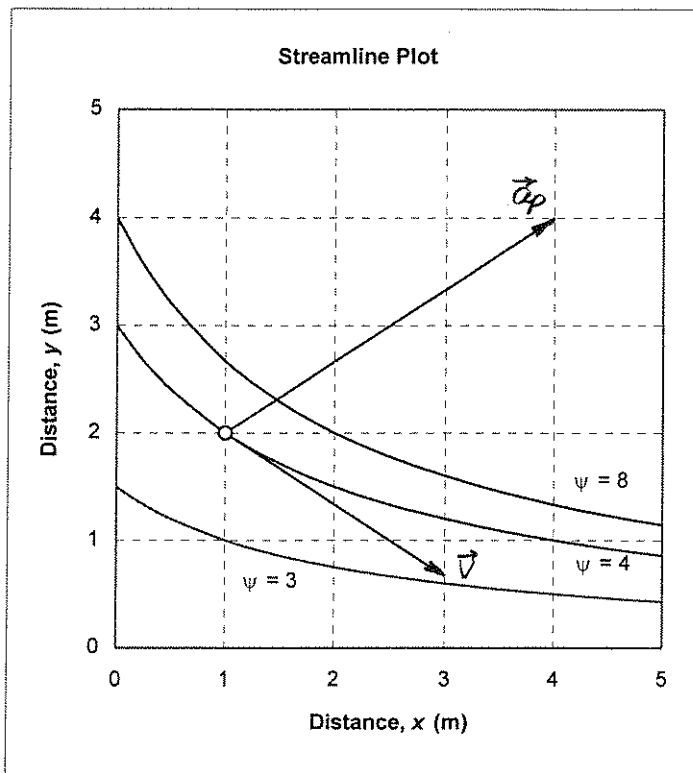
$\vec{a}_t = 1.39\,\hat{e}_t = 1.16\hat{i} - 0.771\hat{j}\ m/s^2$ ← $\vec{a}_{t(1,2)}$

For frictionless flow, Euler's equation along a streamline (neglecting gravity, i.e. assuming flow in horizontal plane) is

$$\frac{\partial p}{\partial s} = -\rho V \frac{\partial V}{\partial s} = -\rho a_t = -1.23\,\frac{kg}{m^3} \times 1.39\,\frac{m}{s^2} \times \frac{N\,s^2}{kg\cdot m}$$

$$\frac{\partial p}{\partial s} = -1.71\ N/m^2/m$$ ← $\frac{\partial p}{\partial s}_{(1,2)}$

Looking at the streamline we would expect $p(2,2)$ to be less than $p(1,1)$ due to streamline curvature; Euler's equation normal to a streamline says

$$\partial p/\partial n = \frac{\rho V^2}{R}$$

Problem 6.31

6.31 A velocity field is given by $\vec{V} = [Ax^3 + Bxy^2]\hat{i} + [Ay^3 + Bx^2y]\hat{j}$; $A = 0.2\ \text{m}^{-2}\cdot\text{s}^{-1}$, B is a constant, and the coordinates are measured in meters. Determine the value and units for B if this velocity field is to represent an incompressible flow. Calculate the acceleration of a fluid particle at point $(x, y) = (2, 1)$. Evaluate the component of particle acceleration normal to the velocity vector at this point.

Given: Velocity field

Find: Constant B for incompressible flow; Acceleration of particle at (2,1); acceleration normal to velocity at (2,1)

Solution:

Basic equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective acceleration}} + \frac{\partial \vec{V}}{\partial t}$$

total acceleration of a particle; convective acceleration; local acceleration

For this flow

$$u(x,y) = A\cdot x^3 + B\cdot x\cdot y^2 \qquad v(x,y) = A\cdot y^3 + B\cdot x^2\cdot y$$

$$\frac{\partial}{\partial x}u(x,y) + \frac{\partial}{\partial y}v(x,y) = \frac{\partial}{\partial x}\left(A\cdot x^3 + B\cdot x\cdot y^2\right) + \frac{\partial}{\partial y}\left(A\cdot y^3 + B\cdot x^2\cdot y\right) = 0$$

$$\frac{\partial}{\partial x}u(x,y) + \frac{\partial}{\partial y}v(x,y) = (3\cdot A + B)\cdot\left(x^2 + y^2\right) = 0$$

Hence $B = -3\cdot A$ $\qquad B = -0.6\frac{1}{m^2\cdot s}$

We can write

$$u(x,y) = A\cdot x^3 - 3\cdot A\cdot x\cdot y^2 \qquad v(x,y) = A\cdot y^3 - 3\cdot A\cdot x^2\cdot y$$

Hence for a_x

$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = \left(A\cdot x^3 - 3\cdot A\cdot x\cdot y^2\right)\cdot\frac{\partial}{\partial x}\left(A\cdot x^3 - 3\cdot A\cdot x\cdot y^2\right) + \left(A\cdot y^3 - 3\cdot A\cdot x^2\cdot y\right)\cdot\frac{\partial}{\partial y}\left(A\cdot x^3 - 3\cdot A\cdot x\cdot y^2\right)$$

$$a_x = 3\cdot A^2\cdot x\cdot\left(x^2 + y^2\right)^2$$

For a_y

$$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial v}v = \left(A\cdot x^3 - 3\cdot A\cdot x\cdot y^2\right)\cdot\frac{\partial}{\partial x}\left(A\cdot y^3 - 3\cdot A\cdot x^2\cdot y\right) + \left(A\cdot y^3 - 3\cdot A\cdot x^2\cdot y\right)\cdot\frac{\partial}{\partial y}\left(A\cdot y^3 - 3\cdot A\cdot x^2\cdot y\right)$$

$$a_y = 3\cdot A^2\cdot y\cdot\left(x^2 + y^2\right)^2$$

Hence at (2,1)

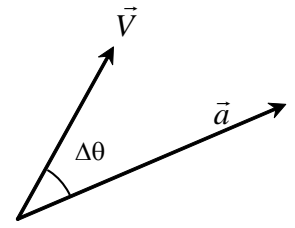
$$a_x = 3\cdot\left(\frac{0.2}{m^2\cdot s}\right)^2 \times 2\cdot m \times \left[(2\cdot m)^2 + (1\cdot m)^2\right]^2 \qquad a_x = 6.00\cdot\frac{m}{s^2}$$

$$a_y = 3\cdot\left(\frac{0.2}{m^2\cdot s}\right)^2 \times 1\cdot m \times \left[(2\cdot m)^2 + (1\cdot m)^2\right]^2 \qquad a_y = 3.00\cdot\frac{m}{s^2}$$

$$a = \sqrt{a_x^{\ 2} + a_y^{\ 2}} \qquad a = 6.71\frac{m}{s^2}$$

We need to find the component of acceleration normal to the velocity vector

At (2,1) the velocity vector is at angle $\theta_{vel} = \operatorname{atan}\left(\frac{v}{u}\right) = \operatorname{atan}\left(\frac{A\cdot y^3 - 3\cdot A\cdot x^2\cdot y}{A\cdot x^3 - 3\cdot A\cdot x\cdot y^2}\right)$

$$\theta_{vel} = \operatorname{atan}\left(\frac{1^3 - 3\cdot 2^2\cdot 1}{2^3 - 3\cdot 2\cdot 1^2}\right) \qquad \theta_{vel} = -79.7\cdot deg$$

At (1,2) the acceleration vector is at angle $\theta_{accel} = \operatorname{atan}\left(\frac{a_y}{a_x}\right)$ $\quad \theta_{accel} = \operatorname{atan}\left(\frac{1}{2}\right)$ $\quad \theta_{accel} = 26.6\cdot deg$

Hence the angle between the acceleration and velocity vectors is $\Delta\theta = \theta_{accel} - \theta_{vel}$ $\quad \Delta\theta = 106\cdot deg$

The component of acceleration normal to the velocity is then $a_n = a\cdot\sin(\Delta\theta) = 6.71\cdot\frac{m}{s^2}\cdot\sin(106\cdot deg)$ $\quad a_n = 6.45\cdot\frac{m}{s^2}$

Problem 6.32

Given: The x component of velocity in a 2-D, incompressible flow field is
$u = Ax^2$ where $A = 1\ ft^{-1} \cdot s^{-1}$ and coordinates are in ft; $w = 0$ and $\partial/\partial z = 0$.

Find: (a) acceleration of fluid particle at $(x, y) = (1, 2)$
(b) radius of curvature of streamline at $(1, 2)$

Plot: streamline through $(1,2)$; show velocity and acceleration vectors on the plot.

Solution:

For 2-D incompressible flow $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, so $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$

$$v = \int \frac{\partial v}{\partial y} dy + f(x) = \int -\frac{\partial u}{\partial x} dy + f(x) = -\int 2Ax\, dy + f(x) = -2Axy + f(x).$$

Choose the simplest solution, $f(x) = 0$, so $v = -2Axy$. Hence

$$\vec{V} = Ax^2\hat{i} - 2Axy\hat{j} = A[x^2\hat{i} - 2xy\hat{j}]$$

The acceleration of a fluid particle is

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = Ax^2[A(2x\hat{i} - 2y\hat{j})] - 2Axy[-2Ax\hat{j}]$$

$$\vec{a}_p = 2A^2x^3\hat{i} + 2A^2x^2y\hat{j} = 2A^2x^2[x\hat{i} + y\hat{j}]$$

At the point $(1, 2)$

$$\vec{a}_p = 2 \times \frac{(1)^2}{ft^2 s^2} \times (1)^2 m^2 [1m\hat{i} + 2m\hat{j}] = 2\hat{i} + 4\hat{j}\ ft/s^2 \longleftarrow \vec{a}_{(1,2)}$$

$$\vec{V} = \frac{1}{ft \cdot s}[(1)^2 m^2 \hat{i} - 2(1m)(2m)\hat{j}] = \hat{i} - 4\hat{j}\ ft/s$$

The unit vector tangent to the streamline is

$$\hat{e}_t = \frac{\vec{V}}{|\vec{V}|} = \frac{\hat{i} - 4\hat{j}}{[(1)^2 + (-4)^2]^{1/2}} = 0.243\hat{i} - 0.970\hat{j}$$

The unit vector normal to the streamline is

$$\hat{e}_n = \hat{e}_t \times \hat{k} = (0.243\hat{i} - 0.970\hat{j}) \times \hat{k} = -0.970\hat{i} - 0.243\hat{j}$$

The normal component of acceleration is

$$a_n = -\frac{V^2}{R} = \vec{a} \cdot \hat{e}_n = (2\hat{i} + 4\hat{j}) \cdot (-0.970\hat{i} - 0.243\hat{j})$$

$$-\frac{V^2}{R} = -2.91\ ft/s^2$$

$$R = \frac{V^2}{2.91} = \frac{17\ ft^2/s^2}{2.91\ ft/s^2} = 5.84\ ft \longleftarrow R$$

The slope of the streamline is given by

$$\left.\frac{dy}{dx}\right)_{s.l} = \frac{v}{u} = \frac{-2Axy}{Ax^2} = \frac{-2y}{x}$$

Thus $\frac{dy}{y} + \frac{2\,dx}{x} = 0$ and $\ln y + \ln x^2 = \ln c$ or

$$x^2 y = c$$

The equation of the streamline through (1,2) is $x^2 y = 2$.

Problem 6.33

Given: Incompressible, 2-D flow with $u = Axy$, $w = 0$; $A = 2\ ft^{-1} s^{-1}$

Find: (a) Acceleration of particle at $(x, y) = (2, 1)$.
(b) Radius of curvature of streamline at that point.
(c) Plot streamline, show velocity vector and acceleration vector.

Solution: For two-d. incompressible flow, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, so

$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -Ay$; Integrating, $v = -\frac{1}{2}Ay^2$; $\vec{V} = Axy\hat{i} - \frac{1}{2}Ay^2\hat{j}$.

The acceleration is

$$a_{p_x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = (Axy)(Ay) + (-\tfrac{1}{2}Ay^2)(Ax) = \tfrac{1}{2}A^2xy^2$$

$$a_{p_y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = (Axy)(0) + (-\tfrac{1}{2}Ay^2)(-Ay) = \tfrac{1}{2}A^2y^3$$

$$\vec{a}_p = \tfrac{1}{2}A^2xy^2\hat{i} + \tfrac{1}{2}A^2y^3\hat{j};\ \text{at } (2,1)\ \ \vec{a}_p = 4\hat{i} + 2\hat{j}\ (ft/s^2)$$

$\vec{a}_p$

Note $a_n = \frac{V^2}{R}$, so $R = \frac{V^2}{a_n}$, where a_n is acceleration normal to $\vec{V}$

At $(2,1)$, $\vec{V} = 4\hat{i} - 1\hat{j}$ ft/s, so $V^2 = (4)^2 + (1)^2 = 17\ ft^2/s^2$

To find a_n, dot $\vec{a}_p$ with $\hat{e}_n$, the unit normal vector. To find $\hat{e}_n$, set

$$\hat{e}_n = -\frac{v}{V}\hat{i} + \frac{u}{V}\hat{j} = \frac{1}{\sqrt{17}}\hat{i} + \frac{4}{\sqrt{17}}\hat{j}$$

$$a_n = \hat{e}_n \cdot \hat{a}_p = \frac{4}{\sqrt{17}} + \frac{8}{\sqrt{17}} = \frac{12}{\sqrt{17}} = 2.91\ ft/s^2$$

Substituting

$$R = \frac{V^2}{a_n} = 17\frac{ft^2}{s^2} \times \frac{s^2}{2.91\ ft} = 5.84\ ft$$

R

The streamline is $\frac{dx}{u} = \frac{dy}{v} = \frac{dx}{Axy} = \frac{dy}{-\frac{1}{2}Ay^2}$ or $\frac{dx}{x} + 2\frac{dy}{y} = 0$

Integrating, $\ln x + 2\ln y = \ln c$ or $xy^2 = c$

For $(x,y) = (2,1)$, then $c = 2\ ft^3$.

The plot and streamlines are on the following page.

Problem 6.33

Components of Velocity and Acceleration:

Input Parameters:

$A =$ 2 $ft^{-1}s^{-1}$

Calculated Values:

$C =$ 2 ft^3

Coord. x	Coord. y	Velocity, V_x	Velocity, V_y	Velocity, V	Accel., a_x	Accel., a_y	Accel., a	Normal Accel., a_n
0.08	5.00							
0.2	3.16							
0.4	2.24							
0.5	2.00	2.00	-4.00	4.47	2.00	16.0	16.1	8.94
0.6	1.83							
0.8	1.58							
1.0	1.41	2.83	-2.00	3.46	2.83	5.66	6.32	6.25
1.5	1.15	3.46	-1.33	3.71	3.46	3.08	4.63	4.12
2.0	1.00	4.00	-1.00	4.12	4.00	2.00	4.47	2.91
2.5	0.89	4.47	-0.80	4.54	4.47	1.43	4.70	2.20
3.0	0.82	4.90	-0.67	4.94	4.90	1.09	5.02	1.74
3.5	0.76	5.29	-0.57	5.32	5.29	0.86	5.36	1.43
4.0	0.71	5.66	-0.50	5.68	5.66	0.71	5.70	1.20
4.5	0.67	6.00	-0.44	6.02	6.00	0.59	6.03	1.03
5.0	0.63	6.32	-0.40	6.34	6.32	0.51	6.34	0.90

Acceleration:

2	1
4	2

Velocity:

2	1
4	0.5

Problem 6.34

6.34 The x component of velocity in a two-dimensional incompressible flow field is given by $u = -\Lambda(x^2 - y^2)/(x^2 + y^2)^2$, where u is in m/s, the coordinates are measured in meters, and $\Lambda = 2\ \mathrm{m^3 \cdot s^{-1}}$. Show that the simplest form of the y component of velocity is given by $v = -2\Lambda xy/(x^2 + y^2)^2$. There is no velocity component or variation in the z direction. Calculate the acceleration of fluid particles at points $(x, y) = (0, 1)$, $(0, 2)$, and $(0, 3)$. Estimate the radius of curvature of the streamlines passing through these points. What does the relation among the three points and their radii of curvature suggest to you about the flow field? Verify this by plotting these streamlines. [Hint: You will need to use an integrating factor.]

Given: x component of velocity field

Find: y component of velocity field; acceleration at several points; estimate radius of curvature; plot streamlines

Solution:

The given data is $\quad \Lambda = 2\cdot\frac{m^3}{s} \qquad u = -\frac{\Lambda\cdot(x^2 - y^2)}{(x^2 + y^2)^2}$

The governing equation (continuity) is $\quad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$

Hence
$$v = -\int \frac{du}{dx}\,dy = -\int \frac{2\cdot\Lambda\cdot x\cdot(x^2 - 3\cdot y^2)}{(x^2 + y^2)^3}\,dy$$

Integrating (using an integrating factor)
$$v = -\frac{2\cdot\Lambda\cdot x\cdot y}{(x^2 + y^2)^2}$$

Alternatively, we could check that the given velocities u and v satisfy continuity

$$u = -\frac{\Lambda\cdot(x^2 - y^2)}{(x^2 + y^2)^2} \qquad \frac{\partial}{\partial x}u = \frac{2\cdot\Lambda\cdot x\cdot(x^2 - 3\cdot y^2)}{(x^2 + y^2)^3} \qquad v = -\frac{2\cdot\Lambda\cdot x\cdot y}{(x^2 + y^2)^2} \qquad \frac{\partial}{\partial y}v = -\frac{2\cdot\Lambda\cdot x\cdot(x^2 - 3\cdot y^2)}{(x^2 + y^2)^3}$$

so $\quad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$

The governing equation for acceleration is
$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\ \text{acceleration}\\ \text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\ \text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\ \text{acceleration}}}$$

For steady, 2D flow this reduces to (after considerable math!)

x - component $\quad a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u$

$$a_x = \left[-\frac{\Lambda\cdot\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\right]\cdot\left[\frac{2\cdot\Lambda\cdot x\cdot\left(x^2-3\cdot y^2\right)}{\left(x^2+y^2\right)^3}\right]+\left[-\frac{2\cdot\Lambda\cdot x\cdot y}{\left(x^2+y^2\right)^2}\right]\cdot\left[\frac{2\cdot\Lambda\cdot y\cdot\left(3\cdot x^2-y^2\right)}{\left(x^2+y^2\right)^3}\right] \qquad a_x = -\frac{2\cdot\Lambda^2\cdot x}{\left(x^2+y^2\right)^3}$$

y - component $\quad a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v$

$$a_y = \left[-\frac{\Lambda\cdot\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\right]\cdot\left[\frac{2\cdot\Lambda\cdot y\cdot\left(3\cdot x^2-y^2\right)}{\left(x^2+y^2\right)^3}\right]+\left[-\frac{2\cdot\Lambda\cdot x\cdot y}{\left(x^2+y^2\right)^2}\right]\cdot\left[\frac{2\cdot\Lambda\cdot y\cdot\left(3\cdot y^2-x^2\right)}{\left(x^2+y^2\right)^3}\right] \qquad a_y = -\frac{2\cdot\Lambda^2\cdot y}{\left(x^2+y^2\right)^3}$$

Evaluating at point (0,1) $\quad u = 2\cdot\frac{m}{s} \quad v = 0\cdot\frac{m}{s} \quad a_x = 0\cdot\frac{m}{s^2} \quad a_y = -8\cdot\frac{m}{s^2}$

Evaluating at point (0,2) $\quad u = 0.5\cdot\frac{m}{s} \quad v = 0\cdot\frac{m}{s} \quad a_x = 0\cdot\frac{m}{s^2} \quad a_y = -0.25\cdot\frac{m}{s^2}$

Evaluating at point (0,3) $\quad u = 0.222\cdot\frac{m}{s} \quad v = 0\cdot\frac{m}{s} \quad a_x = 0\cdot\frac{m}{s^2} \quad a_y = -0.0333\cdot\frac{m}{s^2}$

The instantaneous radius of curvature is obtained from $a_{radial} = -a_y = -\frac{u^2}{r}$ or $r = -\frac{u^2}{a_y}$

For the three points $\quad y = 1\,m \quad r = \frac{\left(2\cdot\frac{m}{s}\right)^2}{8\cdot\frac{m}{s^2}} \quad r = 0.5\,m$

$y = 2\,m \quad r = \frac{\left(0.5\cdot\frac{m}{s}\right)^2}{0.25\cdot\frac{m}{s^2}} \quad r = 1\,m$

$y = 3\,m \quad r = \frac{\left(0.2222\cdot\frac{m}{s}\right)^2}{0.03333\cdot\frac{m}{s^2}} \quad r = 1.5\cdot m$

The radius of curvature in each case is 1/2 of the vertical distance from the origin. The streamlines form circles tangent to the *x* axis

The streamlines are given by $\quad \frac{dy}{dx} = \frac{v}{u} = \frac{-\frac{2\cdot\Lambda\cdot x\cdot y}{\left(x^2+y^2\right)^2}}{-\frac{\Lambda\cdot\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}} = \frac{2\cdot x\cdot y}{\left(x^2-y^2\right)}$

so $\quad -2\cdot x\cdot y\cdot dx + \left(x^2-y^2\right)\cdot dy = 0$

This is an inexact integral, so an integrating factor is needed

First we try $$R = \frac{1}{-2\cdot x\cdot y}\cdot\left[\frac{d}{dx}\left(x^2 - y^2\right) - \frac{d}{dy}(-2\cdot x\cdot y)\right] = -\frac{2}{y}$$

Then the integrating factor is $$F = e^{\int -\frac{2}{y}\,dy} = \frac{1}{y^2}$$

The equation becomes an exact integral $$-2\cdot\frac{x}{y}\cdot dx + \frac{\left(x^2 - y^2\right)}{y^2}\cdot dy = 0$$

So $$u = \int -2\cdot\frac{x}{y}\,dx = -\frac{x^2}{y} + f(y)$$ and $$u = \int \frac{\left(x^2 - y^2\right)}{y^2}\,dy = -\frac{x^2}{y} - y + g(x)$$

Comparing solutions $$\psi = \frac{x^2}{y} + y$$ or $$x^2 + y^2 = \psi\cdot y = \text{const}\cdot y$$

These form circles that are tangential to the *x* axis, as shown in the associated *Excel* workbook

Problem 6.34

6.34 The x component of velocity in a two-dimensional incompressible flow field is given by $u = -\Lambda(x^2 - y^2)/(x^2 + y^2)^2$, where u is in m/s, the coordinates are measured in meters, and $\Lambda = 2\ \mathrm{m^3 \cdot s^{-1}}$. Show that the simplest form of the y component of velocity is given by $v = -2\Lambda xy/(x^2 + y^2)^2$. There is no velocity component or variation in the z direction. Calculate the acceleration of fluid particles at points $(x, y) = (0, 1)$, $(0, 2)$, and $(0, 3)$. Estimate the radius of curvature of the streamlines passing through these points. What does the relation among the three points and their radii of curvature suggest to you about the flow field? Verify this by plotting these streamlines. [Hint: You will need to use an integrating factor.]

Given: x component of velocity field

Find: y component of velocity field; acceleration at several points; estimate radius of curvature; plot streamlines

Solution:

$$\psi = \frac{x^2}{y} + y$$

This function is computed and plotted below

y values

x values	0.10	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.5						
2.50	62.6	25.3	13.0	9.08	7.25	6.25	5.67	5.32	5.13	5.03	5.00	5.02	5.08	5.17	5.2						
2.25	50.7	20.5	10.6	7.50	6.06	5.30	4.88	4.64	4.53	4.50	4.53	4.59	4.69	4.81	4.9						
2.00	40.1	16.3	8.50	6.08	5.00	4.45	4.17	4.04	4.00	4.03	4.10	4.20	4.33	4.48	4.6						
1.75	30.7	12.5	6.63	4.83	4.06	3.70	3.54	3.50	3.53	3.61	3.73	3.86	4.02	4.19	4.3						
1.50	22.6	9.25	5.00	3.75	3.25	3.05	3.00	3.04	3.13	3.25	3.40	3.57	3.75	3.94	4.1						
1.25	15.7	6.50	3.63	2.83	2.56	2.50	2.54	2.64	2.78	2.94	3.13	3.32	3.52	3.73	3.9						
1.00	10.1	4.25	2.50	2.08	2.00	2.05	2.17	2.32	2.50	2.69	2.90	3.11	3.33	3.56	3.7						
0.75	5.73	2.50	1.63	1.50	1.56	1.70	1.88	2.07	2.28	2.50	2.73	2.95	3.19	3.42	3.6						
0.50	2.60	1.25	1.00	1.08	1.25	1.45	1.67	1.89	2.13	2.36	2.60	2.84	3.08	3.33	3.5						
0.25	0.73	0.50	0.63	0.83	1.06	1.30	1.54	1.79	2.03	2.28	2.53	2.77	3.02	3.27	3.52	3.77	4.02	4.26	4.51	4.76	5.01
0.00	0.10	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00

Problem 6.35

Given: The y component of velocity in a 2-D, incompressible flow field is
$v = -Axy$ where $A = 1\ m^{-1}s^{-1}$ and coordinates are in meters; $w = 0$ and $\partial/\partial z = 0$.

Find: (a) acceleration of fluid particle at $(x,y) = (1,2)$
(b) radius of curvature of streamline at $(1,2)$.

Plot: streamline through $(1,2)$; show velocity and acceleration vectors on the plot.

Solution:

For 2-D incompressible flow $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, so $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$.

$$u = \int \frac{\partial u}{\partial x}dx + f(y) = \int -\frac{\partial v}{\partial y}dx + f(y) = -\int(-Ax)\,dx + f(y) = \frac{Ax^2}{2} + f(y)$$

Choose the simplest solution, $f(y) = 0$, so $u = \frac{Ax^2}{2}$. Hence

$$\vec{V} = \frac{Ax^2}{2}\hat{i} - Axy\hat{j} = A\left(\frac{x^2}{2}\hat{i} - xy\hat{j}\right)$$

The acceleration of a fluid particle is

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = \frac{Ax^2}{2}(Ax\hat{i} - Ay\hat{j}) - Axy(-Ax\hat{j})$$

$$\vec{a}_p = \frac{A^2x^3}{2}\hat{i} + \frac{A^2x^2y}{2}\hat{j} = \frac{A^2}{2}(x^3\hat{i} + x^2y\hat{j})$$

At the point $(1,2)$

$$\vec{a}_p = \frac{1}{2}\times(1)\frac{1}{m^2s^2}\left[(1)^3 m^3\hat{i} + (1)^2(2)m^3\hat{j}\right] = 0.5\hat{i} + \hat{j}\ m/s^2 \quad \vec{a}(1,2)$$

$$\vec{V} = \frac{1}{m\cdot s}\left[\frac{1}{2}(1)^2m^2\hat{i} - (1)(2)m^2\hat{j}\right] = 0.5\hat{i} - 2\hat{j}\ m/s$$

The unit vector tangent to the streamline is

$$\hat{e}_t = \frac{\vec{V}}{|\vec{V}|} = \frac{0.5\hat{i} - 2\hat{j}}{[(0.5)^2 + (-2)^2]^{1/2}} = 0.243\hat{i} - 0.970\hat{j}$$

The unit vector normal to the streamline is

$$\vec{e}_n = \hat{e}_t \times \hat{k} = (0.243\hat{i} - 0.970\hat{j}) \times \hat{k} = -0.970\hat{i} - 0.243\hat{j}$$

The normal component of acceleration is

$$a_n = -\frac{V^2}{R} = \vec{a}\cdot\hat{e}_n = (0.5\hat{i} + \hat{j})\cdot(-0.970\hat{i} - 0.243\hat{j})$$

$$-\frac{V^2}{R} = -0.728\ m/s^2$$

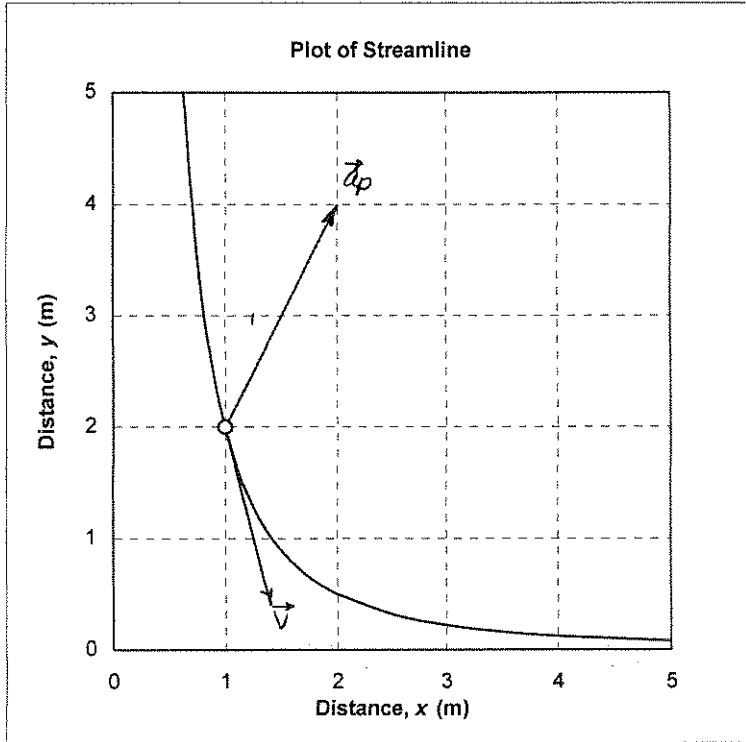
$$R = \frac{V^2}{0.728} = \frac{4.25}{0.728}\frac{m^2/s^2}{m/s^2} = 5.84\ m \quad R(1,2)$$

The slope of the streamlines is given by

$$\left.\frac{dy}{dx}\right)_{sl} = \frac{v}{u} = \frac{-Axy}{Ax^2/2} = -\frac{2y}{x}$$

Thus $\frac{dy}{y} + \frac{2dx}{x} = 0$ and $\ln y + \ln x^2 = \ln c$ or

$$x^2 y = c$$

The equation of the streamline through (1,2) is $x^2 y = 2$

Problem 6.36

6.36 Consider the velocity field $\vec{V} = A[x^4 - 6x^2y^2 + y^4]\hat{i} + B[x^3y - xy^3]\hat{j}$; $A = 2\ \text{m}^{-3}\cdot\text{s}^{-1}$, B is a constant, and the coordinates are measured in meters. Find B for this to be an incompressible flow. Obtain the equation of the streamline through point $(x, y) = (1, 2)$. Derive an algebraic expression for the acceleration of a fluid particle. Estimate the radius of curvature of the streamline at $(x, y) = (1, 2)$.

Given:
Velocity field

Find:
Constant B for incompressible flow; Equation for streamline through (1,2); Acceleration of particle; streamline curvatur

Solution:

Basic equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\text{total acceleration of a particle}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective acceleration}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local acceleration}}$$

For this flow

$$u(x,y) = A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right) \qquad v(x,y) = B\cdot\left(x^3\cdot y - x\cdot y^3\right)$$

$$\frac{\partial}{\partial x}u(x,y) + \frac{\partial}{\partial y}v(x,y) = \frac{\partial}{\partial x}\left[A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)\right] + \frac{\partial}{\partial y}\left[B\cdot\left(x^3\cdot y - x\cdot y^3\right)\right] = 0$$

$$\frac{\partial}{\partial x}u(x,y) + \frac{\partial}{\partial y}v(x,y) = B\cdot\left(x^3 - 3\cdot x\cdot y^2\right) + A\cdot\left(4\cdot x^3 - 12\cdot x\cdot y^2\right) = (4\cdot A + B)\cdot x\cdot\left(x^2 - 3\cdot y^2\right) = 0$$

Hence $\quad B = -4\cdot A \qquad\qquad B = -8\dfrac{1}{m^3\cdot s}$

Hence for a_x

$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)\cdot\frac{\partial}{\partial x}\left[A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)\right] + \left[-4\cdot A\cdot\left(x^3\cdot y - x\cdot y^3\right)\right]\cdot\frac{\partial}{\partial y}\left[A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)\right]$$

$$a_x = 4\cdot A^2\cdot x\cdot\left(x^2 + y^2\right)^3$$

For a_y

$$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v = A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)\cdot\frac{\partial}{\partial x}\left[-4\cdot A\cdot\left(x^3\cdot y - x\cdot y^3\right)\right] + \left[-4\cdot A\cdot\left(x^3\cdot y - x\cdot y^3\right)\right]\cdot\frac{\partial}{\partial y}\left[-4\cdot A\cdot\left(x^3\cdot y - x\cdot y^3\right)\right]$$

$$a_y = 4\cdot A^2\cdot y\cdot\left(x^2 + y^2\right)^3$$

For a streamline $\quad \dfrac{dy}{dx} = \dfrac{v}{u} \quad$ so $\quad \dfrac{dy}{dx} = \dfrac{-4\cdot A\cdot\left(x^3\cdot y - x\cdot y^3\right)}{A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)} = -\dfrac{4\cdot\left(x^3\cdot y - x\cdot y^3\right)}{\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)}$

Let $\quad u = \dfrac{y}{x} \qquad \dfrac{du}{dx} = \dfrac{d\left(\frac{y}{x}\right)}{dx} = \dfrac{1}{x}\cdot\dfrac{dy}{dx} + y\cdot\dfrac{d\left(\frac{1}{x}\right)}{dx} = \dfrac{1}{x}\cdot\dfrac{dy}{dx} - \dfrac{y}{x^2} \qquad \dfrac{dy}{dx} = x\cdot\dfrac{du}{dx} + u$

Hence $$\frac{dy}{dx} = x\cdot\frac{du}{dx} + u = -\frac{4\cdot\left(x^3\cdot y - x\cdot y^3\right)}{\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)} = -\frac{4\cdot\left(1-u^2\right)}{\left(\frac{1}{u} - 6\cdot u + u^3\right)} \quad u + \frac{4\cdot\left(1-u^2\right)}{\left(\frac{1}{u} - 6\cdot u + u^3\right)}$$

$$x\cdot\frac{du}{dx} = -\left[u + \frac{4\cdot\left(1-u^2\right)}{\left(\frac{1}{u} - 6\cdot u + u^3\right)}\right] = -\frac{u\cdot\left(u^4 - 10\cdot u^2 + 5\right)}{u^4 - 6\cdot u^2 + 1}$$

Separating variables $$\frac{dx}{x} = -\frac{u^4 - 6\cdot u^2 + 1}{u\cdot\left(u^4 - 10\cdot u^2 + 5\right)}\cdot du \qquad \ln(x) = -\frac{1}{5}\cdot\ln\left(u^5 - 10\cdot u^3 + 5\cdot u\right) + C$$

$$\left(u^5 - 10\cdot u^3 + 5\cdot u\right)\cdot x^5 = c \qquad y^5 - 10\cdot y^3\cdot x^2 + 5\cdot y\cdot x^4 = \text{const}$$

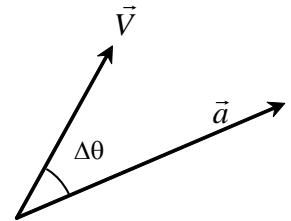
For the streamline through (1,2) $$y^5 - 10\cdot y^3\cdot x^2 + 5\cdot y\cdot x^4 = -38$$

Note that it would be MUCH easier to use the stream function method here!

To find the radius of curvature we use $$a_n = -\frac{V^2}{R} \qquad \text{or} \qquad |R| = \frac{V^2}{a_n}$$

We need to find the component of acceleration normal to the velocity vector

At (1,2) the velocity vector is at angle $$\theta_{vel} = \text{atan}\left(\frac{v}{u}\right) = \text{atan}\left[-\frac{4\cdot\left(x^3\cdot y - x\cdot y^3\right)}{\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)}\right]$$

$$\theta_{vel} = \text{atan}\left[-\frac{4\cdot(2-8)}{1 - 24 + 16}\right] \qquad \theta_{vel} = -73.7\cdot\deg$$

At (1,2) the acceleration vector is at angle $$\theta_{accel} = \text{atan}\left(\frac{a_y}{a_x}\right) = \text{atan}\left[\frac{4\cdot A^2\cdot y\cdot\left(x^2+y^2\right)^3}{4\cdot A^2\cdot x\cdot\left(x^2+y^2\right)^3}\right] = \text{atan}\left(\frac{y}{x}\right)$$

$$\theta_{accel} = \text{atan}\left(\frac{2}{1}\right) \qquad \theta_{accel} = 63.4\cdot\deg$$

Hence the angle between the acceleration and velocity vectors is $$\Delta\theta = \theta_{accel} - \theta_{vel} \qquad \Delta\theta = 137\cdot\deg$$

The component of acceleration normal to the velocity is then $$a_n = a\cdot\sin(\Delta\theta) \quad \text{where} \quad a = \sqrt{a_x^2 + a_y^2}$$

At (1,2) $$a_x = 4\cdot A^2\cdot x\cdot\left(x^2+y^2\right)^3 = 500\cdot m^7 \times A^2 = 500\cdot m^7 \times \left(\frac{2}{m^3\cdot s}\right)^2 = 2000\cdot\frac{m}{s^2}$$

$$a_y = 4\cdot A^2\cdot y\cdot\left(x^2+y^2\right)^3 = 1000\cdot m^7 \times A^2 = 1000\cdot m^7 \times \left(\frac{2}{m^3\cdot s}\right)^2 = 4000\cdot\frac{m}{s^2}$$

$$a = \sqrt{2000^2 + 4000^2}\cdot\frac{m}{s^2} \qquad a = 4472\frac{m}{s^2} \qquad a_n = a\cdot\sin(\Delta\theta) \qquad a_n = 3040\frac{m}{s^2}$$

$$u = A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right) = -14\cdot\frac{m}{s} \qquad v = B\cdot\left(x^3\cdot y - x\cdot y^3\right) = 48\cdot\frac{m}{s} \qquad V = \sqrt{u^2+v^2} = 50\cdot\frac{m}{s}$$

Then $$|R| = \frac{V^2}{a_n} \qquad R = \left(50\cdot\frac{m}{s}\right)^2 \times \frac{1}{3040}\cdot\frac{s^2}{m} \qquad R = 0.822\,m$$

Problem 6.37

6.37 Water flows at a speed of 10 ft/s. Calculate the dynamic pressure of this flow. Express your answer in in. of mercury.

Given: Water at speed 10 ft/s

Find: Dynamic pressure in in. Hg

Solution:

Basic equation $p_{dynamic} = \frac{1}{2}\cdot\rho\cdot V^2$ $\qquad p = \rho_{Hg}\cdot g\cdot\Delta h = SG_{Hg}\cdot\rho\cdot g\cdot\Delta h$

Hence $\Delta h = \frac{\rho\cdot V^2}{2\cdot SG_{Hg}\cdot\rho\cdot g} = \frac{V^2}{2\cdot SG_{Hg}\cdot g}$

$$\Delta h = \frac{1}{2}\times\left(10\cdot\frac{ft}{s}\right)^2\times\frac{1}{13.6}\times\frac{s^2}{32.2\cdot ft}\times\frac{12\cdot in}{1\cdot ft}\qquad \Delta h = 1.37\cdot in$$

Problem 6.38

Given: Standard air

Find: Dynamic pressure that corresponds to $V = 100\ km/hr$

Solution: Dynamic pressure is $p_{dyn} = \frac{1}{2}\rho V^2$

For standard air, $\rho = 1.23\ kg/m^3$

Then $$p_{dyn} = \frac{1}{2} \times 1.23 \frac{kg}{m^3} \times (100)^2 \frac{(km)^2}{(hr)^2} \times (1000)^2 \frac{m^2}{(km)^2} \times \frac{(hr)^2}{(3600)^2 s^2} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$p_{dyn} = 475\ N/m^2$$ ← p_{dyn}

This may be expressed conveniently as a water column height.

$$p_{dyn} = \rho_{water}\, g\, h_{dyn}$$

$$h_{dyn} = \frac{p_{dyn}}{\rho_w g} = 475 \frac{N}{m^2} \times \frac{m^3}{999\ kg} \times \frac{s^2}{9.81\ m} \times \frac{kg \cdot m}{N \cdot s^2}$$

$$h_{dyn} = 0.0484\ m \text{ or } 48.4\ mm$$ ← h_{dyn}

Problem 6.39

6.39 You present your open hand out of the window of an automobile perpendicular to the airflow. Assuming for simplicity that the air pressure on the entire front surface is stagnation pressure (with respect to automobile coordinates), with atmospheric pressure on the rear surface, estimate the net force on your hand when driving at (a) 30 mph and (b) 60 mph. Do these results roughly correspond with your experience? Do the simplifications tend to make the calculated force an over- or underestimate?

Given: Velocity of automobile

Find: Estimates of aerodynamic force on hand

Solution:

For air $\rho = 0.00238 \cdot \frac{slug}{ft^3}$

We need an estimate of the area of a typical hand. Personal inspection indicates that a good approximation is a square of sides 9 cm and 17 cm

$$A = 9 \cdot cm \times 17 \cdot cm \qquad A = 153\,cm^2$$

The governing equation is the Bernoulli equation (in coordinates attached to the vehicle)

$$p_{atm} + \frac{1}{2} \cdot \rho \cdot V^2 = p_{stag}$$

where V is the free stream velocity

Hence, for p_{stag} on the front side of the hand, and p_{atm} on the rear, by assumption,

$$F = \left(p_{stag} - p_{atm}\right) \cdot A = \frac{1}{2} \cdot \rho \cdot V^2 \cdot A$$

(a) $V = 30 \cdot mph$

$$F = \frac{1}{2} \cdot \rho \cdot V^2 \cdot A = \frac{1}{2} \times 0.00238 \cdot \frac{slug}{ft^3} \times \left(30 \cdot mph \cdot \frac{22 \cdot \frac{ft}{s}}{15 \cdot mph}\right)^2 \times 153 \cdot cm^2 \times \left(\frac{\frac{1}{12} \cdot ft}{2.54 \cdot cm}\right)^2 \qquad F = 0.379\,lbf$$

(b) $V = 60 \cdot mph$

$$F = \frac{1}{2} \cdot \rho \cdot V^2 \cdot A = \frac{1}{2} \times 0.00238 \cdot \frac{slug}{ft^3} \times \left(60 \cdot mph \cdot \frac{22 \cdot \frac{ft}{s}}{15 \cdot mph}\right)^2 \times 153 \cdot cm^2 \times \left(\frac{\frac{1}{12} \cdot ft}{2.54 \cdot cm}\right)^2 \qquad F = 1.52\,lbf$$

Problem 6.40

[2]

6.40 A jet of air from a nozzle is blown at right angles against a wall in which two pressure taps are located. A manometer connected to the tap directly in front of the jet shows a head of 0.15 in. of mercury above atmospheric. Determine the approximate speed of the air leaving the nozzle if it is at 50°F and 14.7 psia. At the second tap a manometer indicates a head of 0.10 in. of mercury above atmospheric; what is the approximate speed of the air there?

Given: Air jet hitting wall generating pressures

Find: Speed of air at two locations

Solution:

Basic equation $\quad \frac{p}{\rho_{air}} + \frac{V^2}{2} + g\cdot z = const \qquad \rho_{air} = \frac{p}{R_{air}\cdot T} \qquad \Delta p = \rho_{Hg}\cdot g\cdot \Delta h = SG_{Hg}\cdot\rho\cdot g\cdot\Delta h$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Hence, applying Bernoulli between the jet and where it hits the wall directly

$$\frac{p_{atm}}{\rho_{air}} + \frac{V_j^2}{2} = \frac{p_{wall}}{\rho_{air}} \qquad p_{wall} = \frac{\rho_{air}\cdot V_j^2}{2} \qquad \text{(working in gage pressures)}$$

For air
$$\rho_{air} = 14.7\cdot\frac{lbf}{in^2}\times\frac{144\cdot in^2}{1\cdot ft^2}\times\frac{lbm\cdot R}{53.33\cdot ft\cdot lbf}\times\frac{1\cdot slug}{32.2\cdot lbm}\times\frac{1}{(50+460)\cdot R} \qquad \rho_{air} = 2.42\times10^{-3}\frac{slug}{ft^3}$$

Hence
$$p_{wall} = SG_{Hg}\cdot\rho\cdot g\cdot\Delta h = \frac{\rho_{air}\cdot V_j^2}{2} \qquad \text{so} \qquad V_j = \sqrt{\frac{2\cdot SG_{Hg}\cdot\rho\cdot g\cdot\Delta h}{\rho_{air}}}$$

Hence
$$V_j = \sqrt{2\times13.6\times1.94\cdot\frac{slug}{ft^3}\times\frac{1}{2.42\times10^{-3}}\cdot\frac{ft^3}{slug}\times32.2\cdot\frac{ft}{s^2}\times0.15\cdot in\times\frac{1ft}{12\cdot in}} \qquad V_j = 93.7\frac{ft}{s}$$

Repeating the analysis for the second point

$$\frac{p_{atm}}{\rho_{air}} + \frac{V_j^2}{2} = \frac{p_{wall}}{\rho_{air}} + \frac{V^2}{2} \qquad V = \sqrt{V_j^2 - \frac{2\cdot p_{wall}}{\rho_{air}}} = \sqrt{V_j^2 - \frac{2\cdot SG_{Hg}\cdot\rho\cdot g\cdot\Delta h}{\rho_{air}}}$$

Hence
$$V = \sqrt{\left(93.7\cdot\frac{ft}{s}\right)^2 - 2\times13.6\times1.94\cdot\frac{slug}{ft^3}\times\frac{1}{2.42\times10^{-3}}\cdot\frac{ft^3}{slug}\times32.2\cdot\frac{ft}{s^2}\times0.1\cdot in\times\frac{1ft}{12\cdot in}} \qquad V = 54.1\frac{ft}{s}$$

Problem 6.41

Given: Pitot static probe is used to measure speed in standard air.

$V = 100$ m/s

Find: Manometer deflection in mm H_2O, corresponding to given conditions.

Solution:

Manometer reads $p_0 - p$ in mm of H_2O.

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$ for flow

$\frac{dp}{dz} = -\rho g$ for manometer

Assumptions: (1) steady flow
(2) incompressible flow
(3) flow along a streamline
(4) frictionless deceleration to p_0
(5) ρ = constant for manometer

From the Bernoulli equation

$$\frac{p_0}{\rho} = \frac{p}{\rho} + \frac{V^2}{2}$$

$$p_0 - p = \rho\frac{V^2}{2}$$

For the manometer, $dp = -\rho g\,dz$

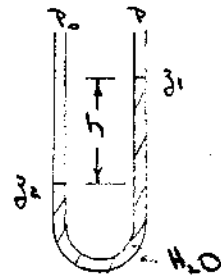
$$p_0 - p = \int_p^{p_0} dp = -\rho g(z_2 - z_1) = \rho g h$$

Then,

$$\rho_{H_2O}\, g h = \rho_{air}\frac{V^2}{2}$$

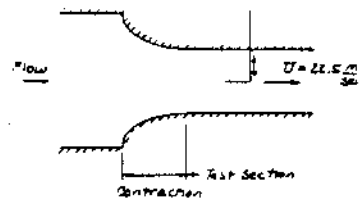
and

$$h = \frac{\rho_{air}}{\rho_{H_2O}}\frac{V^2}{2g} = \frac{1.23}{999} \times (100)^2\frac{m^2}{s^2} \times \frac{1}{2} \times \frac{s^2}{9.81\,m} \times \frac{10^3\,mm}{m} = 628\ mm$$ ← h

Problem 6.42

Given: Wind tunnel with inlet and test section as shown.

$\bar{U} = 22.5$ m/s , $p_{0s} = -6.0$ mm H_2O gage

$p_a = 99.1$ kPa (abs) , $T_a = 23°C$

Find: (a) $p_{dynamic}$ on tunnel centerline
(b) p_{static} " " "
(c) compare p_{static} at tunnel wall with that measured at centerline

Solution:

(a) By definition $\quad p_{dyn} = \frac{1}{2}\rho \bar{U}^2$

Assume: (1) air behaves as an ideal gas, and (2) incompressible flow

Then $\rho = \frac{p}{RT} = 99.1\times10^3 \frac{N}{m^2} \times \frac{kg\cdot K}{287 N\cdot m} \times \frac{1}{(273+23)K} = 1.17\ kg/m^3$

and $p_{dyn} = \frac{1}{2}\rho\bar{U}^2 = \frac{1}{2}\times 1.17\frac{kg}{m^3}\times(22.5)^2\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} = 296\ N/m^2 \longleftarrow p_{dyn}$

(b) By definition $\quad p_0 = p_s + p_{dyn}$

$\therefore p_s = p_0 - p_{dyn} \qquad$ where $p_0 = -6$ mm H_2O gage

then $p_0 - p_a = \rho g \Delta h = 999\frac{kg}{m^3}\times 9.81\frac{m}{s^2}\times -6\times10^{-3} m \times \frac{N\cdot s^2}{kg\cdot m}$

$p_{0gage} = -58.8\ N/m^2$

$\therefore p_s = p_0 - p_{dyn} = -58.8 - 296 = -355\ N/m^2$ gage $\longleftarrow p_{static}$

{ or $p_s = -36.2$ mm H_2O (gage) }

(c) Streamlines in the test section should be straight. Then in the test section the variation of static pressure is given by $\frac{\partial p}{\partial n} = 0$ and $p_{wall} = p_{centerline} \longleftarrow$

In the contraction section the streamlines are curved. The variation of static pressure normal to the streamlines is given by $\frac{\partial p}{\partial n} = \rho\frac{V^2}{R}$

and consequently the static pressure increases toward the centerline, i.e. $p_{wall} < p_{centerline}$

Problem 6.43

[2]

Given: High-pressure hydraulic system subject to small leak.

Plot: Jet speed of a leak vs system-pressure for system pressures up to 40 MPa gage; explain how a high-speed jet of hydraulic fluid can cause injury

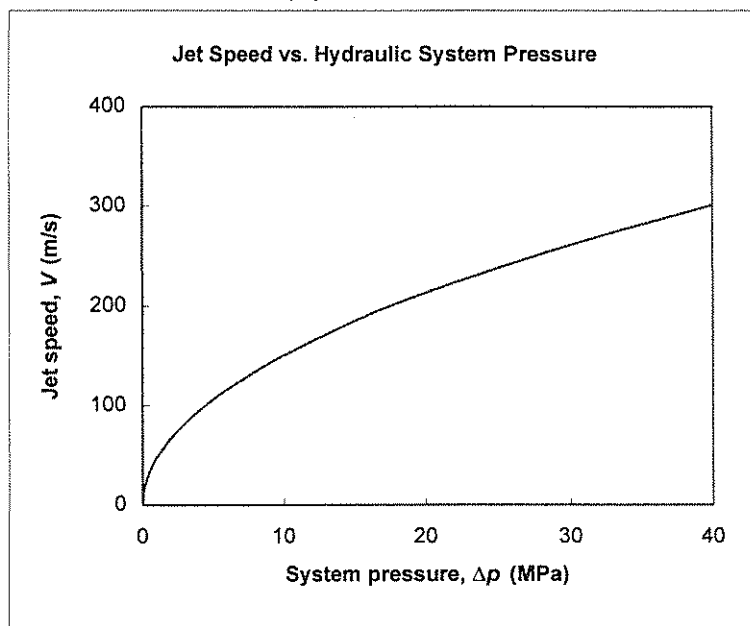
Solution:

Basic equation: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline.

The Bernoulli equation gives

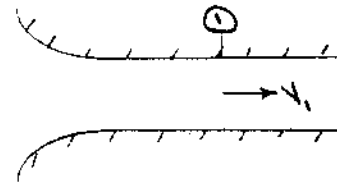
$$V = \left[\frac{2(p_0 - p_{atm})}{\rho} \right]^{1/2}$$

From Table A.2 (Appendix A) for lubricating oil SG = 0.88

The high stagnation pressure ruptures the skin causing the jet to penetrate the tissue.

Given: Air flow in open circuit wind tunnel as shown.

$p_{atm} - p_1 = 45 \text{ mm } H_2O$

$T_0 = 25C$

$p_0 = p_{atm}$

Consider air to be incompressible.

Find: Air speed in tunnel at section ①

Solution:

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$

Assumptions:
(1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) air behaves as an ideal gas
(6) stagnation pressure = p_{atm}

From the Bernoulli equation, $\frac{p_0}{\rho} = \frac{p_1}{\rho} + \frac{V_1^2}{2}$

$$p_0 - p_1 = p_{atm} - p_1 = \frac{1}{2}\rho V_1^2$$

$$V_1 = \left[\frac{2(p_{atm} - p_1)}{\rho}\right]^{1/2}$$

From the manometer reading, $p_{atm} - p_1 = \rho_{H_2O} g h$ then

$$V_1 = \left[\frac{2\rho_{H_2O} g h}{\rho}\right]^{1/2}$$

From the ideal gas equation of state

$$\rho = \frac{p}{RT} = 100 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287\, N \cdot m} \times \frac{1}{298\, K} = 1.17\, kg/m^3$$

$$V_1 = \left[\frac{2\rho_{H_2O}}{\rho} gh\right]^{1/2} = \left[2 \times \frac{999}{1.17} \times 9.81 \frac{m}{s^2} \times 0.045\, m\right]^{1/2} = 27.5\, m/s$$

V_1

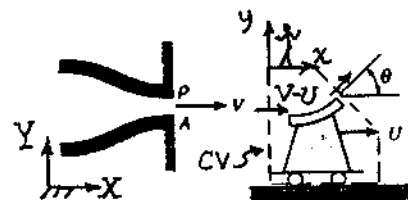
Problem 6.45

Given: Wheeled cart of Problem 4.123;

$V = 40$ m/s

$A = 25$ mm^2

Water, no friction on vane, $\theta = 120°$

Vane accelerates to the right

Find: At instant when $U = 15$ m/s,

(a) stagnation pressure leaving nozzle, relative to fixed observer.

(b) Stagnation pressure leaving nozzle, relative to observer on vane.

(c) Absolute velocity of jet leaving vane.

(d) Stagnation pressure of jet leaving vane, relative to fixed observer.

(e) How would viscous forces increase, decrease, or leave unchanged the stagnation pressure in (d). How can you justify this?

Solution: Stagnation pressure is $p_0 = p + \frac{1}{2}\rho V^2$ or $p_0 - p = \frac{1}{2}\rho V^2$

At jet, $p_{0j} = \frac{1}{2}\rho V^2 = \frac{1}{2} \times 999 \frac{kg}{m^3} \times (40)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 799$ kPa (gage) $\qquad p_{0j}$

At cart, $p_{0,rel} = \frac{1}{2}\rho(V-U)^2 = \frac{1}{2} \times 999 \frac{kg}{m^3} \times (40-15)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 312$ kPa (gage) $\qquad p_{0,rel}$

Leaving vane, $\vec{V}_{abs} = U\hat{i} + (V-U)(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

$$\vec{V}_{abs} = [U + (V-U)\cos\theta]\hat{i} + (V-U)\sin\theta\,\hat{j}$$

$$= \left[15 \frac{m}{s} + (40-15)\frac{m}{s} \times \left(-\frac{1}{2}\right)\right]\hat{i} + (40-15)\frac{m}{s} \times 0.866\,\hat{j}$$

$$\vec{V}_{abs} = 2.5\,\hat{i} + 21.7\,\hat{j} \text{ m/s} \qquad \vec{V}_{abs}$$

The magnitude $|\vec{V}_{abs}| = [(2.5)^2 + (21.7)^2]^{1/2}$ m/s $= 21.8$ m/s

Leaving vane, $p_0 = \frac{1}{2}\rho|\vec{V}_{abs}|^2$, relative to a fixed observer. Thus

$$p_{0,fixed} = \frac{1}{2} \times 999 \frac{kg}{m^3} \times (21.8)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 237 \text{ kPa (gage)} \qquad p_{0,fixed}$$

{The corresponding absolute pressures are 900, 413, and 338 kPa (abs).}

Discussion: Viscous forces would slow the jet speed relative to the vane. The jet would enter the vane with relative speed $(V - U)$; it would leave the vane with speed $\alpha\,(V - U)$, where $\alpha < 1$.

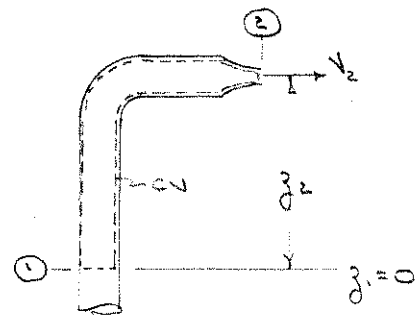
Friction would reduce both components of relative velocity leaving the vane. The absolute velocity of the jet leaving the vane, as seen by a fixed observer, would decrease. Thus the stagnation pressure of the flow leaving the vane, relative to a fixed observer, would decrease.

Problem 6.46

Given: Steady flow of water through elbow and nozzle as shown

$D_1 = 0.1 m$ $D_2 = 0.05 m$

$P_2 = P_{atm}$ $V_2 = 20 m/s$

$z_1 = 0$ $z_2 = 4 m$

Find: Gage pressure, P_1; P_1 if device were inverted

Solution: Apply continuity to cv shown to determine V_1; the Bernoulli equation is then applied along a streamline from ① to ② to determine P_1.

Basic equations:
$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$
$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + g z_2$$

Assumptions:
(1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a steamline
(5) P_2 gage $= 0$
(6) $z_1 = 0$

From the continuity equation, $0 = -|\rho V_1 A_1| + |\rho V_2 A_2|$

then, $V_1^2 = \left(\frac{A_2}{A_1}\right)^2 V_2^2 = \left(\frac{D_2}{D_1}\right)^4 V_2^2$

From the Bernoulli equation

$$P_1 = \rho\left[\frac{V_2^2 - V_1^2}{2} + g z_2\right] = \rho\left[\frac{V_2^2}{2}\left(1 - \frac{V_1^2}{V_2^2}\right) + g z_2\right] = \rho\left[\frac{V_2^2}{2}\left(1 - \left(\frac{D_2}{D_1}\right)^4\right) + g z_2\right]$$

$$P_1 = 999 \frac{kg}{m^3}\left[\frac{1}{2} \times (20)^2 \frac{m^2}{s^2} \times \left(1 - \left(\frac{1}{2}\right)^4\right) + 9.81 \frac{m}{s^2} \times 4 m\right] \times \frac{N \cdot s^2}{kg \cdot m}$$

$P_1 = 227 \; kN/m^2 = 227 \; kPa$ (gage) ← P_1

If device is inverted, $z_2 = -4 m$ with $z_1 = 0$

$$P_1 = \rho\left[\frac{V_2^2}{2}\left\{1 - \left(\frac{D_2}{D_1}\right)^4\right\} + g z_2\right]$$
$$= 999 \frac{kg}{m^3}\left[\frac{1}{2} \times (20)^2 \frac{m^2}{s^2}\left\{1 - \left(\frac{1}{2}\right)^4\right\}\right] + 9.81 \frac{m}{s^2} \times (-4 m)\Big] \frac{N \cdot s^2}{kg \cdot m}$$

$P_1 = 148 \; kN/m^2 = 148 \; kPa$ (gage) ← P_1

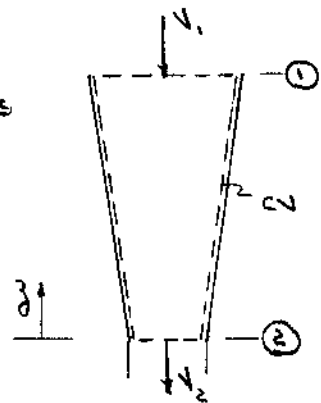
Problem 6.47

Given: Water flow in a circular duct

$D_1 = 0.3\,m \quad P_1 = 260\,kPa\,(gage) \quad \vec{V}_1 = -3\hat{k}\,m/s$

$z_1 = 10\,m$

$z_2 = 0 \quad D_2 = 0.15\,m$

Frictional effects may be neglected.

Find: Pressure, P_2

Solution: Apply continuity to cv shown to determine V_2; the Bernoulli equation is then applied along a streamline from ① to ② to determine P_2

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$ (=0(1))

$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Assumptions:
(1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) uniform flow at sections ① and ②

From the continuity equation

$$0 = -|\rho V_1 A_1| + |\rho V_2 A_2|$$

Then,

$$V_2 = \frac{A_1}{A_2} V_1 = \left(\frac{D_1}{D_2}\right)^2 V_1 = \left(\frac{0.3}{0.15}\right)^2 \times 3\,\frac{m}{s} = 12\,m/s$$

From the Bernoulli equation,

$$P_2 = P_1 + \frac{\rho}{2}(V_1^2 - V_2^2) + \rho g (z_1 - z_2)$$

$$= 260\,\frac{kN}{m^2} + \frac{1}{2} \times 999\,\frac{kg}{m^3} \times \left[(3)^2 - (12)^2\right] \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} + 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 10\,m \times \frac{N \cdot s^2}{kg \cdot m}$$

$P_2 = 291\,kN/m^2 = 291\,kPa\,(gage)$. $\qquad P_2$

Problem 6.48

6.48 You are on a date. Your date runs out of gas unexpectedly. You come to your own rescue by siphoning gas from another car. The height difference for the siphon is about 6 in. The hose diameter is 1 in. What is your gasoline flow rate?

Given: Siphoning of gasoline

Find: Flow rate

Solution:

Basic equation
$$\frac{p}{\rho_{gas}} + \frac{V^2}{2} + g \cdot z = \text{const}$$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Hence, applying Bernoulli between the gas tank free surface and the siphon exit

$$\frac{p_{atm}}{\rho_{gas}} = \frac{p_{atm}}{\rho_{gas}} + \frac{V^2}{2} - g \cdot h$$

where we assume the tank free surface is slowly changing so V_{tank} <<, and h is the difference in levels

Hence
$$V = \sqrt{2 \cdot g \cdot h}$$

The flow rate is then
$$Q = V \cdot A = \frac{\pi \cdot D^2}{4} \cdot \sqrt{2 \cdot g \cdot h}$$

$$Q = \frac{\pi}{4} \times (1 \cdot \text{in})^2 \times \frac{1 \cdot \text{ft}^2}{144 \cdot \text{in}^2} \times \sqrt{2 \times 32.2 \frac{\text{ft}}{\text{s}^2} \times \frac{1}{2} \cdot \text{ft}} \qquad Q = 0.0309 \frac{\text{ft}^3}{\text{s}} \qquad Q = 13.9 \frac{\text{gal}}{\text{min}}$$

Problem 6.49

[2]

6.49 A pipe ruptures and benzene shoots 25 ft into the air. What is the pressure inside the pipe?

Given: Ruptured pipe

Find: Pressure in tank

Solution:

Basic equation $\frac{p}{\rho_{ben}} + \frac{V^2}{2} + g \cdot z = \text{const}$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Hence, applying Bernoulli between the pipe and the rise height of the benzene

$$\frac{p_{pipe}}{\rho_{ben}} = \frac{p_{atm}}{\rho_{ben}} + g \cdot h$$

where we assume V_{pipe} <<, and h is the rise height

Hence $p_{pipe} = \rho_{ben} \cdot g \cdot h = SG_{ben} \cdot \rho \cdot g \cdot h$ where p_{pipe} is now the gage pressure

From Table A.2 $SG_{ben} = 0.879$

Hence $p_{ben} = 0.879 \times 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times 25 \cdot ft \times \frac{lbf \cdot s^2}{slug\,ft}$ $\quad p_{ben} = 1373 \frac{lbf}{ft^2}$ $\quad p_{ben} = 9.53\,psi$ (gage)

Problem 6.50

[2]

6.50 A can of Coke has a small pinhole leak in it. The Coke is being sprayed vertically in the air to a height of 20 in. What is the pressure inside the can of Coke?

Given: Ruptured Coke can

Find: Pressure in can

Solution:

Basic equation
$$\frac{p}{\rho_{Coke}} + \frac{V^2}{2} + g \cdot z = \text{const}$$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Hence, applying Bernoulli between the coke can and the rise height of the coke

$$\frac{p_{can}}{\rho_{Coke}} = \frac{p_{atm}}{\rho_{Coke}} + g \cdot h$$
where we assume V_{Coke} <<, and h is the rise height

Hence
$$p_{Coke} = \rho_{Coke} \cdot g \cdot h = SG_{Coke} \cdot \rho \cdot g \cdot h$$
where p_{pipe} is now the gage pressure

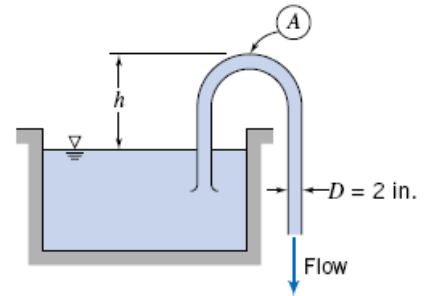
From a web search $SG_{DietCoke} = 1$ $SG_{RegularCoke} = 1.11$

Hence
$$p_{Diet} = 1 \times 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times 20 \cdot in \times \frac{1 \cdot ft}{12 \cdot in} \times \frac{lbf \cdot s^2}{slug\,ft}$$
$p_{Diet} = 104 \cdot \frac{lbf}{ft^2}$ $p_{Diet} = 0.723 \cdot psi$ (gage)

Hence
$$p_{Regular} = 1.11 \times 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times 20 \cdot in \times \frac{1 \cdot ft}{12 \cdot in} \times \frac{lbf \cdot s^2}{slug\,ft}$$
$p_{Regular} = 116 \cdot \frac{lbf}{ft^2}$ $p_{Regular} = 0.803 \cdot psi$ (gage)

Problem 6.51

[2]

6.51 The water flow rate through the siphon is 0.7 ft³/s, its temperature is 70°F, and the pipe diameter is 2 in. Compute the maximum allowable height, *h*, so that the pressure at point *A* is above the vapor pressure of the water. (Assume the flow is frictionless.)

Given:

Flow rate through siphon

Find:

Maximum height h to avoid cavitation

Solution:

Basic equation $\frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{const}$ $\qquad Q = V\cdot A$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

From continuity $V = \frac{Q}{A} = \frac{4\cdot Q}{\pi\cdot D^2}$ $\qquad V = \frac{4}{\pi}\times 0.7\cdot\frac{ft^3}{s}\times\left(\frac{1}{2\cdot in}\right)^2\times\left(\frac{12\cdot in}{1\cdot ft}\right)^2$ $\qquad V = 32.1\frac{ft}{s}$

Hence, applying Bernoulli between the free surface and point A

$$\frac{p_{atm}}{\rho} = \frac{p_A}{\rho} + g\cdot h + \frac{V^2}{2}$$

where we assume $V_{Surface}$ <<

Hence $p_A = p_{atm} - \rho\cdot g\cdot h - \rho\cdot\frac{V^2}{2}$

From the steam tables, at 70ºF the vapor pressure is $\qquad p_v = 0.363\cdot psi$

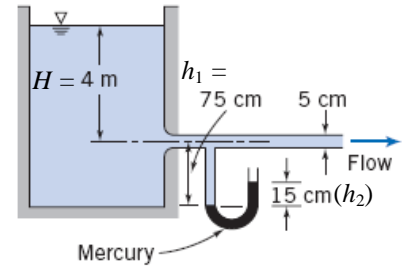
This is the lowest permissible value of p_A

Hence $p_A = p_v = p_{atm} - \rho\cdot g\cdot h - \rho\cdot\frac{V^2}{2}$ $\qquad$ or $\qquad h = \frac{p_{atm} - p_v}{\rho\cdot g} - \frac{V^2}{2\cdot g}$

Hence $h = (14.7 - 0.363)\cdot\frac{lbf}{in^2}\times\left(\frac{12\cdot in}{1\cdot ft}\right)^2\times\frac{1}{1.94}\cdot\frac{ft^3}{slug}\times\frac{s^2}{32.2\cdot ft}\times\frac{slug\cdot ft}{lbf\cdot s^2} - \frac{1}{2}\times\left(32.18\frac{ft}{s}\right)^2\times\frac{s^2}{32.2\cdot ft}$ $\qquad h = 17.0\,ft$

Problem 6.52

[2]

6.52 Water flows from a very large tank through a 5-cm-diameter tube. The dark liquid in the manometer is mercury. Estimate the velocity in the pipe and the rate of discharge from the tank. (Assume the flow is frictionless.)

Given: Flow through tank-pipe system

Find: Velocity in pipe; Rate of discharge

Solution:

Basic equation $\frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{const}$ $\quad \Delta p = \rho\cdot g\cdot \Delta h \quad$ $Q = V\cdot A$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Hence, applying Bernoulli between the free surface and the manometer location

$$\frac{p_{atm}}{\rho} = \frac{p}{\rho} - g\cdot H + \frac{V^2}{2}$$ where we assume $V_{Surface}$ <<, and H = 4 m

Hence $$p = p_{atm} + \rho\cdot g\cdot H - \rho\cdot\frac{V^2}{2}$$

For the manometer $$p - p_{atm} = SG_{Hg}\cdot\rho\cdot g\cdot h_2 - \rho\cdot g\cdot h_1$$ Note that we have water on one side and mercury on the other of the manometer

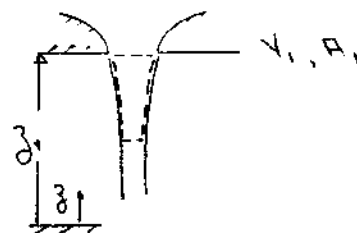
Combining equations $$\rho\cdot g\cdot H - \rho\cdot\frac{V^2}{2} = SG_{Hg}\cdot\rho\cdot g\cdot h_2 - \rho\cdot g\cdot h_1$$ or $$V = \sqrt{2\cdot g\cdot\left(H - SG_{Hg}\cdot h_2 + h_2\right)}$$

Hence $$V = \sqrt{2\times 9.81\cdot\frac{m}{s^2}\times(4 - 13.6\times 0.15 + 0.75)\cdot m}$$ $$V = 7.29\frac{m}{s}$$

The flow rate is $$Q = V\cdot\frac{\pi\cdot D^2}{4}$$ $$Q = \frac{\pi}{4}\times 7.29\cdot\frac{m}{s}\times(0.05\cdot m)^2$$ $$Q = 0.0143\frac{m^3}{s}$$

Problem 6.53

Given: Liquid stream leaving a nozzle pointing downward as shown
Assume uniform flow
Neglect friction

Find: Variation in jet area for $z < z_0$

Solution:

Basic equations: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p}{\rho} + \frac{V^2}{2} + g z$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) $p = p_1 = p_{atm}$
(6) uniform flow at a section

From the Bernoulli equation

$$V^2 = V_1^2 + 2g(z_1 - z)$$

From the continuity equation

$$0 = \int_{CS} \rho \vec{V} \cdot d\vec{A} = -\{|\rho V_1 A_1|\} + \{|\rho V A|\}$$

and

$$V_1 A_1 = V A \quad \text{or} \quad V = V_1 \frac{A_1}{A}$$

Thus

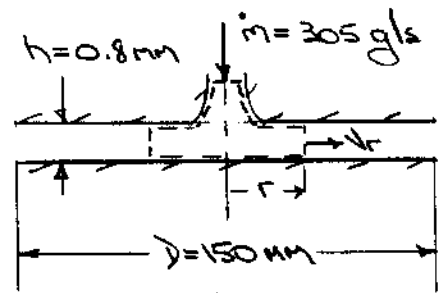
$$V_1^2 \left(\frac{A_1}{A}\right)^2 = V_1^2 + 2g(z_1 - z)$$

Solving for A,

$$A = A_1 \sqrt{\frac{1}{1 + \frac{2g(z_1 - z)}{V_1^2}}}$$

$A(z)$

{Note: jet area decreases as z decreases, owing to the higher velocity}

Given: Water flow between parallel disks discharging to atmosphere as shown.

Find: (a) theoretical static pressure between the disks at r = 50 mm.
(b) in actual laboratory situation, would the pressure be above or below the theoretical value?

Solution:

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Assumptions: (1) steady flow
(2) incompressible flow
(3) flow along a streamline
(4) neglect friction
(5) uniform flow at each section

Apply continuity to the CV shown

$$0 = \{-\dot{m}\} + \{\rho V_r \, 2\pi r h\} \quad \text{so} \quad V = \frac{\dot{m}}{2\pi \rho r h}$$

$$V_1 = V_{r=50mm} = \frac{1}{2\pi} \times 0.305 \frac{kg}{s} \times \frac{m^3}{999 \, kg} \times \frac{1}{0.050 \, m} \times \frac{1}{8 \times 10^{-4} \, m} = 1.21 \, m/s$$

$$V_2 = V_{r=R} = \frac{1}{2\pi} \times 0.305 \frac{kg}{s} \times \frac{m^3}{999 \, kg} \times \frac{1}{0.075 \, m} \times \frac{1}{8 \times 10^{-4} \, m} = 0.810 \, m/s$$

From the Bernoulli equation

$$p_1 - p_2 = p_{r=50mm} - p_{atm} = \frac{1}{2}\rho V_2^2 - \frac{1}{2}\rho V_1^2 = \frac{\rho}{2}(V_2^2 - V_1^2)$$

$$p_{r=50mm} = \frac{1}{2} \times 999 \frac{kg}{m^3} \left[(0.810)^2 - (1.21)^2\right] \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$p_{r=50mm} = -404 \, N/m^2 \text{ (gage)} \qquad p_{r=50mm}$$

Friction would cause a pressure drop in the flow direction. Since the discharge pressure is fixed at p_{atm}, the measured pressure would be greater than the theoretical value.

Problem 6.55

[2]

6.55 Consider steady, frictionless, incompressible flow of air over the wing of an airplane. The air approaching the wing is at 75 kPa (gage), 4°C, and has a speed of 60 m/s relative to the wing. At a certain point in the flow, the pressure is 3 kPa (gage). Calculate the speed of the air relative to the wing at this point.

Given: Air flow over a wing

Find: Air speed relative to wing at a point

Solution:

Basic equation $\frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{const}$ $\qquad p = \rho \cdot R \cdot T$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Hence, applying Bernoulli between the upstream point (1) and the point on the wing (2)

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_2}{\rho} + \frac{V_2^2}{2} \qquad \text{where we ignore gravity effects}$$

Hence $$V_2 = \sqrt{V_1^2 + 2 \cdot \frac{(p_1 - p_2)}{\rho}}$$

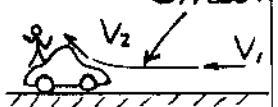
For air $$\rho = \frac{p}{R \cdot T} \qquad \rho = (75 + 101) \times 10^3 \cdot \frac{N}{m^2} \times \frac{kg \cdot K}{286.9 \cdot N \cdot m} \times \frac{1}{(4 + 273) \cdot K} \qquad \rho = 2.21 \frac{kg}{m^3}$$

Then $$V = \sqrt{\left(60 \cdot \frac{m}{s}\right)^2 + 2 \times \frac{m^3}{2.21 \cdot kg} \times (75 - 3) \times 10^3 \cdot \frac{N}{m^2} \times \frac{kg \cdot m}{N \cdot s^2}} \qquad V = 262 \frac{m}{s}$$

NOTE: At this speed, significant density changes will occur, so this result is not very realistic

Problem 6.56

Given: Mercury barometer carried in car on windless day.

Outside: $T = 20°C$, $h_{bar} = 761$ mm Hg (corrected)

Inside: $V = 105$ km/hr, window open, $h_{bar} = 756$ mm Hg

Find: (a) Explain what is happening.
(b) Local speed of air flow past window, relative to car.

Solution: (a) Air speed relative to car is higher than in the freestream, thus lowering the pressure at window.

(b) Apply the Bernoulli equation in frame seen by an observer on the car:

Basic equation: $$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

streamline, V_2, V_1

Assumptions: (1) Steady flow (seen by observer on car)
(2) Incompressible flow
(3) Neglect friction
(4) Flow along a streamline
(5) Neglect Δz

Then $$V_2^2 = \left[V_1^2 + 2\left(\frac{p_1 - p_2}{\rho}\right)\right] \quad \text{or} \quad V_2 = \left[V_1^2 + \frac{2(p_1 - p_2)}{\rho}\right]^{1/2} \qquad (1)$$

From fluid statics

$$p_1 - p_2 = \rho g(h_1 - h_2) = SG\,\rho_{H_2O}\, g\,\Delta h$$

$$= 13.6 \times 1000\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 0.005\,m \times \frac{N \cdot s^2}{kg \cdot m}$$

$$p_1 - p_2 = 667\ N/m^2$$

and from ideal gas

$$\rho = \frac{p}{RT} = 13.6 \times 1000\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 0.761\,m \times \frac{kg \cdot K}{287\ N \cdot m} \times \frac{1}{(273+20)\,K} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$\rho = 1.21\ kg/m^3$$

Substituting into Eq. 1

$$V_2 = \left[\left(105\,\frac{km}{hr} \times 1000\,\frac{m}{km} \times \frac{hr}{3600\,s}\right)^2 + 2 \times 667\,\frac{N}{m^2} \times \frac{m^2}{1.21\,kg} \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2}$$

$V_2 = 44.2$ m/s (159 km/hr) relative to car

V_2

Problem 6.57

[2]

6.57 A fire nozzle is coupled to the end of a hose with inside diameter $D = 3$ in. The nozzle is contoured smoothly and has outlet diameter $d = 1$ in. The design inlet pressure for the nozzle is $p_1 = 100$ psi (gage). Evaluate the maximum flow rate the nozzle could deliver.

Given: Flow through fire nozzle

Find: Maximum flow rate

Solution:

Basic equation

$$\frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{const} \qquad Q = V \cdot A$$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Hence, applying Bernoulli between the inlet (1) and exit (2)

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_2}{\rho} + \frac{V_2^2}{2} \qquad \text{where we ignore gravity effects}$$

But we have

$$Q = V_1 \cdot A_1 = V_1 \cdot \frac{\pi \cdot D^2}{4} = V_2 \cdot A_2 = \frac{\pi \cdot d^2}{4} \qquad \text{so} \qquad V_1 = V_2 \cdot \left(\frac{d}{D}\right)^2$$

$$V_2^2 - V_2^2 \cdot \left(\frac{d}{D}\right)^4 = \frac{2 \cdot (p_2 - p_1)}{\rho}$$

Hence

$$V_2 = \sqrt{\frac{2 \cdot (p_1 - p_2)}{\rho \cdot \left[1 - \left(\frac{d}{D}\right)^4\right]}}$$

Then

$$V_2 = \sqrt{2 \times \frac{ft^3}{1.94 \cdot slug} \times (100 - 0) \cdot \frac{lbf}{in^2} \times \left(\frac{12 \cdot in}{1 \cdot ft}\right)^2 \times \frac{1}{1 - \left(\frac{1}{3}\right)^3} \times \frac{slug\,ft}{lbf \cdot s^2}} \qquad V_2 = 124 \cdot \frac{ft}{s}$$

$$Q = V_2 \cdot \frac{\pi \cdot d^2}{4} \qquad Q = \frac{\pi}{4} \times 124 \cdot \frac{ft}{s} \times \left(\frac{1}{12} \cdot ft\right)^2 \qquad Q = 0.676 \cdot \frac{ft^3}{s} \qquad Q = 304 \cdot \frac{gal}{min}$$

Problem 6.58

Given: Indianapolis race car, $V_0 = 98.3$ m/s, on a straightaway.

Air inlet at location where $V = 25.5$ m/s along body surface.

Find: (a) Static pressure at inlet location.
(b) Express pressure rise as a fraction of the dynamic pressure.

Solution: Apply the Bernoulli equation, relative to the auto.

Basic equation: $$\frac{p_\infty}{\rho} + \frac{V_\infty^2}{2} + g z_\infty = \frac{p}{\rho} + \frac{V^2}{2} + g z$$

Assumptions: (1) Steady flow (as seen by observer on auto)
(2) Incompressible flow ($V_\infty < 100$ m/sec)
(3) No friction
(4) Flow along a streamline
(5) Neglect changes in z
(6) Standard air: $\rho = 1.23$ kg/m^3

Then
$$p - p_\infty = \frac{1}{2}\rho V_\infty^2 - \frac{1}{2}\rho V^2 = \frac{1}{2}\rho V_\infty^2\left[1 - \left(\frac{V}{V_\infty}\right)^2\right] = q\left[1 - \left(\frac{V}{V_\infty}\right)^2\right]$$

$$q = \frac{1}{2}\rho V_\infty^2 = \frac{1}{2} \times 1.23\,\frac{\text{kg}}{\text{m}^3} \times (98.3)^2\,\frac{\text{m}^2}{\text{s}^2} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} = 5.94 \text{ kPa}$$

$$\frac{\Delta p}{q} = 1 - \left(\frac{V}{V_\infty}\right)^2 = 1 - \left(\frac{25.5}{98.3}\right)^2 = 0.933$$ ← $\Delta p / q$

and $\Delta p = 0.933\, q = 0.933 \times 5.94 \text{ kPa} = 5.54 \text{ kPa}$ ← $p - p_\infty$

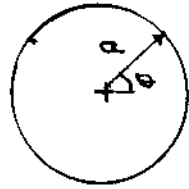
Problem 6.59

Given: Steady, frictionless, incompressible flow over a stationary cylinder of radius, a.

$$\vec{V} = U\left[1-\left(\frac{a}{r}\right)^2\right]\cos\theta\,\hat{e}_r - U\left[1+\left(\frac{a}{r}\right)^2\right]\sin\theta\,\hat{e}_\theta$$

Find: (a) expression for pressure distribution along streamline forming cylinder, $r=a$.
(b) locations on cylinder where $p = p_\infty$.

Solution:

Basic equation: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$

Assumptions: (1) steady flow (given).
(2) incompressible flow (given)
(3) frictionless flow (given)
(4) flow along a streamline.

Along the cylinder surface $r=a$ and $\vec{V} = -2U\sin\theta\,\hat{e}_\theta$

Applying the Bernoulli equation along the streamline $r=a$,

$$\frac{p}{\rho} + \frac{V^2}{2} = \frac{p_\infty}{\rho} + \frac{U^2}{2}$$

$$p = p_\infty + \frac{1}{2}\rho(U^2 - V^2) = p_\infty + \frac{1}{2}\rho(U^2 - 4U^2\sin^2\theta)$$

$$p = p_\infty + \frac{1}{2}\rho U^2(1 - 4\sin^2\theta) \qquad p$$

For $p = p_\infty$, $1 - 4\sin^2\theta = 0$ and $\sin\theta = \pm 0.5$

$$\therefore \theta = 30^\circ, 150^\circ, 210^\circ, 330^\circ \qquad \theta$$

Problem 6.60

6.60 The velocity field for a plane doublet is given in Table 6.2. If $\Lambda = 3\ m^3 \cdot s^{-1}$, the fluid density is $\rho = 1.5\ kg/m^3$, and the pressure at infinity is 100 kPa, plot the pressure along the x axis from $x = -2.0$ m to -0.5 m and $x = 0.5$ m to 2.0 m.

Given: Velocity field for plane doublet

Find: Pressure distribution along x axis; plot distribution

Solution:

The given data is $\Lambda = 3 \cdot \frac{m^3}{s}$ $\qquad \rho = 1000 \cdot \frac{kg}{m^3}$ $\qquad p_0 = 100 \cdot kPa$

From Table 6.1 $\qquad V_r = -\frac{\Lambda}{r^2} \cdot \cos(\theta)$ $\qquad V_\theta = -\frac{\Lambda}{r^2} \cdot \sin(\theta)$

where V_r and V_θ are the velocity components in cylindrical coordinates (r,θ). For points along the x axis, $r = x$, $\theta = 0$, $V_r = u$ and $V_\theta = v = 0$

$$u = -\frac{\Lambda}{x^2} \qquad v = 0$$

The governing equation is the Bernoulli equation

$$\frac{p}{\rho} + \frac{1}{2} \cdot V^2 + g \cdot z = \text{const} \qquad \text{where} \qquad V = \sqrt{u^2 + v^2}$$

so (neglecting gravity) $\qquad \frac{p}{\rho} + \frac{1}{2} \cdot u^2 = \text{const}$

Apply this to point arbitrary point (x,0) on the x axis and at infinity

At $\qquad |x| \rightarrow \qquad u \rightarrow 0 \qquad p \rightarrow p_0$

At point (x,0) $\qquad u = -\frac{\Lambda}{x^2}$

Hence the Bernoulli equation becomes

$$\frac{p_0}{\rho} = \frac{p}{\rho} + \frac{\Lambda^2}{2 \cdot x^4} \qquad \text{or} \qquad p(x) = p_0 - \frac{\rho \cdot \Lambda^2}{2 \cdot x^4}$$

The plot of pressure is shown in the associated *Excel* workbook

Problem 6.60

6.60 The velocity field for a plane doublet is given in Table 6.2. If $\Lambda = 3\ \text{m}^3 \cdot \text{s}^{-1}$, the fluid density is $\rho = 1.5\ \text{kg/m}^3$, and the pressure at infinity is 100 kPa, plot the pressure along the x axis from $x = -2.0$ m to -0.5 m and $x = 0.5$ m to 2.0 m.

Given: Velocity field for plane doublet

Find: Pressure distribution along x axis; plot distribution

Solution:

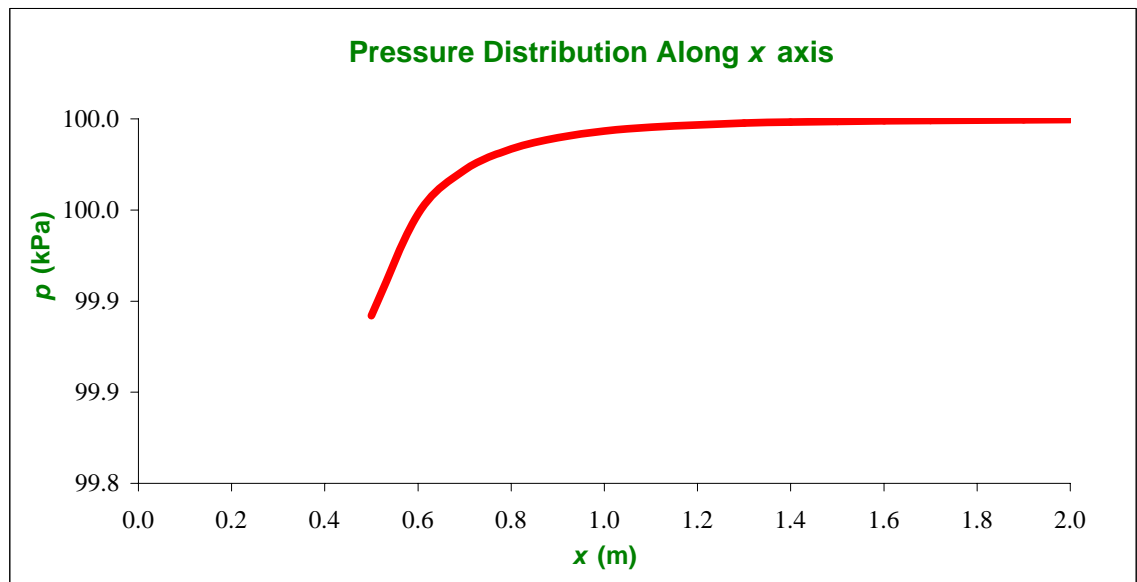
$$p(x) = p_0 - \frac{\rho \cdot \Lambda^2}{2 \cdot x^4}$$

The given data is

$\Lambda =$ 3 m^3/s

$\rho =$ 1.5 kg/m^3

$p_0 =$ 100 kPa

x (m)	p (Pa)
0.5	99.892
0.6	99.948
0.7	99.972
0.8	99.984
0.9	99.990
1.0	99.993
1.1	99.995
1.2	99.997
1.3	99.998
1.4	99.998
1.5	99.999
1.6	99.999
1.7	99.999
1.8	99.999
1.9	99.999
2.0	100.000

Problem 6.61

6.61 The velocity field for a plane source at a distance h above an infinite wall aligned along the x axis was given in Problem 6.8. Using the data from that problem, plot the pressure distribution along the wall from $x = -10h$ to $x = +10h$ (assume the pressure at infinity is atmospheric). Find the net force on the wall if the pressure on the lower surface is atmospheric. Does the force tend to pull the wall towards the source, or push it away?

Given: Velocity field

Find: Pressure distribution along wall; plot distribution; net force on wall

Solution:

The given data is $q = 2\cdot\frac{\frac{m^3}{s}}{m}$ $\quad h = 1\cdot m \quad \rho = 1000\cdot\frac{kg}{m^3}$

$$u = \frac{q\cdot x}{2\cdot\pi\left[x^2+(y-h)^2\right]} + \frac{q\cdot x}{2\cdot\pi\left[x^2+(y+h)^2\right]} \qquad v = \frac{q\cdot(y-h)}{2\cdot\pi\left[x^2+(y-h)^2\right]} + \frac{q\cdot(y+h)}{2\cdot\pi\left[x^2+(y+h)^2\right]}$$

The governing equation is the Bernoulli equation

$$\frac{p}{\rho} + \frac{1}{2}\cdot V^2 + g\cdot z = \text{const} \qquad \text{where} \qquad V = \sqrt{u^2+v^2}$$

Apply this to point arbitrary point (x,0) on the wall and at infinity (neglecting gravity)

At $\quad |x| \to 0 \quad u \to 0 \quad v \to 0 \quad V \to 0$

At point (x,0) $\quad u = \frac{q\cdot x}{\pi\cdot\left(x^2+h^2\right)} \quad v = 0 \quad V = \frac{q\cdot x}{\pi\cdot\left(x^2+h^2\right)}$

Hence the Bernoulli equation becomes $\quad \frac{p_{atm}}{\rho} = \frac{p}{\rho} + \frac{1}{2}\cdot\left[\frac{q\cdot x}{\pi\cdot\left(x^2+h^2\right)}\right]^2$

or (with pressure expressed as gage pressure) $\quad p(x) = -\frac{\rho}{2}\cdot\left[\frac{q\cdot x}{\pi\cdot\left(x^2+h^2\right)}\right]^2$

(Alternatively, the pressure distribution could have been obtained from Problem 6.8, where the momentum equation was used to find the pressure gradient $\frac{\partial}{\partial x}p = \frac{\rho\cdot q^2\cdot x\cdot\left(x^2-h^2\right)}{\pi^2\cdot\left(x^2+h^2\right)^3}$ along the wall. Integration of this with respect to x leads to the same result for $p(x)$)

The plot of pressure is shown in the associated *Excel* workbook. From the plot it is clear that the wall experiences a negative gage pressure on the upper surface (and zero gage pressure on the lower), so the net force on the wall is upwards, towards the source

The force per width on the wall is given by $\quad F = \int_{-10\cdot h}^{10\cdot h}\left(p_{upper}-p_{lower}\right)dx \qquad F = -\frac{\rho\cdot q^2}{2\cdot\pi^2}\cdot\int_{-10\cdot h}^{10\cdot h}\frac{x^2}{\left(x^2+h^2\right)^2}dx$

The integral is

$$\int \frac{x^2}{\left(x^2+h^2\right)^2}\,dx \rightarrow \frac{\operatorname{atan}\left(\frac{x}{h}\right)}{2\cdot h} - \frac{x}{2\cdot h^2 + 2\cdot x^2}$$

so

$$F = -\frac{\rho\cdot q^2}{2\cdot\pi^2\cdot h}\cdot\left(-\frac{10}{101} + \operatorname{atan}(10)\right)$$

$$F = -\frac{1}{2\cdot\pi^2} \times 1000\cdot\frac{kg}{m^3} \times \left(2\cdot\frac{m^2}{s}\right)^2 \times \frac{1}{1\cdot m} \times \left(-\frac{10}{101} + \operatorname{atan}(10)\right) \times \frac{N\cdot s^2}{kg\cdot m} \qquad F = -278\,\frac{N}{m}$$

Problem 6.61

6.61 The velocity field for a plane source at a distance h above an infinite wall aligned along the x axis was given in Problem 6.8. Using the data from that problem, plot the pressure distribution along the wall from $x = -10h$ to $x = +10h$ (assume the pressure at infinity is atmospheric). Find the net force on the wall if the pressure on the lower surface is atmospheric. Does the force tend to pull the wall towards the source, or push it away?

Given: Velocity field

Find: Pressure distribution along wall; plot distribution; net force on wall

Solution:

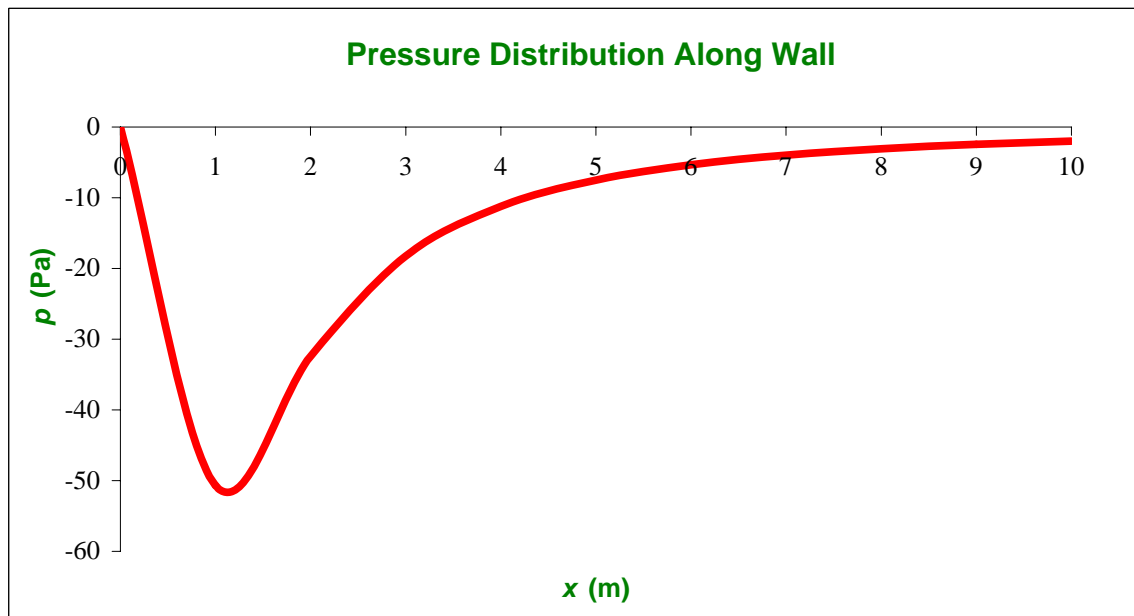
$$p(x) = -\frac{\rho}{2}\cdot\left[\frac{q\cdot x}{\pi\cdot\left(x^2 + h^2\right)}\right]^2$$

The given data is

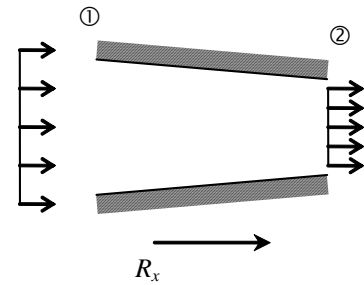
q = 2 m^3/s/m

h = 1 m

ρ = 1000 kg/m^3

x (m)	p (Pa)
0.0	0.00
1.0	-50.66
2.0	-32.42
3.0	-18.24
4.0	-11.22
5.0	-7.49
6.0	-5.33
7.0	-3.97
8.0	-3.07
9.0	-2.44
10.0	-1.99

Problem 6.62

[3]

6.62 A fire nozzle is coupled to the end of a hose with inside diameter $D = 75$ mm. The nozzle is smoothly contoured and its outlet diameter is $d = 25$ mm. The nozzle is designed to operate at an inlet water pressure of 700 kPa (gage). Determine the design flow rate of the nozzle. (Express your answer in L/s.) Evaluate the axial force required to hold the nozzle in place. Indicate whether the hose coupling is in tension or compression.

Given:
Flow through fire nozzle

Find:
Maximum flow rate

Solution:

Basic equation $\quad \frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{const} \qquad Q = V \cdot A \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,d\forall + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Hence, applying Bernoulli between the inlet (1) and exit (2)

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_2}{\rho} + \frac{V_2^2}{2} \qquad \text{where we ignore gravity effects}$$

But we have $\quad Q = V_1 \cdot A_1 = V_1 \cdot \frac{\pi \cdot D^2}{4} = V_2 \cdot \frac{\pi \cdot d^2}{4} \qquad$ so $\qquad V_1 = V_2 \cdot \left(\frac{d}{D}\right)^2$

$$V_2^2 - V_2^2 \cdot \left(\frac{d}{D}\right)^4 = \frac{2 \cdot (p_2 - p_1)}{\rho}$$

Hence $\quad V_2 = \sqrt{\frac{2 \cdot (p_1 - p_2)}{\rho \cdot \left[1 - \left(\frac{d}{D}\right)^4\right]}}$

$$V_2 = \sqrt{2 \times \frac{m^3}{1000 \cdot kg} \times (700 - 0) \times 10^3 \cdot \frac{N}{m^2} \times \frac{1}{1 - \left(\frac{25}{75}\right)^4} \times \frac{kg \cdot m}{N \cdot s^2}} \qquad V_2 = 37.6 \frac{m}{s}$$

Then $\quad Q = V_2 \cdot \frac{\pi \cdot d^2}{4} \qquad Q = \frac{\pi}{4} \times 37.6 \cdot \frac{m}{s} \times (0.025 \cdot m)^2 \qquad Q = 0.0185 \cdot \frac{m^3}{s} \qquad Q = 18.5 \cdot \frac{L}{s}$

From x momentum $\quad R_x + p_1 \cdot A_1 = u_1 \cdot (-\rho \cdot V_1 \cdot A_1) + u_2 \cdot (\rho \cdot V_2 \cdot A_2) \qquad$ using gage pressures

Hence $\quad R_x = -p_1 \cdot \frac{\pi \cdot D^2}{4} + \rho \cdot Q \cdot (V_2 - V_1) = -p_1 \cdot \frac{\pi \cdot D^2}{4} + \rho \cdot Q \cdot V_2 \cdot \left[1 - \left(\frac{d}{D}\right)^2\right]$

$$R_x = -700 \times 10^3 \cdot \frac{N}{m^2} \times \frac{\pi}{4} \cdot (0.075 \cdot m)^2 + 1000 \cdot \frac{kg}{m^3} \times 0.0185 \cdot \frac{m^3}{s} \times 37.6 \cdot \frac{m}{s} \times \left[1 - \left(\frac{25}{75}\right)^3\right] \times \frac{N \cdot s^2}{kg \cdot m} \qquad R_x = -2423\,N$$

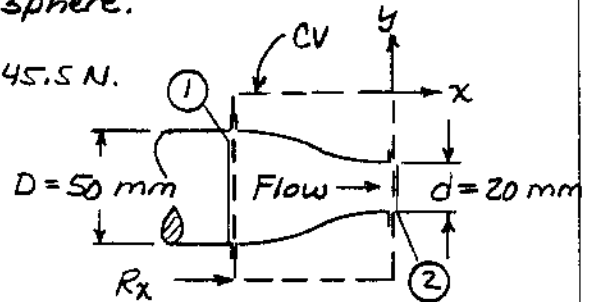
This is the force of the nozzle on the fluid; hence the force of the fluid on the nozzle is 2400 N to the right; the nozzle is in tension

Problem 6.63

Given: Nozzle coupled to straight pipe by flanges, bolts. Water flow discharges to atmosphere.

For steady, inviscid flow, $R_x = -45.5$ N.

Find: Volume flow rate.

Solution: Apply continuity, x momentum, and Bernoulli.

CV, y, x, ① ②, D = 50 mm, Flow →, d = 20 mm, R_x

Basic equation:
$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} \quad (=0(1))$$

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 \quad (p_2 = 0(7))$$

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho\, d\forall + \int_{CS} u \rho \vec{V}\cdot d\vec{A} \quad (F_{B_x} = 0(6),\ =0(1))$$

Assumptions: (1) Steady flow
(2) Uniform flow at each section
(3) Flow along a streamline
(4) Incompressible flow
(5) No friction
(6) Horizontal, $F_{B_x} = 0$, $z_1 = z_2$
(7) Use gage pressures

Then
$$0 = \{-V_1 A_1\} + \{+V_2 A_2\}\ ;\ V_2 = V_1 \frac{A_1}{A_2} = V_1\left(\frac{D}{d}\right)^2\ ;\ Q = V_1 A_1 = V_2 A_2$$

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{V_2^2}{2}\ ;\ p_1 = \rho\left(\frac{V_2^2}{2} - \frac{V_1^2}{2}\right) = \frac{\rho V_1^2}{2}\left[\left(\frac{V_2}{V_1}\right)^2 - 1\right] = \frac{\rho V_1^2}{2}\left[\left(\frac{D}{d}\right)^4 - 1\right]$$

$$R_x + p_1 A_1 - p_2 A_2 = u_1\{-|\rho V_1 A_1|\} + u_2\{+|\rho V_2 A_2|\} = \rho V_1 A_1 (V_2 - V_1) \quad (p_2 = 0(7))$$

$$u_1 = V_1 \qquad u_2 = V_2$$

$$R_x + A_1 \frac{\rho V_1^2}{2}\left[\left(\frac{D}{d}\right)^4 - 1\right] = \rho V_1^2 A_1\left(\frac{V_2}{V_1} - 1\right) = \rho V_1^2 A\left[\left(\frac{D}{d}\right)^2 - 1\right]$$

Thus
$$V_1^2 = \frac{-2R_x}{\rho A_1}\,\frac{1}{\left(\frac{D}{d}\right)^4 - 2\left(\frac{D}{d}\right)^2 + 1} \quad \text{so} \quad V_1 = \sqrt{\frac{-2R_x}{\rho A_1}}\,\frac{1}{\left(\frac{D}{d}\right)^2 - 1}$$

$$V_1 = \left[-2 \times -45.5\,\text{N} \times \frac{\text{m}^3}{999\,\text{kg}} \times \frac{4}{\pi (0.050)^2\,\text{m}^2} \times \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right]^{\frac{1}{2}} \frac{1}{\left(\frac{50}{20}\right)^2 - 1} = 1.30\ \text{m/s}$$

Finally,

$$Q = V_1 A_1 = 1.30\,\frac{\text{m}}{\text{s}} \times \frac{\pi}{4}(0.050)^2\,\text{m}^2 = 2.55 \times 10^{-3}\ \text{m}^3/\text{s}$$

Q

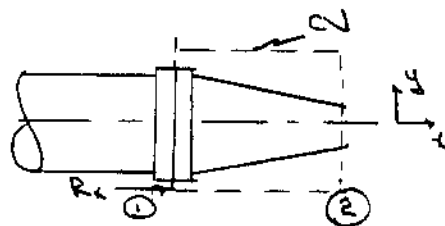
{Note: It is necessary to recognize that $R_x < 0$ for a nozzle, see Example Problem 4.7.}

Problem 6.64

Given: Water flows steadily through a pipe with diameter $D = 3.25$ in. and discharges through a nozzle ($d = 1.25$ in) to atmosphere. The flow rate is $Q = 24.5$ gal/min.

Find: (a) the minimum static pressure required in the pipe to produce this flowrate
(b) the horizontal force of the nozzle assembly on the pipe flange.

Solution:

Apply the Bernoulli equation along the central streamline between sections ① and ②

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Assumptions: (1) steady flow (2) incompressible flow
(3) frictionless flow (4) flow along a streamline.
(5) $\Delta z = 0$ (6) uniform flow at each section

Then $$p_1 = p_2 + \frac{\rho}{2}(V_2^2 - V_1^2) = p_2 + \frac{\rho V_2^2}{2}\left[1 - \left(\frac{V_1}{V_2}\right)^2\right]$$

$p_2 = p_{atm}$ and from continuity, $A_2V_2 = A_1V_1$.

$$\therefore p_{1g} = \frac{\rho}{2}V_2^2\left[1 - \left(\frac{A_2}{A_1}\right)^2\right] = \frac{\rho V_2^2}{2}\left[1 - \left(\frac{D_2}{D_1}\right)^4\right]$$

$$V_2 = \frac{Q}{A} = \frac{4Q}{\pi d^2} = \frac{4}{\pi} \times 24.5\frac{\text{gal}}{\text{min}} \times \frac{\text{ft}^3}{7.48\,\text{gal}} \times \frac{\text{min}}{60\,\text{s}} \times \frac{1}{(1.25)^2\,\text{in}^2} \times \frac{144\,\text{in}^2}{\text{ft}^2}$$

$V_2 = 6.41$ ft/s and

$$p_{1g} = \frac{1}{2} \times 1.94\frac{\text{slug}}{\text{ft}^3} \times (6.41)^2\frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}}\left[1 - \left(\frac{1.25}{3.25}\right)^4\right] = 39.0 \text{ psf gage}$$ p_1

(b) Apply the x momentum equation to the cv

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

$$R_x + p_{1g}A_1 = u_1\{-\dot{m}\} + u_2\{\dot{m}\} = -V_1\dot{m} + V_2\dot{m}$$

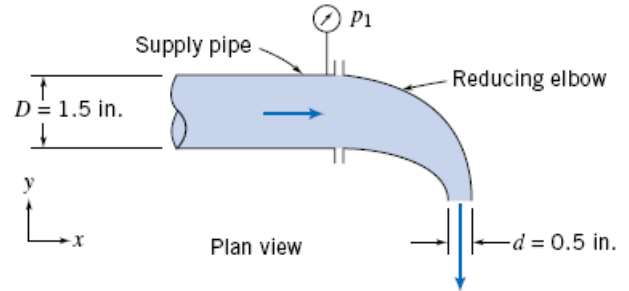
$$R_x = -p_{1g}A_1 + \dot{m}(V_2 - V_1) = -p_{1g}A_1 + \rho Q V_2\left(1 - \frac{V_1}{V_2}\right)$$

$$= -39\frac{\text{lbf}}{\text{ft}^2} \times \frac{\pi}{4}\left(\frac{3.25}{12}\right)^2\text{ft}^2 + 1.94\frac{\text{slug}}{\text{ft}^3} \times 24.5\frac{\text{gal}}{\text{min}} \times \frac{\text{ft}^3}{7.48\,\text{gal}} \times \frac{\text{min}}{60\,\text{s}} \times 6.41\frac{\text{ft}}{\text{s}}\left[1 - \left(\frac{1.25}{3.25}\right)^2\right]\frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}}$$

$R_x = -2.25 + 0.58 = -1.67$ lbf

Force of nozzle on flange $K_x = -R_x = 1.67$ lbf K_x

Problem 6.65

6.65 Water flows steadily through the reducing elbow shown. The elbow is smooth and short, and the flow accelerates, so the effect of friction is small. The volume flow rate is $Q = 20$ gpm. The elbow is in a horizontal plane. Estimate the gage pressure at section (1). Calculate the x component of the force exerted *by* the reducing elbow *on* the supply pipe.

Given:

Flow through reducing elbow

Find:

Mass flow rate in terms of Δp, T_1 and D_1 and D_2

Solution:

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{const}$ $\qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV\!\!\!/ + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$ $\qquad Q = V\cdot A$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline 5) Ignore elevation change 6) $p_2 = p_{atm}$

Available data: $Q = 20\cdot\text{gpm}$ $\quad Q = 0.0446\frac{ft^3}{s}$ $\quad D = 1.5\cdot\text{in}$ $\quad d = 0.5\cdot\text{in}$ $\quad \rho = 1.94\cdot\frac{slug}{ft^3}$

From contnuity $\quad V_1 = \frac{Q}{\left(\frac{\pi\cdot D^2}{4}\right)}$ $\quad V_1 = 3.63\frac{ft}{s}$ $\quad V_2 = \frac{Q}{\left(\frac{\pi\cdot d^2}{4}\right)}$ $\quad V_2 = 32.7\frac{ft}{s}$

Hence, applying Bernoulli between the inlet (1) and exit (2) $\quad \frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_2}{\rho} + \frac{V_2^2}{2}$

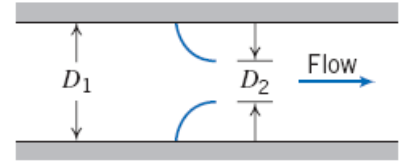
or, in gage pressures $\quad p_{1g} = \frac{\rho}{2}\cdot\left(V_2^2 - V_1^2\right)$ $\quad p_{1g} = 7.11\,\text{psi}$

From x-momentum $\quad R_x + p_{1g}\cdot A_1 = u_1\cdot\left(-m_{rate}\right) + u_2\cdot\left(m_{rate}\right) = -m_{rate}\cdot V_1 = -\rho\cdot Q\cdot V_1$ $\quad$ because $\quad u_1 = V_1 \quad u_2 = 0$

$$R_x = -p_{1g}\cdot\frac{\pi\cdot D^2}{4} - \rho\cdot Q\cdot V_1 \qquad R_x = -12.9\,\text{lbf}$$

The force on the supply pipe is then $\quad K_x = -R_x \quad K_x = 12.9\,\text{lbf}$ $\quad$ on the pipe to the right

Problem 6.66

6.66 A flow nozzle is a device for measuring the flow rate in a pipe. This particular nozzle is to be used to measure low-speed air flow for which compressibility may be neglected. During operation, the pressures p_1 and p_2 are recorded, as well as upstream temperature, T_1. Find the mass flow rate in terms of $\Delta p = p_2 - p_1$ and T_1, the gas constant for air, and device diameters D_1 and D_2. Assume the flow is frictionless. Will the actual flow be more or less than this predicted flow? Why?

Given: Flow nozzle

Find: Mass flow rate in terms of Δp, T_1 and D_1 and D_2

Solution:

Basic equation $\quad \frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{const} \qquad Q = V\cdot A$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Hence, applying Bernoulli between the inlet (1) and exit (2)

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_2}{\rho} + \frac{V_2^2}{2} \qquad \text{where we ignore gravity effects}$$

But we have $\quad Q = V_1\cdot A_1 = V_1\cdot\frac{\pi\cdot D_1^2}{4} = V_2\cdot\frac{\pi\cdot D_2^2}{4} \qquad$ so $\qquad V_1 = V_2\cdot\left(\frac{D_2}{D_1}\right)^2$

Note that we assume the flow at D_2 is at the same pressure as the entire section 2; this will be true if there is turbulent mixing

Hence $\quad V_2^2 - V_2^2\cdot\left(\frac{D_2}{D_1}\right)^4 = \frac{2\cdot(p_2 - p_1)}{\rho}$

$$V_2 = \sqrt{\frac{2\cdot(p_1 - p_2)}{\rho\cdot\left[1 - \left(\frac{D_2}{D_1}\right)^4\right]}}$$

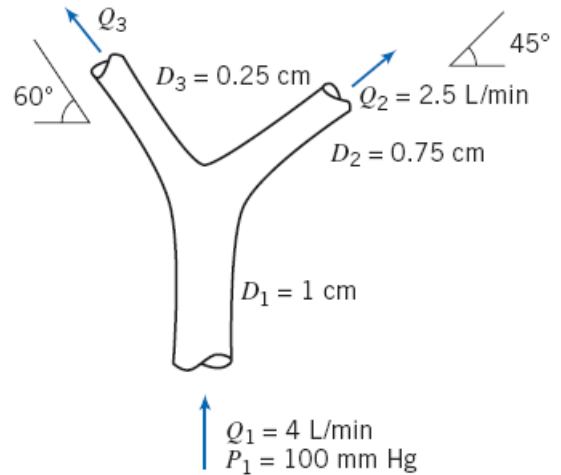
Then the mass flow rate is $m_{flow} = \rho\cdot V_2\cdot A_2 = \rho\cdot\frac{\pi\cdot D_2^2}{4}\cdot\sqrt{\frac{2\cdot(p_1 - p_2)}{\rho\cdot\left[1 - \left(\frac{D_2}{D_1}\right)^4\right]}} = \frac{\pi\cdot D_2^2}{2\cdot\sqrt{2}}\cdot\sqrt{\frac{\Delta p\cdot\rho}{\left[1 - \left(\frac{D_2}{D_1}\right)^4\right]}}$

Using $\quad p = \rho\cdot R\cdot T \qquad m_{flow} = \frac{\pi\cdot D_2^2}{2\cdot\sqrt{2}}\cdot\sqrt{\frac{\Delta p\cdot p_1}{R\cdot T_1\cdot\left[1 - \left(\frac{D_2}{D_1}\right)^4\right]}}$

For a flow nozzle $\quad m_{flow} = k\cdot\sqrt{\Delta p} \qquad$ where $\qquad k = \frac{\pi\cdot D_2^2}{2\cdot\sqrt{2}}\cdot\sqrt{\frac{p_1}{R\cdot T_1\cdot\left[1 - \left(\frac{D_2}{D_1}\right)^4\right]}}$

We can expect the actual flow will be less because there is actually significant loss in the device. Also the flow will experience a vena co that the minimum diameter is actually smaller than D_2. We will discuss this device in Chapter 8.

Problem 6.67

6.67 The branching of a blood vessel is shown. Blood at a pressure of 100 mm Hg flows in the main vessel at 4 L/min. Estimate the blood pressure in each branch, assuming that blood vessels behave as rigid tubes, that we have frictionless flow, and that the vessel lies in the horizontal plane. What is the force generated at the branch by the blood? You may approximate blood to have the same density as water.

Given: Flow through branching blood vessel

Find: Blood pressure in each branch; force at branch

Solution:

Basic equation $\quad \dfrac{p}{\rho} + \dfrac{V^2}{2} + g\cdot z = \text{const} \qquad \sum_{CV} Q = 0 \qquad Q = V\cdot A \qquad \Delta p = \rho\cdot g\cdot \Delta h$

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, d\forall + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

$$F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\,\rho\, d\forall + \int_{CS} v\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

For Q_3 we have $\quad \sum_{CV} Q = -Q_1 + Q_2 + Q_3 = 0 \quad$ so $\quad Q_3 = Q_1 - Q_2 \qquad Q_3 = 1.5\cdot\dfrac{L}{min}$

We will need each velocity

$$V_1 = \frac{Q_1}{A_1} = \frac{4\cdot Q_1}{\pi\cdot D_1^{\,2}} \qquad V_1 = \frac{4}{\pi}\times 4\cdot\frac{L}{min}\times\frac{0.001\cdot m^3}{1\cdot L}\times\frac{1\cdot min}{60\cdot s}\times\left(\frac{1}{0.01\cdot m}\right)^2 \qquad V_1 = 0.849\frac{m}{s}$$

Similarly $\quad V_2 = \dfrac{4\cdot Q_2}{\pi\cdot D_2^{\,2}} \qquad V_2 = 0.943\dfrac{m}{s} \qquad V_3 = \dfrac{4\cdot Q_3}{\pi\cdot D_3^{\,2}} \qquad V_3 = 5.09\dfrac{m}{s}$

Hence, applying Bernoulli between the inlet (1) and exit (2)

$$\frac{p_1}{\rho} + \frac{V_1^{\,2}}{2} = \frac{p_2}{\rho} + \frac{V_2^{\,2}}{2} \qquad \text{where we ignore gravity effects}$$

$$p_2 = p_1 + \frac{\rho}{2}\cdot\left(V_1^{\,2} - V_2^{\,2}\right)$$

$$p_1 = SG_{Hg}\cdot\rho\cdot g\cdot h_1 \qquad \text{where } h_1 = 100 \text{ mm Hg}$$

$$p_1 = 13.6\times 1000\cdot\frac{kg}{m^3}\times 9.81\cdot\frac{m}{s^2}\times 0.1\cdot m\times\frac{N\cdot s^2}{kg\cdot m} \qquad p_1 = 13.3\cdot kPa$$

Hence $$p_2 = 13300\cdot\frac{N}{m^2} + \frac{1}{2}\cdot 1000\cdot\frac{kg}{m^3}\times\left(0.849^2 - 0.943^2\right)\cdot\left(\frac{m}{s}\right)^2\times\frac{N\cdot s^2}{kg\cdot m} \qquad p_2 = 13.2\cdot kPa$$

In mm Hg $$h_2 = \frac{p_2}{SG_{Hg}\cdot\rho\cdot g} \qquad h_2 = \frac{1}{13.6}\times\frac{1}{1000}\cdot\frac{m^3}{kg}\times\frac{s^2}{9.81\cdot m}\times 13200\cdot\frac{N}{m^2}\times\frac{kg\cdot m}{s^2\cdot N} \qquad h_2 = 98.9\cdot mm$$

Similarly for exit (3) $$p_3 = p_1 + \frac{\rho}{2}\cdot\left(V_1^{\ 2} - V_3^{\ 2}\right)$$

$$p_3 = 13300\cdot\frac{N}{m^2} + \frac{1}{2}\cdot 1000\cdot\frac{kg}{m^3}\times\left(0.849^2 - 5.09^2\right)\cdot\left(\frac{m}{s}\right)^2\times\frac{N\cdot s^2}{kg\cdot m} \qquad p_3 = 706\cdot Pa$$

In mm Hg $$h_3 = \frac{p_3}{SG_{Hg}\cdot\rho\cdot g} \qquad h_3 = \frac{1}{13.6}\times\frac{1}{1000}\cdot\frac{m^3}{kg}\times\frac{s^2}{9.81\cdot m}\times 706\cdot\frac{N}{m^2}\times\frac{kg\cdot m}{s^2\cdot N} \qquad h_3 = 5.29\cdot mm$$

Note that all pressures are gage.

For x momentum $$R_x + p_3\cdot A_3\cdot\cos(60\cdot deg) - p_2\cdot A_2\cdot\cos(45\cdot deg) = u_3\cdot\left(\rho\cdot Q_3\right) + u_2\cdot\left(\rho\cdot Q_2\right)$$

$$R_x = p_2\cdot A_2\cdot\cos(45\cdot deg) - p_3\cdot A_3\cdot\cos(60\cdot deg) + \rho\cdot\left(Q_2\cdot V_2\cdot\cos(45\cdot deg) - Q_3\cdot V_3\cdot\cos(60\cdot deg)\right)$$

$$R_x = 13200\cdot\frac{N}{m^2}\times\frac{\pi\cdot(0.0075\cdot m)^2}{4}\times\cos(45\cdot deg) - 706\cdot\frac{N}{m^2}\times\frac{\pi\cdot(0.0025\cdot m)^2}{4}\times\cos(60\cdot deg)\ \ldots$$

$$+ 1000\cdot\frac{kg}{m^3}\cdot\left(2.5\cdot\frac{L}{min}\cdot 0.943\cdot\frac{m}{s}\cdot\cos(45\cdot deg) - 1.5\cdot\frac{L}{min}\cdot 5.09\cdot\frac{m}{s}\cdot\cos(60\cdot deg)\right)\times\frac{10^{-3}\cdot m^3}{1\cdot L}\times\frac{1\cdot min}{60\cdot s}\times\frac{N\cdot s^2}{kg\times m} \qquad R_x = 0.375N$$

For y momentum $$R_y - p_3\cdot A_3\cdot\sin(60\cdot deg) - p_2\cdot A_2\cdot\sin(45\cdot deg) = v_3\cdot\left(\rho\cdot Q_3\right) + v_2\cdot\left(\rho\cdot Q_2\right)$$

$$R_y = p_2\cdot A_2\cdot\sin(45\cdot deg) + p_3\cdot A_3\cdot\sin(60\cdot deg) + \rho\cdot\left(Q_2\cdot V_2\cdot\sin(45\cdot deg) + Q_3\cdot V_3\cdot\sin(60\cdot deg)\right)$$

$$R_y = 13200\cdot\frac{N}{m^2}\times\frac{\pi\cdot(0.0075\cdot m)^2}{4}\times\sin(45\cdot deg) + 706\cdot\frac{N}{m^2}\times\frac{\pi\cdot(0.0025\cdot m)^2}{4}\cdot\sin(60\cdot deg)\ \ldots$$

$$+ 1000\cdot\frac{kg}{m^3}\cdot\left(2.5\cdot\frac{L}{min}\cdot 0.943\cdot\frac{m}{s}\cdot\sin(45\cdot deg) + 1.5\cdot\frac{L}{min}\cdot 5.09\cdot\frac{m}{s}\cdot\sin(60\cdot deg)\right)\times\frac{10^{-3}\cdot m^3}{1\cdot L}\times\frac{1\cdot min}{60\cdot s}\times\frac{N\cdot s^2}{kg\times m} \qquad R_y = 0.553N$$

Problem 6.68

Given: A water jet is directed upward from a well-designed nozzle of area $A_1 = 600\ mm^2$; $V_1 = 6.3\ m/s$. The flow is steady and liquid stream does not break up. Point ② is $H = 1.55\ m$ above nozzle exit.

Find: (a) V_2 (b) p_{0_2}
(c) force on flat plate placed normal to the flow at ②
(d) Sketch pressure distribution on the plate

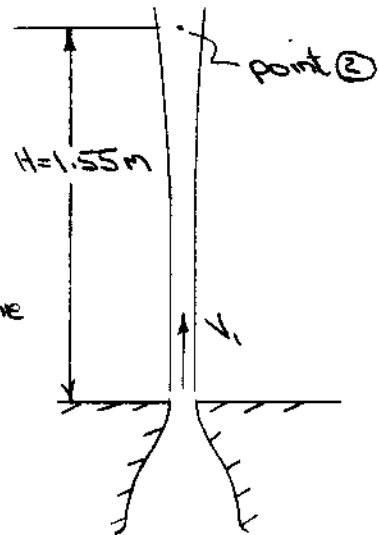
Solution: Apply Bernoulli and then y-momentum equation

Basic eq.: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) $p_1 = p_2 = p_{atm}$

point ②

$H = 1.55\ m$

V_1

Then

$$V_2 = \left[V_1^2 + 2g(z_1 - z_2)\right]^{1/2}$$

$$V_2 = \left[(6.3)^2 \frac{m^2}{s^2} + 2 \times 9.81 \frac{m}{s^2}(-1.55\ m)\right]^{1/2}$$

$V_2 = 3.05\ m/s$ ← V_2

By definition, $p_{0_2} = p_2 + \frac{1}{2}\rho V_2^2 = p_{atm} + \frac{1}{2}\rho V_2^2$, so

$$p_{0_2\,gage} = \frac{1}{2} \times 999 \frac{kg}{m^3} \times (3.05)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 4.65\ kPa\,(g)$$ ← p_{0_2}

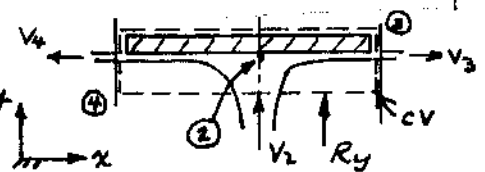
Apply y-momentum equation to CV surrounding plate

Basic eq.: $F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\rho\, dV + \int_{CS} v\rho \vec{V} \cdot d\vec{A}$ ($F_{B_y} \approx 0$ (6); $\frac{\partial}{\partial t} = 0$ (1))

Assumptions: (6) neglect mass in CV
(7) V_2 enters CV uniformly
(8) $v_3 = v_4 = 0$

V_4, V_3, ③, ④, ②, CV, V_2, R_y, y, x

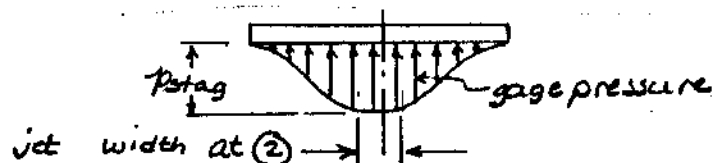
Then

$$R_y = v_2\{-\rho V_1 A_1\} + v_3\{\dot{m}_3\} + v_4\{\dot{m}_4\} = -\rho V_1 A_1 V_2$$ and ($v_3 = 0$ (8), $v_4 = 0$ (8))

$$K_y = -R_y = \rho V_1 A_1 V_2 = 999 \frac{kg}{m^3} \times 6.3 \frac{m}{s} \times 600\ mm^2 \times 3.05 \frac{m}{s} \times \frac{m^2}{10^6\ mm^2} \times \frac{N \cdot s^2}{kg \cdot m}$$

$K_y = 11.5\ N$ (force up) ← K_y

The pressure distribution on the plate is as shown.

p_{stag} — gage pressure

jet width at ②

Given: A flat object moves downward, at speed $U = 5$ ft/sec, into the water jet of the spray system shown. The spray system, of mass $M = 0.200$ lbm and internal volume $\forall = 12$ in^3, operates under steady conditions

Find: (a) the minimum supply pressure required to produce the jet of the spray system.
(b) the maximum pressure exerted by the jet on the object when the object is at $z = 1.5$ ft.

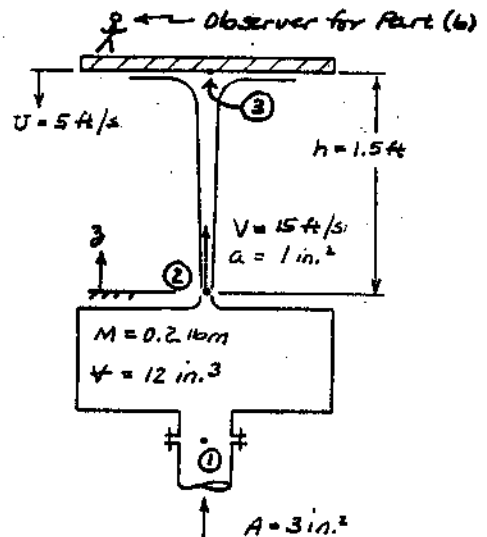
Solution:

Observer for Part (b)

$U = 5$ ft/s; $h = 1.5$ ft; $V = 15$ ft/s; $a = 1$ in.^{2}; $M = 0.2$ lbm; $\forall = 12$ in.^{3}; $A = 3$ in.^{2}

(a)
The minimum pressure occurs when friction is neglected, and so we apply the Bernoulli equation

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1^{\,(5)} = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2^{\,(5)}$$

Assume: (1) steady flow
(2) incompressible flow
(3) no friction
(4) flow along a streamline
(5) neglect $z_2 - z_1$
(6) $p_2 = p_{atm}$
(7) uniform flow at ① & ②

Then

$$p_{1g} = p_1 - p_{atm} = \frac{\rho}{2}\left(V_2^2 - V_1^2\right) = \frac{\rho V_2^2}{2}\left[1 - \left(\frac{V_1}{V_2}\right)^2\right]$$

From continuity, $A_1V_1 = A_2V_2$, and $\frac{V_1}{V_2} = \frac{A_2}{A_1} = \frac{a}{A}$. Then,

$$p_{1g} = \frac{\rho V_2^2}{2}\left[1 - \left(\frac{a}{A}\right)^2\right] = \frac{1}{2} \times 1.94\,\frac{\text{slug}}{\text{ft}^3} \times (15)^2\,\frac{\text{ft}^2}{\text{s}^2}\left[1 - \left(\frac{1}{3}\right)^2\right] \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}} \times \frac{\text{ft}^2}{144\,\text{in}^2} = 1.35\ \text{psig} \qquad p_{1g}$$

Frictional effects would cause this value to be higher.

(b) The maximum pressure of the jet on the object is the stagnation pressure

$$p_0 = p + \frac{1}{2}\rho V_r^2$$

where V is the velocity of the impinging jet relative to the object

At $z = 1.5$ ft, the jet velocity, V_4, in the absence of the object can be calculated from

$$\frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 = \frac{p_4}{\rho} + \frac{V_4^2}{2} + g z_4$$

$$V_4 = \left[V_2^2 - 2g(z_4 - z_2)\right]^{1/2} = \left[(15)^2\,\frac{\text{ft}^2}{\text{s}^2} - 2 \times 32.2\,\frac{\text{ft}}{\text{s}^2}\,(1.5)\,\text{ft}\right]^{1/2} = 11.3\ \text{ft/s}$$

Then

$$V_{rel} = V_4 - (-U) = (11.3 + 5)\ \text{ft/s} = 16.3\ \text{ft/s}$$

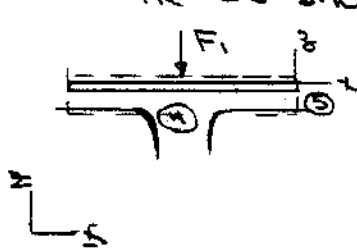
and

$$p_0 - p_{atm} = p_{0g} = \frac{1}{2}\rho V_r^2 = \frac{1}{2} \times 1.94\,\frac{\text{slug}}{\text{ft}^3} \times (16.3)^2\,\frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}} \times \frac{\text{ft}^2}{144\,\text{in}^2} = 1.79\ \text{psig} \qquad p_{0g}$$

Problem 6.69

(c) To determine the force of the water on the object we apply the z component of the momentum equation to the CV shown

$$F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{CV} w_{xyz}\,\rho\, d\forall + \int_{CS} w_{xyz}\,(\rho \vec{V}_{xyz}\cdot d\vec{A})$$

(F_{B_z} = 0 (9); $\frac{\partial}{\partial t}\int_{CV}$ term = 0 (8))

Assumptions: (8) neglect $\frac{\partial}{\partial t}\int_{CV}$
(9) neglect body forces
(10) uniform radial flow at ⑤
(11) uniform vertical flow at ④ with $z_4 = 1.5$ ft

Then $-F_1 = -w_{4xyz}\,|\rho V_{4xyz} A_4|$

where F_1 is applied force necessary to maintain motion of plate at constant speed U

$$V_{4xyz} = V_4 - (-U) = V_4 + U$$

$$w_{4xyz} = V_{4xyz} = V_4 + U$$

$$\therefore\ F_1 = \rho (V_4 + U)^2 A_4$$

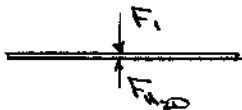
From continuity $A_2 V_2 = A_4 V_4$

and $A_4 = \frac{V_2}{V_4} A_2 = \frac{15}{11.3} \times 1\,in^2 = 1.33\,in^2$

Then

$$F_1 = \rho (V_4 + U)^2 A_4 = 1.94\,\frac{slug}{ft^3}\,(11.3 + 5)^2\,\frac{ft^2}{s^2} \times 1.33\,in^2 \times \frac{ft^2}{144\,in^2} \times \frac{lbf \cdot s^2}{ft \cdot slug}$$

$F_1 = 4.76$ lbf (in the direction shown)

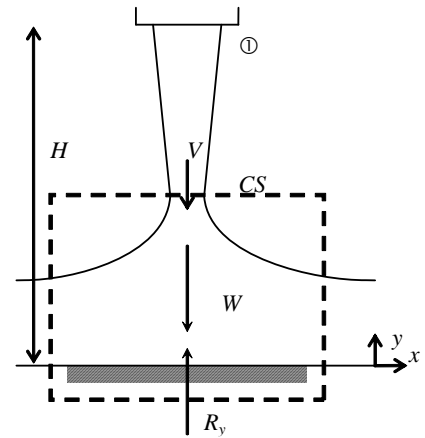
Since the plate is moving at constant speed, then

$$\sum \vec{F}_{plate} = m\vec{a} = 0 \quad \text{and}$$

neglecting the weight of the plate then

$$F_{H_2O} = F_1 = 4.76\ lbf$$

$$\vec{F}_{H_2O} = 4.76\,\hat{k}\ lbf$$

Problem 6.70

6.70 Water flows out of a kitchen faucet of 1.25 cm diameter at the rate of 0.1 L/s. The bottom of the sink is 45 cm below the faucet outlet. Will the cross-sectional area of the fluid stream increase, decrease, or remain constant between the faucet outlet and the bottom of the sink? Explain briefly. Obtain an expression for the stream cross section as a function of distance y above the sink bottom. If a plate is held directly under the faucet, how will the force required to hold the plate in a horizontal position vary with height above the sink? Explain briefly.

Given: Flow through kitchen faucet

Find: Area variation with height; force to hold plate as function of height

Solution:

Basic equation $\quad \frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{const} \qquad Q = V \cdot A \qquad F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\,\rho\, dV + \int_{CS} v\,\rho\,\vec{V}\cdot d\vec{A}$

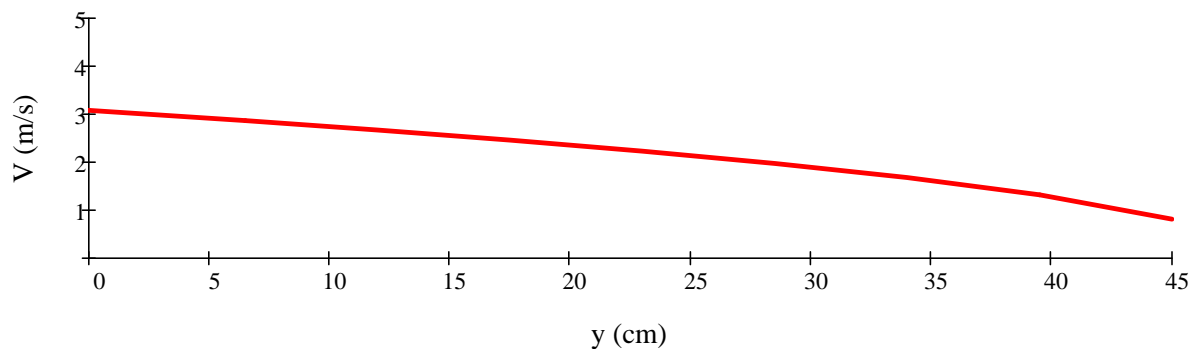
Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Hence, applying Bernoulli between the faucet (1) and any height y

$$\frac{V_1^2}{2} + g \cdot H = \frac{V^2}{2} + g \cdot y$$

where we assume the water is at p_{atm}

Hence $\quad V(y) = \sqrt{V_1^2 + 2 \cdot g \cdot (H - y)}$

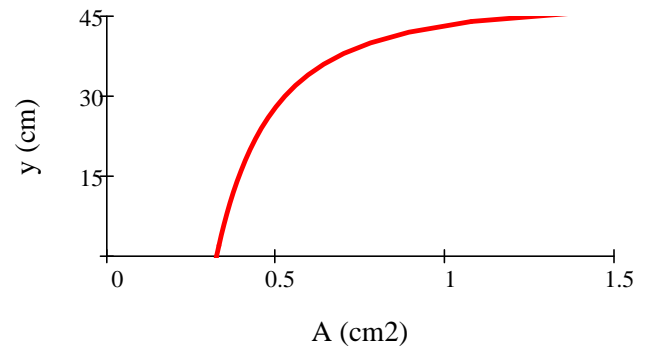
The problem doesn't require a plot, but it looks like $\quad V_1 = 0.815 \frac{m}{s} \qquad V(0 \cdot m) = 3.08 \frac{m}{s}$

The speed increases as y decreases because the fluid particles "trade" potential energy for kinetic, just as a falling solid particle does!

But we have $\quad Q = V_1 \cdot A_1 = V_1 \cdot \frac{\pi \cdot D^2}{4} = V \cdot A$

Hence $\quad A = \frac{V_1 \cdot A_1}{V} \qquad A(y) = \frac{\pi \cdot D_1^2 \cdot V_1}{4 \cdot \sqrt{V_1^2 + 2 \cdot g \cdot (H - y)}}$

The problem doesn't require a plot, but it looks like

$$A(H) = 1.23\,\text{cm}^2$$

$$A(0) = 0.325\,\text{cm}^2$$

The area decreases as the speed increases. If the stream falls far enough the flow will change to turbulent.

For the CV above $\quad R_y - W = u_{in}\cdot\left(-\rho\cdot V_{in}\cdot A_{in}\right) = -V\cdot(-\rho\cdot Q)$

$$R_y = W + \rho\cdot V^2\cdot A = W + \rho\cdot Q\cdot\sqrt{V_1^{\,2} + 2\cdot g\cdot(H - y)}$$

Hence R_y increases in the same way as V as the height y varies; the maximum force is when y = $R_{ymax} = W + \rho\cdot Q\cdot\sqrt{V_1^{\,2} + 2\cdot g\cdot H}$

Problem 6.71

[4]

An old magic trick uses an empty thread spool and a playing card. The playing card is placed against the bottom of the spool. Contrary to intuition, when one blows downward through the central hole in the spool, the card is not blown away. Instead it is ‘‘sucked’’ up against the spool. Explain.

Open-Ended Problem Statement: An old magic trick uses an empty thread spool and a playing card. The playing card is placed against the bottom of the spool. Contrary to intuition, when one blows downward through the central hole in the spool, the card is not blown away. Instead it is ‘‘sucked’’ up against the spool. Explain.

Discussion: The secret to this "parlor trick" lies in the velocity distribution, and hence the pressure distribution, that exists between the spool and the playing cards.

Neglect viscous effects for the purposes of discussion. Consider the space between the end of the spool and the playing card as a pair of parallel disks. Air from the hole in the spool enters the annular space surrounding the hole, and then flows radially outward between the parallel disks. For a given flow rate of air the edge of the hole is the cross-section of minimum flow area and therefore the location of maximum air speed.

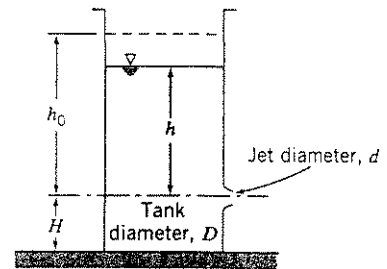
After entering the space between the parallel disks, air flows radially outward. The flow area becomes larger as the radius increases. Thus the air slows and its pressure increases. The largest flow area, slowest air speed, and highest pressure between the disks occur at the outer periphery of the spool where the air is discharged from an annular area.

The air leaving the annular space between the disk and card must be at atmospheric pressure. This is the location of the highest pressure in the space between the parallel disks. Therefore pressure at smaller radii between the disks must be lower, and hence the pressure between the disks is sub-atmospheric. Pressure above the card is less than atmospheric pressure; pressure beneath the card is atmospheric. Each portion of the card experiences a pressure difference acting upward. This causes a net pressure force to act upward on the whole card. The upward pressure force acting on the card tends to keep it from blowing off the spool when air is introduced through the central hole in the spool.

Viscous effects are present in the narrow space between the disk and card. However, they only reduce the pressure rise as the air flows outward, they do not dominate the flow behavior.

Problem 6.72

Given: Tank shown has well-rounded nozzle.
At time $t=0$, water level is h_0

Find: expression for h/h_0 as a function of time.

Plot: (a) h/h_0 vs t for $D/d = 10$, with h_0 as a parameter for $0.1 \le h/h_0 \le 1\,m$

(b) h/h_0 vs t for $h_0 = 1\,m$, with D/d as a parameter for $2 \le D/d \le 10$.

Solution:

Apply the Bernoulli equation along a streamline between the surface and the jet

Basic equation: $$\frac{p_s}{\rho} + \frac{V_t^2}{2} + gz_s = \frac{p_j}{\rho} + \frac{V_j^2}{2} + gz_j$$

Assumptions: (1) quasi-steady flow, i.e. neglect acceleration in tank.
(2) incompressible flow
(3) neglect frictional effects
(4) flow along a streamline
(5) $p_t = p_j = p_{atm}$.

From continuity, $V_t A_t = V_j A_j$ or $V_j = V_t \frac{A_t}{A_j} = V_t \left(\frac{D}{d}\right)^2$

Solving,
$$\frac{V_t^2}{2} - \frac{V_j^2}{2} = \frac{V_t^2}{2}\left[1 - \left(\frac{V_j}{V_t}\right)^2\right] = g(z_j - z_s) = g\left[H - (H+h)\right] = -gh$$

Then
$$V_t = \left[\frac{2gh}{(V_j/V_t)^2 - 1}\right]^{1/2} = \left[\frac{2gh}{(A_t/A_j)^2 - 1}\right]^{1/2} = \left[\frac{2gh}{(D/d)^4 - 1}\right]^{1/2} = -\frac{dh}{dt}$$

Separating variables,
$$\frac{dh}{h^{1/2}} = -\left[\frac{2g}{(D/d)^4 - 1}\right]^{1/2} dt$$

Integrating,
$$2h^{1/2} = -\left[\frac{2g}{(D/d)^4 - 1}\right]^{1/2} t + c$$

At $t=0$, $h = h_0$, so $c = 2h_0^{1/2}$ and
$$h = \left\{ h_0^{1/2} - \frac{1}{2}\left[\frac{2g}{(D/d)^4 - 1}\right]^{1/2} t \right\}^2$$

Nondimensionalize (divide by h_0) to obtain

$$\frac{h}{h_0} = \left\{1 - \sqrt{\frac{g}{2h_0\{(D/d)^4 - 1\}}}\, t\right\}^2$$

h/h_0

Draining of a cylindrical liquid tank:

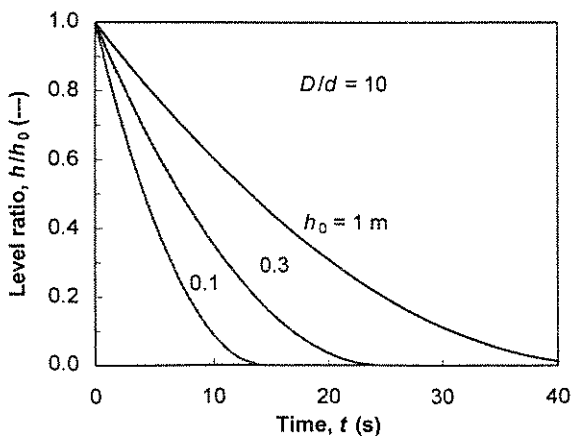
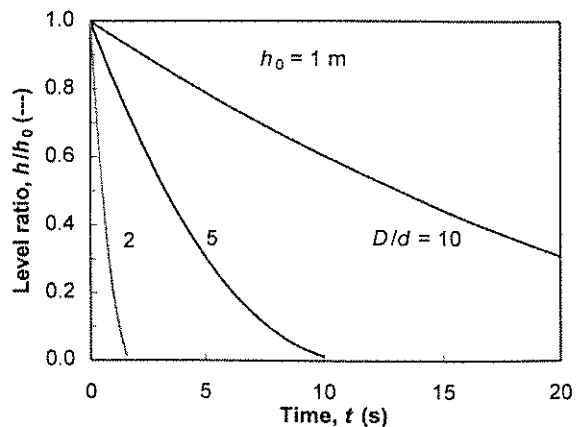
Plot of h/h_0 vs. t for $0.1 < h_0 < 1$ m

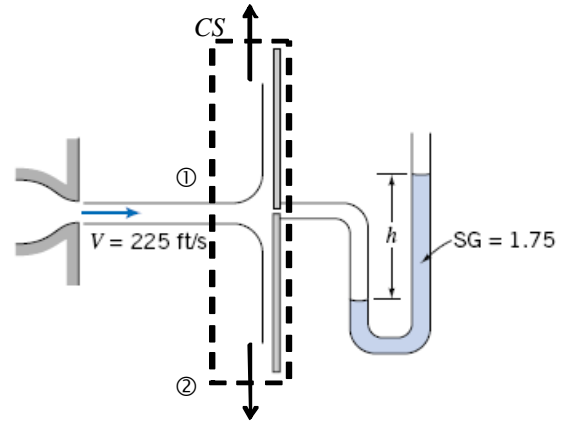
Input Data: D = 50 mm, d = 5 mm

h_0 (m) =	0.1	0.3	1
Time, t (s)	**h/h_0 (---)**	**h/h_0 (---)**	**h/h_0 (---)**
0	1.00	1.00	1.00
2	0.739	0.845	0.913
4	0.518	0.703	0.831
6	0.336	0.574	0.752
8	0.193	0.458	0.677
10	0.090	0.355	0.606
12	0.025	0.265	0.539
14	0.000	0.188	0.476
16		0.125	0.417
18		0.074	0.362
20		0.037	0.310
22		0.012	0.263
24		0.001	0.219
26			0.180
28			0.144
30			0.113
32			0.085
34			0.061
36			0.041
38			0.025
40			0.013
45			0.000

Plot of h/h_0 vs. t for $10 < D/d < 2$

h_0 = 1 m

D/d (---) =	2	5	10
Time, t (s)	**h/h_0 (---)**	**h/h_0 (---)**	**h/h_0 (---)**
0	1.00	1.00	1.00
0.5	0.523	0.913	0.978
1	0.199	0.831	0.956
1.5	0.029	0.752	0.935
1.6	0.013	0.737	0.930
3		0.539	0.872
4		0.417	0.831
5		0.310	0.791
6		0.219	0.752
7		0.144	0.714
8		0.085	0.677
9		0.041	0.641
10		0.013	0.606
12			0.539
14			0.476
16			0.417
18			0.362
20			0.310

Problem 6.73

6.73 A horizontal axisymmetric jet of air with 0.4 in. diameter strikes a stationary vertical disk of 7.5 in. diameter. The jet speed is 225 ft/s at the nozzle exit. A manometer is connected to the center of the disk. Calculate (a) the deflection, if the manometer liquid has SG = 1.75, (b) the force exerted by the jet on the disk, and (c) the force exerted on the disk if it is assumed that the stagnation pressure acts on the entire forward surface of the disk. Sketch the streamline pattern and plot the distribution of pressure on the face of the disk.

Given: Air jet striking disk

Find: Manometer deflection; Force to hold disk; Force assuming p_0 on entire disk; plot pressure distribution

Solution:

Basic equations: Hydrostatic pressure, Bernoulli, and momentum flux in x direction

$$\Delta p = SG\cdot\rho\cdot g\cdot\Delta h \qquad \frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{constant} \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible 3) No friction 4) Flow along streamline 5) Uniform flow 6) Horizontal flow ($g_x = 0$)

Applying Bernoulli between jet exit and stagnation point

$$\frac{p_{atm}}{\rho_{air}} + \frac{V^2}{2} = \frac{p_0}{\rho_{air}} + 0 \qquad p_0 - p_{atm} = \frac{1}{2}\cdot\rho_{air}\cdot V^2$$

But from hydrostatics $\quad p_0 - p_{atm} = SG\cdot\rho\cdot g\cdot\Delta h \quad$ so

$$\Delta h = \frac{\frac{1}{2}\cdot\rho_{air}\cdot V^2}{SG\cdot\rho\cdot g} = \frac{\rho_{air}\cdot V^2}{2\cdot SG\cdot\rho\cdot g}$$

$$\Delta h = 0.002377\cdot\frac{slug}{ft^3}\times\left(225\cdot\frac{ft}{s}\right)^2\times\frac{1}{2\cdot 1.75}\times\frac{ft^3}{1.94\cdot slug}\times\frac{s^2}{32.2\cdot ft} \qquad \Delta h = 0.55\cdot ft \qquad \Delta h = 6.60\cdot in$$

For x momentum

$$R_x = V\cdot\left(-\rho_{air}\cdot A\cdot V\right) = -\rho_{air}\cdot V^2\cdot\frac{\pi\cdot d^2}{4}$$

$$R_x = -0.002377\cdot\frac{slug}{ft^3}\times\left(225\cdot\frac{ft}{s}\right)^2\times\frac{\pi\cdot\left(\frac{0.4}{12}\cdot ft\right)^2}{4}\times\frac{lbf\cdot s^2}{slug\cdot ft} \qquad R_x = -0.105\cdot lbf$$

The force of the jet on the plate is then $\quad F = -R_x \qquad F = 0.105\cdot lbf$

The stagnation pressure is $\quad p_0 = p_{atm} + \frac{1}{2}\cdot\rho_{air}\cdot V^2$

The force on the plate, assuming stagnation pressure on the front face, is

$$F = \left(p_0 - p\right)\cdot A = \frac{1}{2}\cdot\rho_{air}\cdot V^2\cdot\frac{\pi\cdot D^2}{4}$$

$$F = \frac{\pi}{8} \times 0.002377 \cdot \frac{slug}{ft^3} \times \left(225 \cdot \frac{ft}{s}\right)^2 \times \left(\frac{7.5}{12} \cdot ft\right)^2 \times \frac{lbf \cdot s^2}{slug \cdot ft} \qquad F = 18.5\,lbf$$

Obviously this is a huge overestimate!

For the pressure distribution on the disk, we use Bernoulli between the disk outside edge any radius r for radial flow

$$\frac{p_{atm}}{\rho_{air}} + \frac{1}{2} \cdot v_{edge}^2 = \frac{p}{\rho_{air}} + \frac{1}{2} \cdot v^2$$

We need to obtain the speed v as a function of radius. If we assume the flow remains constant thickness h, then

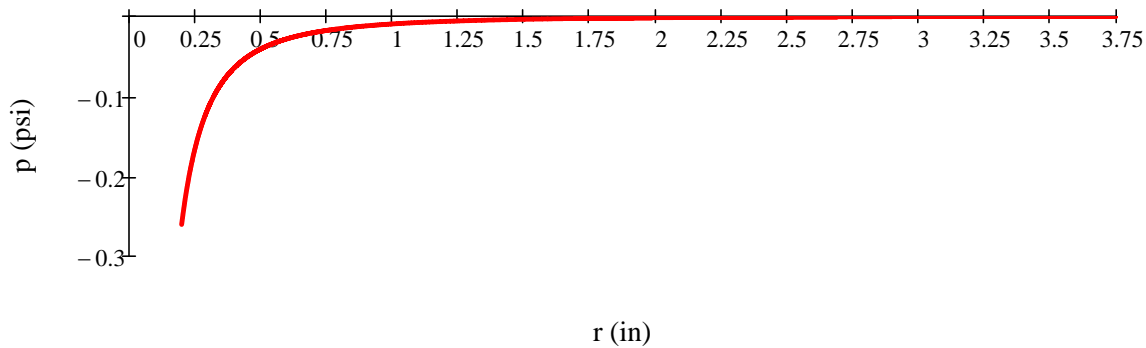
$$Q = v \cdot 2 \cdot \pi \cdot r \cdot h = V \cdot \frac{\pi \cdot d^2}{4} \qquad v(r) = V \cdot \frac{d^2}{8 \cdot h \cdot r}$$

We need an estimate for h. As an approximation, we assume that h = d (this assumption will change the scale of p(r) but not the basic shap

Hence $\qquad v(r) = V \cdot \frac{d}{8 \cdot r}$

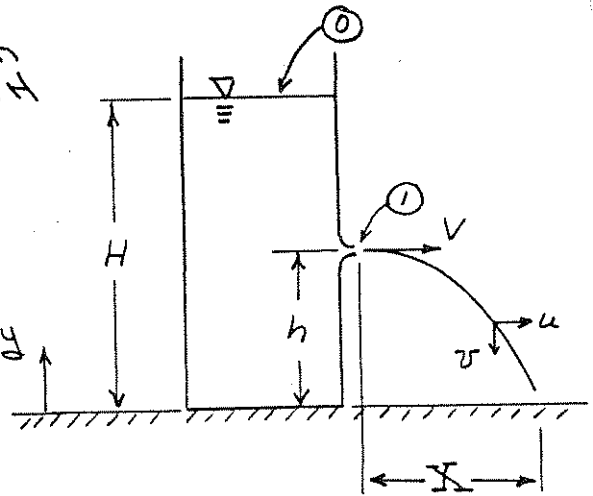
Using this in Bernoulli $\qquad p(r) = p_{atm} + \frac{1}{2} \cdot \rho_{air} \cdot \left(v_{edge}^2 - v(r)^2\right) = p_{atm} + \frac{\rho_{air} \cdot V^2 \cdot d^2}{128} \cdot \left(\frac{4}{D^2} - \frac{1}{r^2}\right)$

Expressed as a gage pressure $\qquad p(r) = \frac{\rho_{air} \cdot V^2 \cdot d^2}{128} \cdot \left(\frac{4}{D^2} - \frac{1}{r^2}\right)$

Problem 6.74

Given: Water level in tank shown is maintained at height H

Find: Elevation h to maximize range, X, of jet.

Plot: Jet speed, V, & distance, X as function of h, for $0 < h < H$.

Solution:

Apply Bernoulli equation between tank surface and jet.

Basic equation: $\frac{p_0}{\rho} + \frac{V_0^2}{2} + gy_0 = \frac{p_1}{\rho} + \frac{V_1^2}{2} + gy_1$

Assumptions: (1) steady flow (2) incompressible flow (3) flow along streamline (4) no friction

Then

$$gH = \frac{V^2}{2} + gh \quad \text{or} \quad V = \sqrt{2g(H-h)} \qquad (1)$$

Assume no air resistance in the stream. Then u = constant, and $X = ut = \sqrt{2g(H-h)}\, t$ — (2)

The only force acting on the stream is gravity

$$\Sigma F_y = -mg = ma_y = m\frac{dv}{dt}\ ; \text{ thus } \frac{dv}{dt} = -g$$

Integrating we obtain $v = v_0 - gt$ and

$$y = y_0 + v_0 t - \frac{1}{2}gt^2$$

Solving for t, $t = \left[\frac{2(y_0 - y)}{g}\right]^{1/2}$

The time of flight is then $t = \sqrt{\frac{2y_0}{g}} = \sqrt{\frac{2h}{g}}$

Substituting into Eq. 2

$$X = \sqrt{2g(H-h)}\sqrt{\frac{2h}{g}} = 2\sqrt{h(H-h)} \qquad (3)$$

X will be maximized when $h(H-h)$ is maximized, or when

$$\frac{d}{dh}\left[h(H-h)\right] = 0 = (H-h) + h(-1) = H - 2h \quad \text{or} \quad h = H/2 \qquad h$$

The corresponding range is

$$X = 2\sqrt{\frac{H}{2}\cdot\frac{H}{2}} = H$$

See the next page for plots

Problem 6.74

From Eq.1, $\frac{V}{\sqrt{2gH}} = \sqrt{1 - \frac{h}{H}}$

From Eq.2, $\frac{X}{H} = 2\sqrt{\frac{h}{H}\left(1 - \frac{h}{H}\right)}$

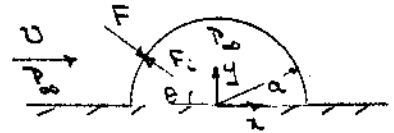
Exit velocity and throw distance from orifice in side of tank, versus height *h/H*

h/H	$V/(2gH)^{1/2}$	*X/H*
0.00	1.00	0.000
0.01	0.995	0.199
0.02	0.990	0.280
0.03	0.985	0.341
0.04	0.980	0.392
0.05	0.975	0.436
0.10	0.949	0.600
0.15	0.922	0.714
0.20	0.894	0.800
0.25	0.866	0.866
0.30	0.837	0.917
0.35	0.806	0.954
0.40	0.775	0.980
0.45	0.742	0.995
0.50	0.707	1.000
0.55	0.671	0.995
0.60	0.632	0.980
0.65	0.592	0.954
0.70	0.548	0.917
0.75	0.500	0.866
0.80	0.447	0.800
0.85	0.387	0.714
0.90	0.316	0.600
0.95	0.224	0.436
0.96	0.200	0.392
0.97	0.173	0.341
0.98	0.141	0.280
0.99	0.100	0.199
1.00	0.00	0.00

Problem 6.75

Given: Flow over a Quonset hut may be approximated by the velocity field

$$\vec{V} = U\left[1-\left(\frac{a}{r}\right)^2\right]\cos\theta\,\hat{e}_r - U\left[1+\left(\frac{a}{r}\right)^2\right]\sin\theta\,\hat{e}_\theta$$

with $0 \le \theta \le 2\pi$

The hut has a diameter, $D = 6\,m$, and a length, $L = 18\,m$

During a storm, $U = 100\,km/hr$, $p_\infty = 720\,mm\,Hg$, $T_\infty = 5C$

Find: The net force tending to lift the hut off its foundation.

Solution:

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{const}$ $\qquad F = \int p\,dA$

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline

Along the top half of the cylinder, $r = a$ and $\vec{V} = -2U\sin\theta\,\hat{e}_\theta$, $0 \le \theta \le \pi$

Applying the Bernoulli equation along the streamline $(r = a)$

$$\frac{p}{\rho} + \frac{V^2}{2} = \frac{p_\infty}{\rho} + \frac{V_\infty^2}{2}$$

$$p - p_\infty = \frac{\rho}{2}(V_\infty^2 - V^2) = \frac{\rho}{2}(U^2 - 4U^2\sin^2\theta) = \rho\frac{U^2}{2}(1 - 4\sin^2\theta)$$

$$F_{R_y} = \int_A (p_\infty - p)\,dA\,\sin\theta = \int_0^\pi (p_\infty - p)\sin\theta\,La\,d\theta$$

$$= \int_0^\pi \rho\frac{U^2}{2}(4\sin^2\theta - 1)\sin\theta\,La\,d\theta = \rho\frac{U^2}{2}aL\left\{4\left[\frac{\cos^3\theta}{3} - \cos\theta\right]_0^\pi + \cos\theta\Big]_0^\pi\right\}$$

$$= \rho\frac{U^2}{2}aL\left\{4\left[\left(-\frac{1}{3}+1\right) - \left(\frac{1}{3}-1\right)\right] + (-1-1)\right\}$$

$$F_{R_y} = \rho\frac{U^2}{2}aL\left(\frac{10}{3}\right) = \frac{5}{3}\rho U^2 aL$$

From the ideal gas equation of state

$$\rho = \frac{p}{RT} = 720\,mm\,Hg \times \frac{atm}{760\,mm} \times 1.01\times10^5\frac{N}{m^2\cdot atm} \times \frac{kg\cdot K}{287\,N\cdot m} \times \frac{1}{278\,K} = 1.20\,kg/m^3$$

$$F_{R_y} = \frac{5}{3}\rho U^2 aL = \frac{5}{3}\times 1.20\frac{kg}{m^3}\times(10^5)^2\frac{m^2}{hr^2}\times\frac{hr^2}{(3600)^2 s^2}\times 3\,m\times 18\,m\times\frac{N\cdot s^2}{kg\cdot m}$$

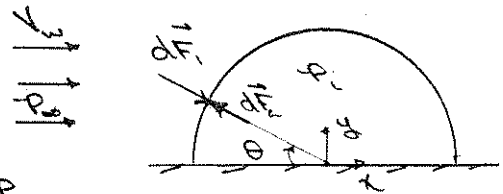
$F_{R_y} = 83.3\,kN$ $\qquad F_{R_y}$

Comment: The actual pressure distribution over the rear portion of the hut is not modelled well by ideal flow. The force calculated here is lower than the actual force.

Problem 6.76

Given: Inflatable "bubble" structure modelled as circular semi-cylinder.
diameter, $D = 30\text{ m}$
length, $L = 70\text{ m}$
Pressure inside is $p_i = p_\infty + \Delta p$
where $\Delta p = \rho_{H_2O} g \Delta h$ and $\Delta h = 10\text{ mm}$.
Pressure distribution over outer surface is given by

$$\frac{p - p_\infty}{\frac{1}{2}\rho V_w^2} = 1 - 4\sin^2\theta \qquad V_w = 60\text{ km/hr}$$

V_w, p_∞, $d\vec{F}_1$, $d\vec{F}_2$, p_i, θ, x, y

Find: net vertical force exerted on the structure.

Solution:

The force due to pressure is $F = \int p\,dA$.

The vertical component of dF_1 is $dF_{1v} = -p\,dA\sin\theta = -p\,RL\,d\theta\sin\theta$

The vertical component of dF_2 is $dF_{2v} = p_i\,dA\sin\theta = p_i RL\,d\theta\sin\theta$

Then, neglecting end effects

$$dF_{v\,net} = (p_i - p)RL\sin\theta\,d\theta = (p_\infty + \Delta p - p)RL\sin\theta\,d\theta$$

$$F_v = \int dF_v = \int_0^\pi \left[\Delta p - (p - p_\infty)\right] RL\sin\theta\,d\theta$$

$$= \int_0^\pi \left[\Delta p - \tfrac{1}{2}\rho V_w^2(1 - 4\sin^2\theta)\right] RL\sin\theta\,d\theta$$

$$= RL\left\{\Delta p\left[-\cos\theta\right]_0^\pi - \tfrac{1}{2}\rho V_w^2\left[-\cos\theta + 4\left(\cos\theta - \frac{\cos^3\theta}{3}\right)\right]_0^\pi\right.$$

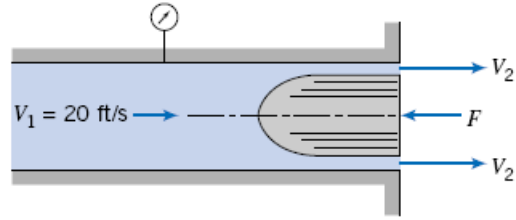
$$= RL\left\{2\Delta p - \tfrac{1}{2}\rho V_w^2\left[2 + 4\left(-2 + \tfrac{2}{3}\right)\right]\right.$$

$$F_v = RL\left\{2\Delta p + \tfrac{5}{3}\rho V_w^2\right\} = RL\left\{2\rho_{H_2O} g\Delta h + \tfrac{5}{3}\rho V_w^2\right\}$$

$$F_v = 15\text{ m}\times 70\text{ m}\left\{2\times 999\frac{\text{kg}}{\text{m}^3}\times 9.81\frac{\text{m}}{\text{s}^2}\times 0.01\text{ m} + \frac{5}{3}\times 1.23\frac{\text{kg}}{\text{m}^3}\times(60)^2\frac{\text{km}^2}{\text{hr}^2}\times 10^6\frac{\text{m}^2}{\text{km}^2}\times\frac{\text{hr}^2}{(3600\text{ s})^2}\right\}\times\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$F_{v\,net} = 804\text{ kN} \qquad F_{v\,net}$$

Problem 6.77

6.77 Water flows at low speed through a circular tube with inside diameter of 2 in. A smoothly contoured body of 1.5 in. diameter is held in the end of the tube where the water discharges to atmosphere. Neglect frictional effects and assume uniform velocity profiles at each section. Determine the pressure measured by the gage and the force required to hold the body.

Given: Water flow out of tube

Find: Pressure indicated by gage; force to hold body in place

Solution:

Basic equations: Bernoulli, and momentum flux in x direction

$$\frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{constant} \qquad Q = V \cdot A \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible 3) No friction 4) Flow along streamline 5) Uniform flow 6) Horizontal flow (g_x = 0)

Applying Bernoulli between jet exit and stagnation point

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_2}{\rho} + \frac{V_2^2}{2} = \frac{V_2^2}{2} \qquad \text{where we work in gage pressure}$$

$$p_1 = \frac{\rho}{2} \cdot \left(V_2^2 - V_1^2\right)$$

But from continuity $\quad Q = V_1 \cdot A_1 = V_2 \cdot A_2 \qquad V_2 = V_1 \cdot \frac{A_1}{A_2} = V_1 \cdot \frac{D^2}{D^2 - d^2}$ $\qquad$ where D = 2 in and d = 1.5 in

$$V_2 = 20 \cdot \frac{ft}{s} \cdot \left(\frac{2^2}{2^2 - 1.5^2}\right) \qquad V_2 = 45.7 \frac{ft}{s}$$

Hence $\quad p_1 = \frac{1}{2} \times 1.94 \cdot \frac{slug}{ft^3} \times \left(45.7^2 - 20^2\right) \cdot \left(\frac{ft}{s}\right)^2 \times \frac{lbf \cdot s^2}{slug\,ft} \qquad p_1 = 1638 \frac{lbf}{ft^2} \qquad p_1 = 11.4\,psi \qquad$ (gage)

The x mometum is $\quad -F + p_1 \cdot A_1 - p_2 \cdot A_2 = u_1 \cdot \left(-\rho \cdot V_1 \cdot A_1\right) + u_2 \cdot \left(\rho \cdot V_2 \cdot A_2\right)$

$$F = p_1 \cdot A_1 + \rho \cdot \left(V_1^2 \cdot A_1 - V_2^2 \cdot A_2\right) \qquad \text{using gage pressures}$$

$$F = 11.4 \cdot \frac{lbf}{in^2} \times \frac{\pi \cdot (2 \cdot in)^2}{4} + 1.94 \cdot \frac{slug}{ft^3} \times \left[\left(20 \cdot \frac{ft}{s}\right)^2 \times \frac{\pi \cdot (2 \cdot in)^2}{4} - \left(45.7 \cdot \frac{ft}{s}\right)^2 \times \frac{\pi \cdot \left[(2 \cdot in)^2 - (1.5 \cdot in)^2\right]}{4}\right] \times \left(\frac{1 \cdot ft}{12 \cdot in}\right)^2 \times \frac{lbf \cdot s^2}{slug\,ft}$$

$$F = 14.1\,lbf \qquad \text{in the direction shown}$$

Given: High-pressure air forces a stream of water from a tiny, rounded orifice, of area A, in a tank. The air expands slowly so the expansion may be considered isothermal.

Find: (a) algebraic expression for $\dot{m}$ leaving the tank
(b) " " " dM/dt in tank.
(c) expression for $M_w(t)$
(d) plot $M_w(t)$ for $0 < t < 40\,min$ if $\forall_0 = 5\,m^3$, $\forall_t = 10\,m^3$, $A = 25\,mm^2$, $p_0 = 1\,MPa$

Solution:

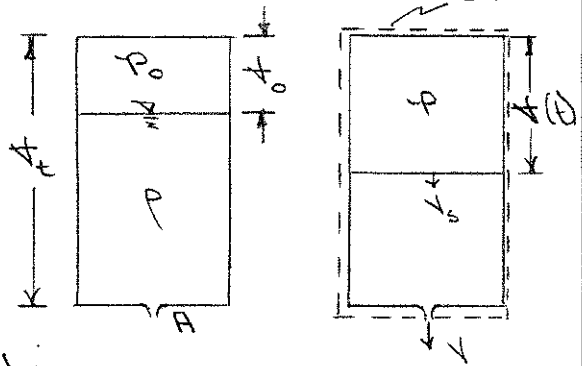
Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = const$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) quasi steady flow $V_s << V$
(2) frictionless
(3) incompressible
(4) flow along a streamline
(5) uniform flow at outlet.
(6) neglect gravity
(7) $p \gg p_{atm}$ $\therefore p_{abs} \approx p_{gage}$

Apply Bernoulli equation between liquid surface and orifice

$$V_j = \left[\frac{2(p - p_{atm})}{\rho}\right]^{1/2} \approx \sqrt{\frac{2p}{\rho}}$$

$$\dot{m} = \rho A V_j = \rho A \sqrt{\frac{2p}{\rho}} = \sqrt{2p\rho}\, A \qquad \leftarrow \dot{m}$$

Rate of change of mass in tank is $\frac{dM}{dt} = \frac{\partial}{\partial t}\int \rho\, d\forall$

$$\frac{dM}{dt} = \rho_w \frac{d\forall_w}{dt} = -\rho_w \frac{d\forall_{air}}{dt} \qquad (\forall_t = \forall_{air} + \forall_w) \qquad \leftarrow \frac{dM}{dt}$$

For isothermal flow, $\frac{p}{\rho} = RT = constant = \frac{p_0}{\rho_0}$

where ρ is the air density and $\rho = M_{air}/\forall_{air}$

thus

$$p\forall = p_0 \forall_0 \quad \text{or} \quad p = p_0 \frac{\forall_0}{\forall}$$

From continuity

$$0 = \rho_w \frac{d\forall_w}{dt} + \dot{m}$$

and

$$0 = -\rho_w \frac{d\forall_{air}}{dt} + \sqrt{2p\rho_w}\, A$$

$$\frac{d\forall}{dt} = \sqrt{\frac{2p}{\rho_w}} = \sqrt{\frac{2p_0 \forall_0}{\rho_w \forall}}$$

Problem 6.78

Separating variables, $\forall^{1/2}\, d\forall = \sqrt{\dfrac{2 p_0 \forall_0}{\rho}}\, A\, dt$

Integrating $\dfrac{2}{3}\forall^{3/2}\Big]_{\forall_0}^{\forall} = \sqrt{\dfrac{2 p_0 \forall_0}{\rho_w}}\, At$

$$\frac{2}{3}\left(\forall^{3/2} - \forall_0^{3/2}\right) = \frac{2\forall_0^{3/2}}{3}\left[\left(\frac{\forall}{\forall_0}\right)^{3/2} - 1\right] = \sqrt{\frac{2 p_0 \forall_0}{\rho_w}}\, At$$

Then

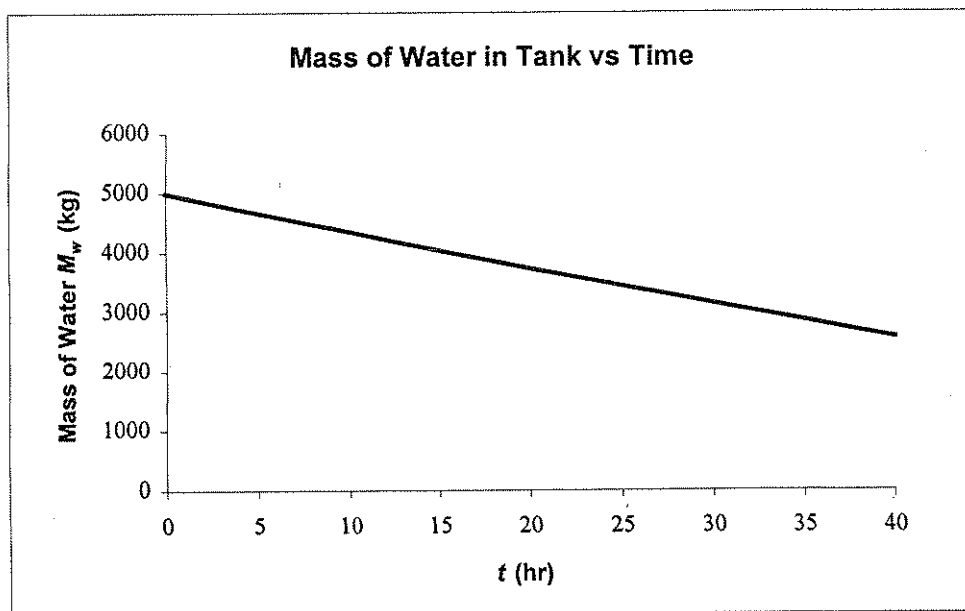
$$\left(\frac{\forall}{\forall_0}\right)^{3/2} = \left[1 + \frac{3}{2\forall_0^{3/2}}\sqrt{\frac{2 p_0 \forall_0}{\rho_w}}\, At\right]$$

$$\frac{\forall}{\forall_0} = \left[1 + 1.5\sqrt{\frac{2 p_0}{\rho_w}}\, \frac{At}{\forall_0}\right]^{2/3}$$

But $M_w = \rho_w(\forall_t - \forall) = \rho\forall_0\left\{\dfrac{\forall_t}{\forall_0} - \dfrac{\forall}{\forall_0}\right\}$

$$\therefore M_w = \rho_w\forall_0\left\{\frac{\forall_t}{\forall_0} - \left[1 + 1.5\sqrt{\frac{2 p_0}{\rho_w}}\,\frac{At}{\forall_0}\right]^{2/3}\right\}$$

M_w

t (s)	M_w (kg)
0	4995
2	4862
4	4730
6	4600
8	4472
10	4345
12	4220
14	4096
16	3973
18	3851
20	3731
22	3612
24	3494
26	3377
28	3260
30	3145
32	3031
34	2918
36	2806
38	2695
40	2584

Given: High-pressure air forces a stream of water from a tiny rounded orifice, of area A, in a tank. The air expands rapidly so the expansion may be treated as adiabatic.

Find: (a) algebraic expression for $\dot{m}$ leaving the tank
(b) " " dM/dt in the tank
(c) expression for $M_w(t)$; plot $M_w(t)$ for $0 < t < 40$ min if $\forall_0 = 5\,m^3$, $\forall_t = 10\,m^3$, $A = 25\,mm^2$, $p_0 = 1$ MPa

Solution:

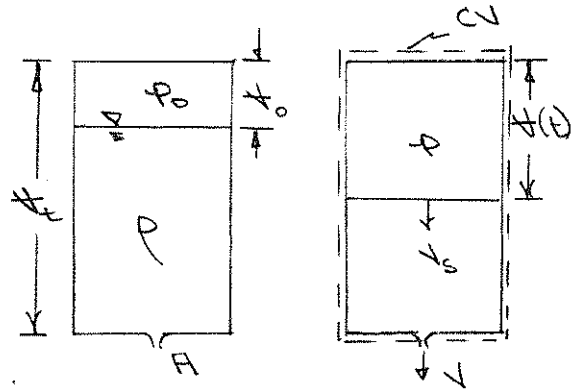
Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{const}$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) quasi steady flow $V_s \ll V$
(2) frictionless
(3) incompressible
(4) flow along a streamline
(5) uniform flow at outlet
(6) neglect gravity
(7) $p \gg p_{atm}$ $\therefore$ $p_{abs} = p_{gage}$

Apply Bernoulli equation between liquid surface and orifice

$$V_j = \left[\frac{2(p - p_{atm})}{\rho}\right]^{1/2} \approx \sqrt{\frac{2p}{\rho}}$$

$$\dot{m} = \rho A V_j = \rho A \sqrt{\frac{2p}{\rho}} = \sqrt{2p\rho}\, A \qquad \dot{m}$$

Rate of change of mass in tank is $\frac{dM}{dt} = \frac{\partial}{\partial t}\int \rho\, d\forall$

$$\frac{dM}{dt} = \rho_w \frac{d\forall_w}{dt} = -\rho_w \frac{d\forall_{air}}{dt} \quad (\forall_t = \forall_{air} + \forall_w) \qquad \frac{dM}{dt}$$

For adiabatic expansion of air $p/\rho^k = \text{constant}$

Since mass of air is constant, $p_0 \forall_0^k = p \forall^k$

From continuity, $-\rho_w \frac{d\forall_{air}}{dt} + \sqrt{2p\rho_w}\, A = 0$

$$\frac{d\forall_{air}}{dt} = \frac{A\sqrt{2}}{\sqrt{\rho_w}}\, p^{1/2} = A\sqrt{\frac{2}{\rho_w}}\left[\frac{p_0 \forall_0^k}{\forall^k}\right]^{1/2} = A\sqrt{\frac{2 p_0 \forall_0^k}{\rho_w}}\, \forall^{-k/2}$$

$$\forall^{k/2}\, d\forall = A\sqrt{\frac{2 p_0 \forall_0^k}{\rho_w}}\, dt = c\, dt \quad \text{where } c = A\sqrt{\frac{2 p_0 \forall_0^k}{\rho_w}}$$

Integrating

$$\frac{2}{(k+2)} \forall^{\frac{k}{2}+1}\Big]_{\forall_0}^{\forall} = ct$$

Problem 6.79

$$V^{\frac{k+2}{2}} - V_0^{\frac{k+2}{2}} = \frac{k+2}{2} c t$$

$$\left(\frac{V}{V_0}\right)^{\frac{k+2}{2}} = 1 + \frac{(k+2)}{2} c t \cdot \frac{1}{V_0^{\frac{k+2}{2}}}$$

$$= 1 + \frac{(k+2)}{2} A \left[\frac{2 p_0 V_0^{k}}{\rho_w}\right]^{1/2} \cdot \left[\frac{1}{V_0^{(k+2)}}\right]^{1/2} t$$

$$\left(\frac{V}{V_0}\right)^{\frac{k+2}{2}} = 1 + \frac{(k+2)}{2} A \left[\frac{2 p_0}{\rho_w V_0^2}\right]^{1/2} t = 1 + \frac{(k+2)}{2} \frac{A}{V_0} \left[\frac{2 p_0}{\rho_w}\right]^{1/2} t$$

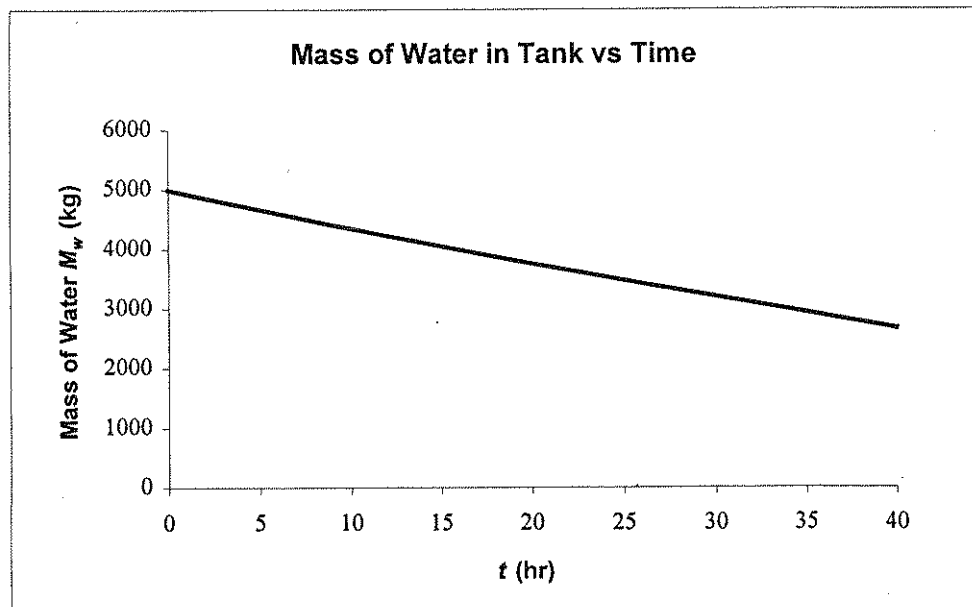
$$\frac{V}{V_0} = \left[1 + \frac{A}{V_0}\sqrt{\frac{2 p_0}{\rho_w}}\left(\frac{k+2}{2}\right) t\right]^{2/k+2}$$

$$M_w = \rho_w (V_t - V) = \rho V_0 \left\{\frac{V_t}{V_0} - \frac{V}{V_0}\right\}$$

$$M_w = \rho_w V_0 \left\{\frac{V_t}{V_0} - \left[1 + \frac{A}{V_0}\sqrt{\frac{2 p_0}{\rho_w}}\left(\frac{k+2}{2}\right) t\right]^{2/k+2}\right\}$$

$$M_w = \rho_w V_0 \left\{\frac{V_t}{V_0} - \left[1 + 1.70\sqrt{\frac{2 p_0}{\rho_w}}\frac{A t}{V_0}\right]^{0.588}\right\}$$

M_w

t (s)	M_w (kg)
0	4995
2	4862
4	4732
6	4603
8	4477
10	4353
12	4231
14	4110
16	3991
18	3874
20	3759
22	3645
24	3532
26	3420
28	3310
30	3202
32	3094
34	2988
36	2882
38	2778
40	2675

Problem 6.80

[5]

Describe the pressure distribution on the exterior of a multistory building in a steady wind. Identify the locations of the maximum and minimum pressures on the outside of the building. Discuss the effect of these pressures on infiltration of outside air into the building.

Open-Ended Problem Statement: Describe the pressure distribution on the exterior of a multistory building in a steady wind. Identify the locations of the maximum and minimum pressures on the outside of the building. Discuss the effect of these pressures on infiltration of outside air into the building.

Discussion: A multi-story building acts as a bluff-body obstruction in a thick atmospheric boundary layer. The boundary-layer velocity profile causes the air speed near the top of the building to be highest and that toward the ground to be lower.

Obstruction of air flow by the building causes regions of stagnation pressure on upwind surfaces. The stagnation pressure is highest where the air speed is highest. Therefore the maximum surface pressure occurs near the roof on the upwind side of the building. Minimum pressure on the upwind surface of the building occurs near the ground where the air speed is lowest.

The minimum pressure on the entire building will likely be in the low-speed, low-pressure wake region on the downwind side of the building.

Static pressure inside the building will tend to be an average of all the surface pressures that act on the outside of the building. It is never possible to seal all openings completely. Therefore air will tend to infiltrate into the building in regions where the outside surface pressure is above the interior pressure, and will tend to pass out of the building in regions where the outside surface pressure is below the interior pressure. Thus generally air will tend to move through the building from the upper floors toward the lower floors, and from the upwind side to the downwind side.

Problem 6.81

[5]

Imagine a garden hose with a stream of water flowing out through a nozzle. Explain why the end of the hose may be unstable when held a half meter or so from the nozzle end.

Open-Ended Problem Statement: Imagine a garden hose with a stream of water flowing out through a nozzle. Explain why the end of the hose may be unstable when held a half meter or so from the nozzle end.

Discussion: Water flowing out of the nozzle tends to exert a thrust force on the end of the hose. The thrust force is aligned with the flow from the nozzle and is directed toward the hose.

Any misalignment of the hose will lead to a tendency for the thrust force to bend the hose further. This will quickly become unstable, with the result that the free end of the hose will "flail" about, spraying water from the nozzle in all directions.

This instability phenomenon can be demonstrated easily in the backyard. However, it will tend to do least damage when the person demonstrating it is wearing a bathing suit!

Problem 6.82

[5]

An aspirator provides suction by using a stream of water flowing through a venturi. Analyze the shape and dimensions of such a device. Comment on any limitations on its use.

Open-Ended Problem Statement: An aspirator provides suction by using a stream of water flowing through a venturi. Analyze the shape and dimensions of such a device. Comment on any limitations on its use.

Discussion: The basic shape of the aspirator channel should be a converging nozzle section to reduce pressure followed by a diverging diffuser section to promote pressure recovery. The basic shape is that of a venturi flow meter.

If the diffuser exhausts to atmosphere, the exit pressure will be atmospheric. The pressure rise in the diffuser will cause the pressure at the diffuser inlet (venturi throat) to be below atmospheric.

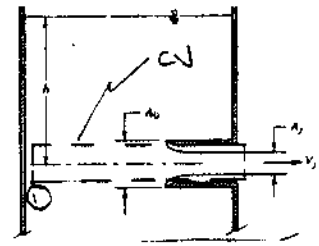
A small tube can be brought in from the side of the throat to aspirate another liquid or gas into the throat as a result of the reduced pressure there.

The following comments can be made about limitations on the aspirator:

1. It is desirable to minimize the area of the aspirator tube compared to the flow area of the venturi throat. This minimizes the disturbance of the main flow through the venturi and promotes the best possible pressure recovery in the diffuser.
2. It is desirable to avoid cavitation in the throat of the venturi. Cavitation alters the effective shape of the flow channel and destroys the pressure recovery in the diffuser. To avoid cavitation, the reduced pressure must always be above the vapor pressure of the driver liquid.
3. It is desirable to limit the flow rate of gas into the venturi throat. A large amount of gas can alter the flow pattern and adversely affect pressure recovery in the diffuser.

The best combination of specific dimensions could be determined experimentally by a systematic study of aspirator performance. A good starting point probably would be to use dimensions similar to those of a commercially available venturi flow meter.

Problem 6.83

Given: Reentrant orifice in the side of a large tank. Pressure along the tank walls is essentially hydrostatic.

Find: the contraction coefficient, $C_c = A_j / A_o$

Solution:

Apply the x-component of the momentum equation to the CV shown

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) steady flow
(2) uniform flow at jet exit.
(3) hydrostatic pressure variation across CS ①, $V_1 \approx 0$
(4) x momentum flux across horizontal portion of CS is negligible.
(5) ρ = constant

Then

$$\int_{A_o} p \, dA_1 = \dot{m} V_j = \rho V_j A_j V_j = \rho A_j V_j^2$$

$$\bar{p}_1 A_o = \rho g h A_o = \rho A_j V_j^2$$

$$\therefore \frac{A_o}{A_j} = \frac{V_j^2}{gh}$$

Apply the Bernoulli equation along the central streamline from ① to the jet exit.

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_j}{\rho} + \frac{V_j^2}{2} + g z_j$$

Assumptions: (6) frictionless flow

$$p_1 = \rho g h = \rho \frac{V_j^2}{2}$$

$$\therefore \frac{V_j^2}{2} = gh$$

and

$$\frac{A_o}{A_j} = \frac{V_j^2}{gh} = 2$$

$$\therefore C_c = \frac{A_j}{A_o} = \frac{1}{2}$$

C_c

Problem 6.84

[2]

Carefully sketch the energy grade lines (EGL) and hydraulic grade lines (HGL) for the system shown in Fig. 6.6 if the pipe is horizontal (i.e., the outlet is at the base of the reservoir), and a water turbine (extracting energy) is located at (a) point ②, or (b) at point ③. In Chapter 8 we will investigate the effects of friction on internal flows. Can you anticipate and sketch the effect of friction on the EGL and HGL for cases (a) and (b)?

(a) Note that the effect of friction would be that the EGL would tend to drop: suddenly at the contraction, gradually in the large pipe, more steeply in the small pipe. The HGL would then "hang" below the HGL in a manner similar to that shown.

(b) Note that the effect of friction would be that the EGL would tend to drop: suddenly at the contraction, gradually in the large pipe, more steeply in the small pipe. The HGL would then "hang" below the HGL in a manner similar to that shown.

Problem 6.85

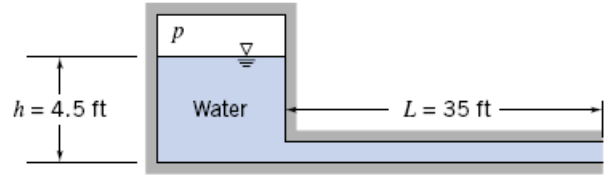
Carefully sketch the energy grade lines (EGL) and hydraulic grade lines (HGL) for the system shown in Fig. 6.6 if a pump (adding energy to the fluid) is located at (a) point ②, or (b) at point ③, such that flow is into the reservoir. In Chapter 8 we will investigate the effects of friction on internal flows. Can you anticipate and sketch the effect of friction on the EGL and HGL for cases (a) and (b)?

(a) Note that the effect of friction would be that the EGL would tend to drop from right to left: steeply in the small pipe, gradually in the large pipe, and suddenly at the expansion. The HGL would then "hang" below the HGL in a manner similar to that shown.

(b) Note that the effect of friction would be that the EGL would tend to drop from right to left: steeply in the small pipe, gradually in the large pipe, and suddenly at the expansion. The HGL would then "hang" below the HGL in a manner similar to that shown.

Problem *6.86

[2]

*6.86 Compressed air is used to accelerate water from a tube. Neglect the velocity in the reservoir and assume the flow in the tube is uniform at any section. At a particular instant, it is known that $V = 6$ ft/s and $dV/dt = 7.5$ ft/s^2. The cross-sectional area of the tube is $A = 32$ in.^{2}. Determine the pressure in the tank at this instant.

Given: Unsteady water flow out of tube

Find: Pressure in the tank

Solution:

Basic equation: Unsteady Bernoulli

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 + \int_1^2 \frac{\partial V}{\partial t} ds$$

Assumptions: 1) Unsteady flow 2) Incompressible 3) No friction 4) Flow along streamline 5) Uniform flow 6) Horizontal flow ($g_x = 0$)

Applying unsteady Bernoulli between reservoir and tube exit

$$\frac{p}{\rho} + g \cdot h = \frac{V^2}{2} + \int_1^2 \frac{\partial}{\partial t} V\, ds = \frac{V^2}{2} + \frac{dV}{dt} \cdot \int_1^2 1\, ds$$

where we work in gage pressure

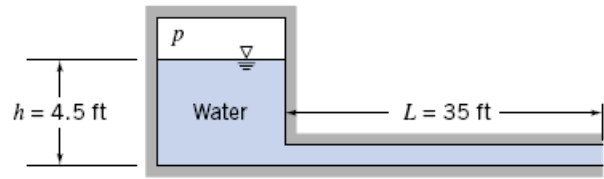
Hence $$p = \rho \cdot \left(\frac{V^2}{2} - g \cdot h + \frac{dV}{dt} \cdot L \right)$$

Hence $$p = 1.94 \cdot \frac{slug}{ft^3} \times \left(\frac{6^2}{2} - 32.2 \times 4.5 + 7.5 \times 35 \right) \cdot \left(\frac{ft}{s} \right)^2 \times \frac{lbf \cdot s^2}{slug\, ft}$$

$$p = 263 \cdot \frac{lbf}{ft^2} \qquad p = 1.83 \cdot psi \qquad \text{(gage)}$$

Problem *6.87

[2]

*6.87 If the water in the pipe in Problem 6.86 is initially at rest and the air pressure is 3 psig, what will be the initial acceleration of the water in the pipe?

Given: Unsteady water flow out of tube

Find: Initial acceleration

Solution:

Basic equation: Unsteady Bernoulli

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 + \int_1^2 \frac{\partial V}{\partial t} ds$$

Assumptions: 1) Unsteady flow 2) Incompressible 3) No friction 4) Flow along streamline 5) Uniform flow 6) Horizontal flow ($g_x = 0$)

Applying unsteady Bernoulli between reservoir and tube exit

$$\frac{p}{\rho} + g \cdot h = \int_1^2 \frac{\partial}{\partial t} V\, ds = \frac{dV}{dt} \cdot \int_1^2 1\, ds = a_x \cdot L$$

where we work in gage pressure

Hence $$a_x = \frac{1}{L} \cdot \left(\frac{p}{\rho} + g \cdot h \right)$$

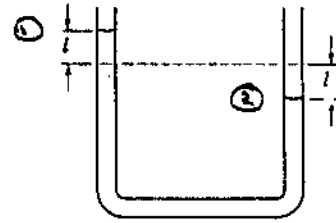
Hence $$a_x = \frac{1}{35 \cdot ft} \times \left[3 \cdot \frac{lbf}{in^2} \times \left(\frac{12 \cdot in}{1 \cdot ft} \right)^2 \times \frac{ft^3}{1.94 \cdot slug} \times \frac{slug\, ft}{s^2 \cdot lbf} + 32.2 \cdot \frac{ft}{s^2} \times 4.5 \cdot ft \right]$$

$$a_x = 10.5 \cdot \frac{ft}{s^2}$$

Note that we obtain the same result if we treat the water in the pipe as a single body at rest with gage pressure p + ρgh at the left end!

Problem *6.88

[5]______

Given: U-tube manometer of constant area as shown
Manometer fluid is initially deflected and then released

Find: a differential equation for l as a function of time

Solution

Basic equation: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \int_1^2 \frac{\partial V_s}{\partial t} ds$

Assumptions: (1) incompressible flow
(2) frictionless flow
(3) flow along a streamline

Since $p_1 = p_2 = p_{atm}$ and $V_1^2 = V_2^2$, then

$$g(z_1 - z_2) = \int_1^2 \frac{\partial V_s}{\partial t} ds$$

Let L = total length of column
l = deflection

Then $ds = dL$
$V_s = V = \frac{dl}{dt}$

$$\therefore \; 2gl = \int_1^2 \frac{\partial V}{\partial t} dL = \frac{\partial V}{\partial t}\int_1^2 dL = L\frac{\partial V}{\partial t}$$

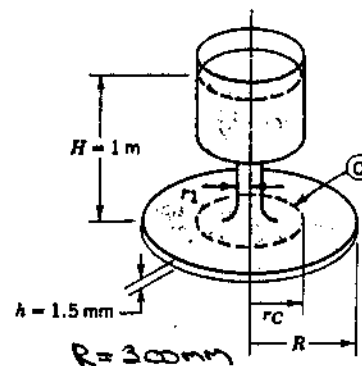
Since $V = -\frac{dl}{dt}$

$$2gl = L\frac{\partial V}{\partial t} = -L\frac{d^2 l}{dt^2}$$

Finally $$\frac{d^2 l}{dt^2} + \frac{2g}{L}l = 0$$

Problem *6.89

Given: Flow between parallel disks shown is started from rest at $t=0$. The reservoir level is maintained constant; $r_1 = 50mm$.

Find: Rate of change of volume flow, dQ/dt, at $t=0$

Solution:

Apply the unsteady Bernoulli equation from the surface to the exit.

$$\frac{p_s}{\rho} + \frac{V_s^2}{2} + gz_s = \frac{p_e}{\rho} + \frac{V_e^2}{2} + gz_e + \int_1^2 \frac{\partial V_s}{\partial t} ds$$

$$gH = \frac{V_e^2}{2} + \int_1^2 \frac{\partial V_s}{\partial t} ds$$

Assumptions: (1) frictionless flow
(2) incompressible flow
(3) flow along a streamline

For uniform flow at any section between the plates, for $r \geq r_1$, the volume flow rate is given by

$$Q = \int \vec{V} \cdot d\vec{A} = V_r 2\pi r h \quad \text{and } V_r = \frac{Q}{2\pi r h}$$

At the exit $V_e = Q/2\pi R h$

Assume that the rate of change of fluid velocity in the reservoir (out to $r = r_1$) is negligible. Then

$$\int_1^2 \frac{\partial V_s}{\partial t} ds = \frac{\partial}{\partial t}\int_1^2 V_r\, dr = \frac{\partial}{\partial t}\int_1^2 \frac{Q}{2\pi h}\frac{dr}{r} = \frac{\ln R/r_1}{2\pi h}\frac{dQ}{dt}$$

Then substituting into the unsteady Bernoulli equation, we obtain

$$gH = \frac{Q^2}{8\pi^2 R^2 h^2} + \frac{\ln R/r_1}{2\pi h}\frac{dQ}{dt}$$

At $t=0$, $Q=0$ and

$$\frac{dQ}{dt} = \frac{2\pi h g H}{\ln R/r_1}$$

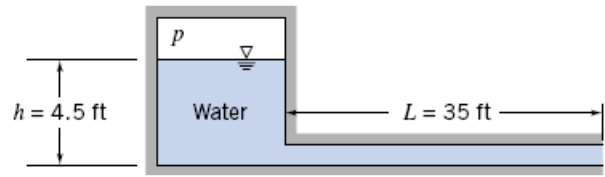
$$= 2\pi \times 0.0015m \times 9.81\frac{m}{s^2} \times 1m \times \frac{1}{\ln\frac{300}{50}}$$

$$\frac{dQ}{dt} = 0.0516\ m^3/s/s$$

$\left.\frac{dQ}{dt}\right)_{t=0}$

Problem *6.90

[4]

*6.90 If the water in the pipe of Problem 6.86 is initially at rest, and the air pressure is maintained at 1.5 psig, derive a differential equation for the velocity V in the pipe as a function of time, integrate, and plot V versus t for $t = 0$ to 5 s.

Given: Unsteady water flow out of tube

Find: Differential equation for velocity; Integrate; Plot v versus time

Solution:

Basic equation: Unsteady Bernoulli $\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 + \int_1^2 \frac{\partial V}{\partial t} ds$

Assumptions: 1) Unsteady flow 2) Incompressible 3) No friction 4) Flow along streamline 5) Uniform flow 6) Horizontal flow ($g_x = 0$)

Applying unsteady Bernoulli between reservoir and tube exit

$$\frac{p}{\rho} + g \cdot h = \frac{V^2}{2} + \int_1^2 \frac{\partial}{\partial t} V\, ds = \frac{V^2}{2} + \frac{dV}{dt} \cdot \int_1^2 1\, ds = \frac{V^2}{2} + \frac{dV}{dt} \cdot L$$ where we work in gage pressure

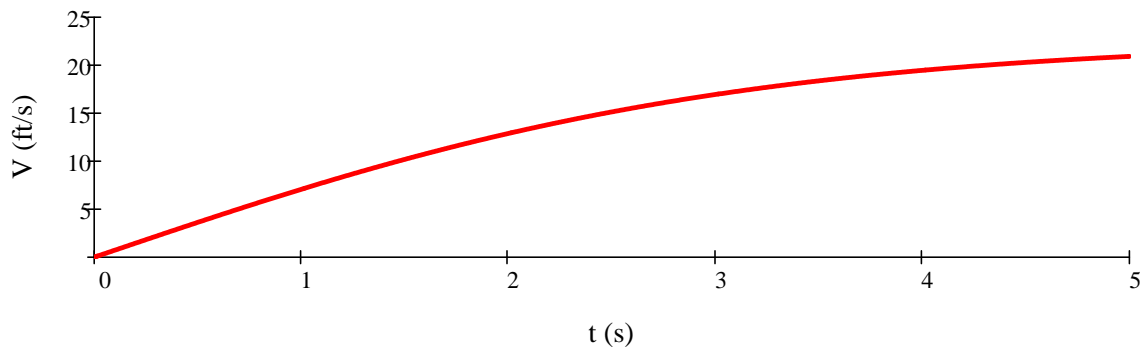
Hence $$\frac{dV}{dt} + \frac{V^2}{2 \cdot L} = \frac{1}{L} \cdot \left(\frac{p}{\rho} + g \cdot h\right)$$ is the differential equation for the flow

Separating variables $$\frac{L \cdot dV}{\frac{p}{\rho} + g \cdot h - \frac{V^2}{2}} = dt$$

Integrating and using limits $V(0) = 0$ and $V(t) = V$

$$V(t) = \sqrt{2 \cdot \left(\frac{p}{\rho} + g \cdot h\right)} \cdot \tanh\left(\sqrt{\frac{\frac{p}{\rho} + g \cdot h}{2 \cdot L^2}} \cdot t\right)$$

V (ft/s)

t (s)

This graph is suitable for plotting in *Excel*

For large times $$V = \sqrt{2 \cdot \left(\frac{p}{\rho} + g \cdot h\right)} \qquad V = 22.6 \frac{ft}{s}$$

Problem *6.91

Given: A cylindrical tank of diameter, $D = 50$ mm, drains through an opening, $d = 5$ mm, in the body of the tank. If the flow is assumed to be quasi-steady, the speed of the liquid leaving the tank may be approximated by $V = \sqrt{2gy}$, where y is the height from the tank bottom to the free surface.

Find: Using the Bernoulli equation for unsteady flow along a streamline, evaluate the minimum diameter ratio, D/d, required to justify the assumption that flow from the tank is quasi-steady.

Solution:

For incompressible, frictionless flow along a streamline, the unsteady Bernoulli equation is

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gy_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gy_2 + \int_1^2 \frac{\partial V_s}{\partial t}\,dy$$

$p_1 = p_2 = p_{atm}$, $y_2 = 0$

From continuity $V_1 A_1 = V_2 A_2 = V_j A_j$

$$\therefore \quad \frac{1}{2}V_j^2\left(\frac{A_j}{A_1}\right) + gy_1 = \frac{1}{2}V_j^2 + \int_1^2 \frac{\partial V_s}{\partial t}\,dy$$

or,

$$gy_1 = \frac{1}{2}V_j^2\left[1 - \left(\frac{A_j}{A_1}\right)^2\right] + \int_1^2 \frac{\partial V_s}{\partial t}\,dy$$

If we assume quasi-steady flow, we say that $\int_1^2 \frac{\partial V_s}{\partial t}\,dy$ is negligible and hence $\frac{2gy}{V_j^2[1-AR^2]} = 1$ where $AR = \frac{A_j}{A_1}$

Now, $\int_1^2 \frac{\partial V_s}{\partial t} \approx y\frac{dV_s}{dt} = y\frac{dV_1}{dt} = y\frac{d}{dt}\left(V_j\frac{A_j}{A_1}\right) = y\frac{A_j}{A_1}\frac{dV_j}{dt}$

Thus for the assumption to be reasonable we must have

$$\left|y\frac{A_j}{A_1}\frac{dV_j}{dt}\right| \ll gy \qquad \text{or} \qquad \left|\frac{A_j}{A_1}\frac{dV_j}{dt}\right| \ll g$$

Under the assumption of quasi-steady flow

$$V_j = \left[2gy\frac{1}{(1-AR^2)}\right]^{1/2} \quad \text{where } AR = A_j/A_1$$

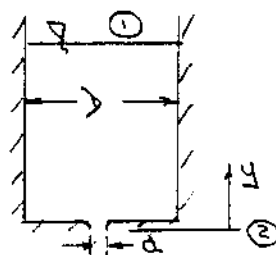
then,

$$\frac{dV_j}{dt} = \sqrt{\frac{2g}{(1-AR^2)}}\,\frac{1}{2\sqrt{y}}\,\frac{dy}{dt} = \frac{dy}{dt}\sqrt{\frac{g}{2y(1-AR^2)}}$$

Since $\frac{dy}{dt} = -V_1 = -V_j\frac{A_j}{A_1}$, then

$$\frac{dV_j}{dt} = -V_j\frac{A_j}{A_1}\sqrt{\frac{g}{2y(1-AR^2)}} = -\frac{A_j}{A_1}\sqrt{\frac{V_j^2(1-AR^2)}{2gy}}\;\frac{g}{(1-AR^2)}$$

and

$$\frac{dV_j}{dt} = -\frac{A_j}{A_1}\frac{g}{(1-AR^2)}$$

Problem *6.91

For $\left|\frac{A_j}{A_1}\frac{dV_j}{dt}\right| \ll g$, then $\left(\frac{A_j}{A_1}\right)^2 \frac{1}{(1-AR^2)} \ll 1$

If we take
$$\left(\frac{A_j}{A_1}\right)^2 \frac{1}{(1-AR^2)} \approx 0.01$$

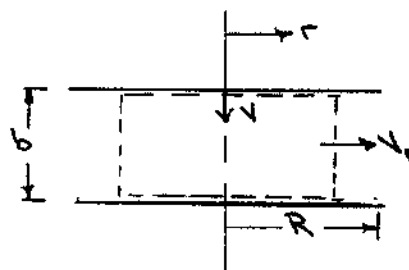
then,
$$\left(\frac{A_j}{A_1}\right)^2 = 0.01\,(1-AR^2) = 0.01\left[1-\left(\frac{A_j}{A_1}\right)^2\right]$$

and
$$1.01\left(\frac{A_j}{A_1}\right)^2 = 0.01$$
$$\frac{A_j}{A_1} = 0.0995$$

or
$$\frac{D_j}{D_1} = \left(\frac{A_j}{A_1}\right)^{1/2} = 0.32 \longleftarrow$$

In problem 4.44, $D_j/D_1 = d/D = 0.1$ and hence the assumption of quasi-steady flow is valid.

Problem *6.92

Given: Two circular discs of radius, R, are separated by a distance, b.
Upper disc moves toward the lower one at speed, V.
Fluid between discs is incompressible and is squeezed out radially.
Assume frictionless flow and uniform radial flow and any radial section.
Pressure surrounding disc is at P_{atm}

Find: gage pressure at r=0

Solution:

Basic equation: $\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \int_1^2 \frac{\partial V_s}{\partial t} ds$

$$0 = \frac{\partial}{\partial t}\int_{cv} \rho d\forall + \int_{cs} \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) incompressible flow
(2) frictionless flow
(3) flow along a streamline
(4) uniform radial flow at any r
(5) neglect elevation changes.

Then,

$$0 = \frac{\partial}{\partial t}\int_{cv} \rho d\forall + \int_{cs} \rho\vec{V}\cdot d\vec{A} = \frac{\partial}{\partial t}(\rho \pi r^2 b) + \rho V_r 2\pi r b$$

$$= \rho \pi r^2 \frac{\partial b}{\partial t} + \rho V_r 2\pi r b. \quad \text{But } \frac{\partial b}{\partial t} = -V$$

$$\therefore 0 = -\rho\pi r^2 V + \rho V_r 2\pi r b \qquad \text{and } V_r = V\frac{r}{2b}$$

Applying the Bernoulli equation between point ① (r=r) and point ② (r=R)

$$P_1 - P_2 = \frac{\rho}{2}\left[V_2^2 - V_1^2\right] + \int_r^R \rho \frac{\partial V_r}{\partial t} dr \quad \text{Now, } \frac{\partial V_r}{\partial t} = \frac{\partial}{\partial t}\left(V\frac{r}{2b}\right) = \frac{rV}{2}\left(-\frac{1}{b^2}\frac{db}{dt}\right) = \frac{V^2 r}{2b^2}$$

$$= \frac{\rho}{2}\left[\left(\frac{VR}{2b}\right)^2 - \left(\frac{Vr}{2b}\right)^2\right] + \int_r^R \rho \frac{V^2 r}{2b^2} dr$$

$$= \frac{\rho V^2}{8b^2}\left[R^2 - r^2\right] + \frac{\rho V^2}{4b^2} r^2\Big]_r^R = \frac{\rho V^2}{8b^2}\left[R^2 - r^2\right] + \frac{\rho V^2}{4b^2}\left[R^2 - r^2\right]$$

$$P_1 - P_{atm} = \frac{3}{8}\frac{\rho V^2}{b^2}\left[R^2 - r^2\right] = \frac{3}{8}\rho\frac{V^2R^2}{b^2}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

When r = 0 $\quad P_1 = P_0$

$$\therefore P_0 - P_{atm} = \frac{3}{8}\rho\frac{V^2R^2}{b^2}$$

Problem *6.93

Given: Two vortex flows with velocity fields

$$\vec{V}_1 = \omega r \hat{e}_\theta \qquad \vec{V}_2 = \frac{K}{2\pi r}\hat{e}_\theta$$

Determine: if the Bernoulli equation can be applied between different radii for each flow.

Solution: Since $V_r = 0$, the streamlines are concentric circles.
In order for it to be possible to apply the Bernoulli equation between different radii, it is necessary that the flow be irrotational.

Basic equation: $\vec{\omega} = \frac{1}{2}\nabla \times \vec{V}$

Flow (1)

$$\nabla \times \vec{V}_1 = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z}\right) \times \omega r \hat{e}_\theta$$

$$= \hat{e}_r \times \hat{e}_\theta \frac{\partial}{\partial r}(\omega r) + \hat{e}_r \times \omega r \frac{\partial \hat{e}_\theta}{\partial r}^{\,0} + \hat{e}_\theta \times \hat{e}_\theta^{\,0} \frac{1}{r}\frac{\partial(\omega r)}{\partial \theta} + \hat{e}_\theta \times \frac{\omega r}{r}\frac{\partial \hat{e}_\theta}{\partial \theta}$$

$$= \hat{k}\omega + \hat{e}_\theta \times \omega(-\hat{e}_r)$$

$$\nabla \times \vec{V}_1 = 2\omega \hat{k}$$

$\therefore$ Flow (1) is rotational and Bernoulli equation cannot be applied between different radii.

Flow (2)

$$\nabla \times \vec{V}_2 = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z}\right) \times \frac{K}{2\pi r}\hat{e}_\theta$$

$$= \hat{e}_r \times \hat{e}_\theta \frac{\partial}{\partial r}\left(\frac{K}{2\pi r}\right) + \hat{e}_r \times \left(\frac{K}{2\pi r}\right)\frac{\partial \hat{e}_\theta}{\partial r}^{\,0} + \hat{e}_\theta \times \hat{e}_\theta^{\,0} \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{K}{2\pi r}\right) + \hat{e}_\theta \times \frac{1}{r}\left(\frac{K}{2\pi r}\right)\frac{\partial \hat{e}_\theta}{\partial \theta}$$

$$= -\hat{k}\frac{K}{2\pi r^2} + \hat{e}_\theta \frac{K}{2\pi r^2} \times (-\hat{e}_r)$$

$$= -\hat{k}\frac{K}{2\pi r^2} + \hat{k}\frac{K}{2\pi r^2}$$

$$\nabla \times \vec{V}_2 = 0$$

Since the flow field is irrotational, Bernoulli equation can be applied between different radii if the flow is also incompressible and frictionless.

Problem *6.94

***6.94** Consider the flow represented by the stream function $\psi = Ax^2y$, where A is a dimensional constant equal to 2.5 $m^{-1} \cdot s^{-1}$. The density is 1200 kg/m^3. Is the flow rotational? Can the pressure difference between points $(x, y) = (1, 4)$ and $(2, 1)$ be evaluated? If so, calculate it, and if not, explain why.

Given: Stream function

Find: If the flow is irrotational; Pressure difference between points (1,4) and (2,1)

Solution:

Basic equations: Incompressibility because ψ exists $\quad u = \frac{\partial}{\partial y}\psi \quad v = -\frac{\partial}{\partial x}\psi \quad$ Irrotationality $\quad \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

$$\psi(x,y) = A \cdot x^2 \cdot y$$

$$u(x,y) = \frac{\partial}{\partial y}\psi(x,y) = \frac{\partial}{\partial y}\left(A \cdot x^2 \cdot y\right) \qquad u(x,y) = A \cdot x^2$$

$$v(x,y) = -\frac{\partial}{\partial x}\psi(x,y) = -\frac{\partial}{\partial x}\left(A \cdot x^2 \cdot y\right) \qquad v(x,y) = -2 \cdot A \cdot x \cdot y$$

Hence $\quad \frac{\partial}{\partial x}v(x,y) - \frac{\partial}{\partial y}u(x,y) \rightarrow -2 \cdot A \cdot y \qquad \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u \neq 0 \quad$ so flow is NOT IRROTATIONAL

Since flow is rotational, we must be on same streamline to be able to use Bernoulli

At point (1,4) $\quad \psi(1,4) = 4A \qquad$ and at point (2,1) $\quad \psi(2,1) = 4A$

Hence these points are on same streamline so Bernoulli can be used. The velocity at a point is $\quad V(x,y) = \sqrt{u(x,y)^2 + v(x,y)^2}$

Hence at (1,4) $\quad V_1 = \sqrt{\left[\frac{2.5}{m \cdot s} \times (1 \cdot m)^2\right]^2 + \left(-2 \times \frac{2.5}{m \cdot s} \times 1 \cdot m \times 4 \cdot m\right)^2} \qquad V_1 = 20.2 \frac{m}{s}$

Hence at (2,1) $\quad V_2 = \sqrt{\left[\frac{2.5}{m \cdot s} \times (2 \cdot m)^2\right]^2 + \left(-2 \times \frac{2.5}{m \cdot s} \times 2 \cdot m \times 1 \cdot m\right)^2} \qquad V_2 = 14.1 \frac{m}{s}$

Using Bernoulli $\quad \frac{p_1}{\rho} + \frac{1}{2} \cdot V_1^2 = \frac{p_2}{\rho} + \frac{1}{2} \cdot V_2^2 \qquad \Delta p = \frac{\rho}{2} \cdot \left(V_2^2 - V_1^2\right)$

$$\Delta p = \frac{1}{2} \times 1200 \cdot \frac{kg}{m^3} \times \left(14.1^2 - 20.2^2\right) \cdot \left(\frac{m}{s}\right)^2 \times \frac{N \cdot s^2}{kg \cdot m} \qquad \Delta p = -126 \cdot kPa$$

Given: Two-dimensional flow represented by the velocity field $\vec{V} = (Ax - By)t\hat{i} - (Bx + Ay)t\hat{j}$, where $A = 1\,s^{-2}$, $B = 2\,s^{-2}$, t is in s, and coordinates are in meters.

Find: (a) Is this a possible incompressible flow?
(b) Is the flow steady or unsteady?
(c) Show that the flow is irrotational
(d) Derive an expression for the velocity potential

Solution: For incompressible flow, $\nabla \cdot \vec{V} = 0$

For given flow $\nabla \cdot \vec{V} = \frac{\partial}{\partial x}(Ax - By)t - \frac{\partial}{\partial y}(Bx + Ay)t = At - At = 0$

$\therefore$ velocity field represents a possible incompressible flow

The flow is unsteady since $\vec{V} = \vec{V}(x, y, t)$

The rotation is given by $\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$

$$\vec{\omega} = \frac{1}{2}\left[\frac{\partial}{\partial x} - (Bx + Ay)t - \frac{\partial}{\partial y}(Ax - By)t\right] = -Bt + Bt = 0$$

$\vec{\omega} = 0$, so flow is irrotational

From the definition of the velocity potential, $\vec{V} = -\nabla \phi$

$u = -\frac{\partial \phi}{\partial x}$ and $\phi = \int -u\,dx + f(y,t) = \int -(Ax - By)t\,dx + f(y,t)$

$$\phi = \left(-A\frac{x^2}{2} + Bxy\right)t + f(y,t)$$

$v = -\frac{\partial \phi}{\partial y}$ and $\phi = \int -v\,dy + g(x,t) = \int (Bx + Ay)t\,dy + g(x,t)$

$$\phi = \left(Bxy + A\frac{y^2}{2}\right)t + g(x,t)$$

Comparing the two expressions for ϕ we conclude

$$f(y,t) = \frac{A}{2}y^2 t \quad \text{and} \quad g(x,t) = -\frac{A}{2}x^2 t$$

Hence,

$$\phi = \left\{\frac{A}{2}(y^2 - x^2) + Bxy\right\}t$$ ϕ

Problem *6.96

*6.96 Using Table 6.2, find the stream function and velocity potential for a plane source, of strength q, near a 90° corner. The source is equidistant h from each of the two infinite planes that make up the corner. Find the velocity distribution along one of the planes, assuming $p = p_0$ at infinity. By choosing suitable values for q and h, plot the streamlines and lines of constant velocity potential. (Hint: Use the *Excel* workbook of Example 6.10.)

Given: Data from Table 6.2

Find: Stream function and velocity potential for a source in a corner; plot; velocity along one plane

Solution:

From Table 6.2, for a source at the origin $\psi(r,\theta) = \frac{q}{2\cdot\pi}\cdot\theta$ $\qquad \phi(r,\theta) = -\frac{q}{2\cdot\pi}\cdot\ln(r)$

Expressed in Cartesian coordinates $\psi(x,y) = \frac{q}{2\cdot\pi}\cdot\operatorname{atan}\left(\frac{y}{x}\right)$ $\qquad \phi(x,y) = -\frac{q}{4\cdot\pi}\cdot\ln\left(x^2+y^2\right)$

To build flow in a corner, we need image sources at three locations so that there is symmetry about both axes. We need sources at (h,h), (h,- h), (- h,h), and (- h,- h)

Hence the composite stream function and velocity potential are

$$\psi(x,y) = \frac{q}{2\cdot\pi}\cdot\left(\operatorname{atan}\left(\frac{y-h}{x-h}\right)+\operatorname{atan}\left(\frac{y+h}{x-h}\right)+\operatorname{atan}\left(\frac{y+h}{x+h}\right)+\operatorname{atan}\left(\frac{y-h}{x+h}\right)\right)$$

$$\phi(x,y) = -\frac{q}{4\cdot\pi}\cdot\ln\left[\left[(x-h)^2+(y-h)^2\right]\cdot\left[(x-h)^2+(y+h)^2\right]\right] - \frac{q}{4\cdot\pi}\cdot\left[(x+h)^2+(y+h)^2\right]\cdot\left[(x+h)^2+(y-h)^2\right]$$

By a similar reasoning the horizontal velocity is given by

$$u = \frac{q\cdot(x-h)}{2\cdot\pi\left[(x-h)^2+(y-h)^2\right]}+\frac{q\cdot(x-h)}{2\cdot\pi\left[(x-h)^2+(y+h)^2\right]}+\frac{q\cdot(x+h)}{2\cdot\pi\left[(x+h)^2+(y+h)^2\right]}+\frac{q\cdot(x+h)}{2\cdot\pi\left[(x+h)^2+(y+h)^2\right]}$$

Along the horizontal wall (y = 0)

$$u = \frac{q\cdot(x-h)}{2\cdot\pi\left[(x-h)^2+h^2\right]}+\frac{q\cdot(x-h)}{2\cdot\pi\left[(x-h)^2+h^2\right]}+\frac{q\cdot(x+h)}{2\cdot\pi\left[(x+h)^2+h^2\right]}+\frac{q\cdot(x+h)}{2\cdot\pi\left[(x+h)^2+h^2\right]}$$

or $\qquad u(x) = \frac{q}{\pi}\cdot\left[\frac{x-h}{(x-h)^2+h^2}+\frac{x+h}{(x+h)^2+h^2}\right]$

Problem *6.96

***6.96** Using Table 6.2, find the stream function and velocity potential for a plane source, of strength q, near a 90° corner. The source is equidistant h from each of the two infinite planes that make up the corner. Find the velocity distribution along one of the planes, assuming $p = p_0$ at infinity. By choosing suitable values for q and h, plot the streamlines and lines of constant velocity potential. (Hint: Use the *Excel* workbook of Example 6.10.)

Given: Data from Table 6.2

Find: Stream function and velocity potential for a source in a corner; plot; velocity along one plane

Solution:

$$\psi(x,y) = \frac{q}{2\cdot\pi}\cdot\left(\text{atan}\left(\frac{y-h}{x-h}\right) + \text{atan}\left(\frac{y+h}{x-h}\right) + \text{atan}\left(\frac{y+h}{x+h}\right) + \text{atan}\left(\frac{y-h}{x+h}\right)\right)$$

$$\phi(x,y) = -\frac{q}{4\cdot\pi}\cdot\ln\left[\left[(x-h)^2+(y-h)^2\right]\cdot\left[(x-h)^2+(y+h)^2\right]\right] - \frac{q}{4\cdot\pi}\cdot\left[(x+h)^2+(y+h)^2\right]\cdot\left[(x+h)^2+(y-h)^2\right]$$

#NAME?

Stream Function

#NAME?

Velocity Potential

Note that the plot is from $x = 0$ to 5 and $y = 0$ to 5

Stream Function

Velocity Potential

Problem *6.97

***6.97** The flow field for a plane source at a distance h above an infinite wall aligned along the x axis is given by

$$\vec{V} = \frac{q}{2\pi[x^2 + (y-h)^2]}[x\hat{i} + (y-h)\hat{j}] + \frac{q}{2\pi[x^2 + (y+h)^2]}[x\hat{i} + (y+h)\hat{j}]$$

where q is the strength of the source. The flow is irrotational and incompressible. Derive the stream function and velocity potential. By choosing suitable values for q and h, plot the streamlines and lines of constant velocity potential. (Hint: Use the *Excel* workbook of Example 6.10.)

Given: Velocity field of irrotational and incompressible flow

Find: Stream function and velocity potential; plot

Solution:

The velocity field is
$$u = \frac{q\cdot x}{2\cdot\pi\left[x^2 + (y-h)^2\right]} + \frac{q\cdot x}{2\cdot\pi\left[x^2 + (y+h)^2\right]} \qquad v = \frac{q\cdot(y-h)}{2\cdot\pi\left[x^2 + (y-h)^2\right]} + \frac{q\cdot(y+h)}{2\cdot\pi\left[x^2 + (y+h)^2\right]}$$

The governing equations are
$$u = \frac{\partial}{\partial y}\psi \qquad v = -\frac{\partial}{\partial x}\psi \qquad u = -\frac{\partial}{\partial x}\phi \qquad v = -\frac{\partial}{\partial y}\phi$$

Hence for the stream function
$$\psi = \int u(x,y)\,dy = \frac{q}{2\cdot\pi}\cdot\left(\operatorname{atan}\left(\frac{y-h}{x}\right) + \operatorname{atan}\left(\frac{y+h}{x}\right)\right) + f(x)$$

$$\psi = -\int v(x,y)\,dx = \frac{q}{2\cdot\pi}\cdot\left(\operatorname{atan}\left(\frac{y-h}{x}\right) + \operatorname{atan}\left(\frac{y+h}{x}\right)\right) + g(y)$$

The simplest expression for ψ is
$$\psi(x,y) = \frac{q}{2\cdot\pi}\cdot\left(\operatorname{atan}\left(\frac{y-h}{x}\right) + \operatorname{atan}\left(\frac{y+h}{x}\right)\right)$$

For the stream function
$$\phi = -\int u(x,y)\,dx = -\frac{q}{4\cdot\pi}\cdot\ln\left[\left[x^2 + (y-h)^2\right]\cdot\left[x^2 + (y+h)^2\right]\right] + f(y)$$

$$\phi = -\int v(x,y)\,dy = -\frac{q}{4\cdot\pi}\cdot\ln\left[\left[x^2 + (y-h)^2\right]\cdot\left[x^2 + (y+h)^2\right]\right] + g(x)$$

The simplest expression for φ is
$$\phi(x,y) = -\frac{q}{4\cdot\pi}\cdot\ln\left[\left[x^2 + (y-h)^2\right]\cdot\left[x^2 + (y+h)^2\right]\right]$$

Problem *6.97

*6.97 The flow field for a plane source at a distance h above an infinite wall aligned along the x axis is given by

$$\vec{V} = \frac{q}{2\pi[x^2 + (y-h)^2]}[x\hat{i} + (y-h)\hat{j}] + \frac{q}{2\pi[x^2 + (y+h)^2]}[x\hat{i} + (y+h)\hat{j}]$$

where q is the strength of the source. The flow is irrotational and incompressible. Derive the stream function and velocity potential. By choosing suitable values for q and h, plot the streamlines and lines of constant velocity potential. (Hint: Use the *Excel* workbook of Example 6.10.)

Given: Velocity field of irrotational and incompressible flow

Find: Stream function and velocity potential; plot

Solution:

$$\psi(x,y) = \frac{q}{2\cdot\pi}\cdot\left(\operatorname{atan}\left(\frac{y-h}{x}\right) + \operatorname{atan}\left(\frac{y+h}{x}\right)\right)$$

$$\phi(x,y) = -\frac{q}{4\cdot\pi}\cdot\ln\left[\left[x^2 + (y-h)^2\right]\cdot\left[x^2 + (y+h)^2\right]\right]$$

#NAME?

Stream Function

#NAME?

Velocity Potential

Note that the plot is from x = -2.5 to 2.5 and y = 0 to 5

Stream Function

Velocity Potential

Problem *6.98

***6.98** Using Table 6.2, find the stream function and velocity potential for a plane vortex, of strength K, near a 90° corner. The vortex is equidistant h from each of the two infinite planes that make up the corner. Find the velocity distribution along one of the planes, assuming $p = p_0$ at infinity. By choosing suitable values for K and h, plot the streamlines and lines of constant velocity potential. (Hint: Use the *Excel* workbook of Example Problem 6.10.)

Given: Data from Table 6.2

Find: Stream function and velocity potential for a vortex in a corner; plot; velocity along one plane

Solution:

From Table 6.2, for a vortex at the origin $\phi(r,\theta) = \frac{K}{2\cdot\pi}\cdot\theta$ $\qquad \psi(r,\theta) = -\frac{K}{2\cdot\pi}\cdot\ln(r)$

Expressed in Cartesian coordinates $\phi(x,y) = \frac{q}{2\cdot\pi}\cdot\operatorname{atan}\left(\frac{y}{x}\right)$ $\qquad \psi(x,y) = -\frac{q}{4\cdot\pi}\cdot\ln\left(x^2 + y^2\right)$

To build flow in a corner, we need image vortices at three locations so that there is symmetry about both axes. We need vortices at (h,h), (h,- h), (- h,h), and (- h,- h). Note that some of them must have strengths of - K!

Hence the composite velocity potential and stream function are

$$\phi(x,y) = \frac{K}{2\cdot\pi}\cdot\left(\operatorname{atan}\left(\frac{y-h}{x-h}\right) - \operatorname{atan}\left(\frac{y+h}{x-h}\right) + \operatorname{atan}\left(\frac{y+h}{x+h}\right) - \operatorname{atan}\left(\frac{y-h}{x+h}\right)\right)$$

$$\psi(x,y) = -\frac{K}{4\cdot\pi}\cdot\ln\left[\frac{(x-h)^2+(y-h)^2}{(x-h)^2+(y+h)^2}\cdot\frac{(x+h)^2+(y+h)^2}{(x+h)^2+(y-h)^2}\right]$$

By a similar reasoning the horizontal velocity is given by

$$u = -\frac{K\cdot(y-h)}{2\cdot\pi\left[(x-h)^2+(y-h)^2\right]} + \frac{K\cdot(y+h)}{2\cdot\pi\left[(x-h)^2+(y+h)^2\right]} - \frac{K\cdot(y+h)}{2\cdot\pi\left[(x+h)^2+(y+h)^2\right]} + \frac{K\cdot(y-h)}{2\cdot\pi\left[(x+h)^2+(y-h)^2\right]}$$

Along the horizontal wall (y = 0)

$$u = \frac{K\cdot h}{2\cdot\pi\left[(x-h)^2+h^2\right]} + \frac{K\cdot h}{2\cdot\pi\left[(x-h)^2+h^2\right]} - \frac{K\cdot h}{2\cdot\pi\left[(x+h)^2+h^2\right]} - \frac{K\cdot h}{2\cdot\pi\left[(x+h)^2+h^2\right]}$$

or

$$u(x) = \frac{K\cdot h}{\pi}\cdot\left[\frac{1}{(x-h)^2+h^2} - \frac{1}{(x+h)^2+h^2}\right]$$

Problem *6.98

*6.98 Using Table 6.2, find the stream function and velocity potential for a plane vortex, of strength K, near a 90° corner. The vortex is equidistant h from each of the two infinite planes that make up the corner. Find the velocity distribution along one of the planes, assuming $p = p_0$ at infinity. By choosing suitable values for K and h, plot the streamlines and lines of constant velocity potential. (Hint: Use the *Excel* workbook of Example Problem 6.10.)

Given: Data from Table 6.2

Find: Stream function and velocity potential for a vortex in a corner; plot; velocity along one plane

Solution:

$$\phi(x,y) = \frac{K}{2\cdot\pi}\cdot\left(\operatorname{atan}\left(\frac{y-h}{x-h}\right) - \operatorname{atan}\left(\frac{y+h}{x-h}\right) + \operatorname{atan}\left(\frac{y+h}{x+h}\right) - \operatorname{atan}\left(\frac{y-h}{x+h}\right)\right)$$

$$\psi(x,y) = -\frac{K}{4\cdot\pi}\cdot\ln\left[\frac{(x-h)^2+(y-h)^2}{(x-h)^2+(y+h)^2}\cdot\frac{(x+h)^2+(y+h)^2}{(x+h)^2+(y-h)^2}\right]$$

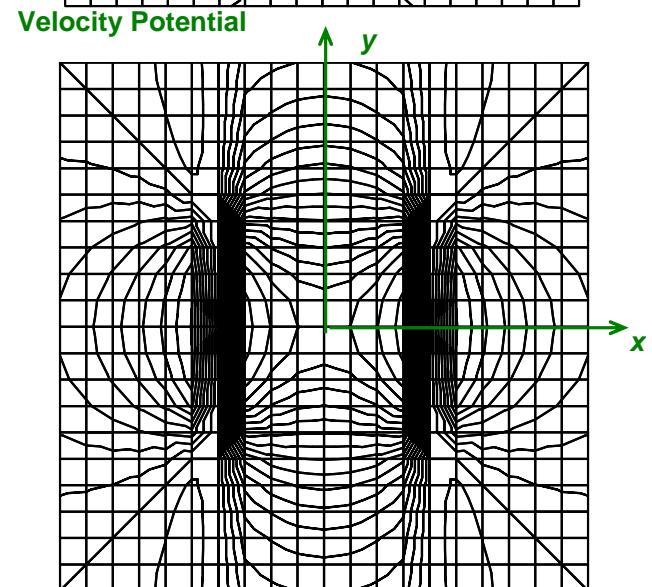
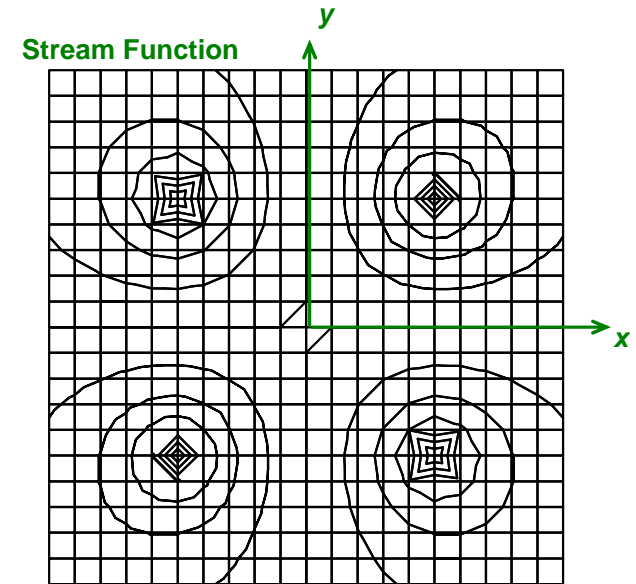
#NAME?

Stream Function

#NAME?

#NAME?

Velocity Potential

Note that the plot is
from x = -5 to 5 and y = -5 to 5

Given: Flow field represented by $\psi = Ax^2y - By^3$, where $A = 1\ m^{-1}\cdot s^{-1}$, $B = \frac{1}{3}\ m^{-1}\cdot s^{-1}$, and coordinates are in meters.

Find: an expression for the velocity potential, ϕ

Solution:

The velocity field is determined from the stream function

$$\left.\begin{array}{l} u = \partial\psi/\partial y = Ax^2 - 3By^2 \\ v = -\partial\psi/\partial x = -2Axy \end{array}\right\} \therefore \vec{V} = (Ax^2 - 3By^2)\hat{i} - 2Axy\hat{j}$$

The rotation is given by $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

$$\omega_z = \frac{1}{2}(-2Ay + 6By) = \frac{1}{2}\left(-2\times 1\,y + 6\times\frac{1}{3}y\right) = 0$$

Since $\omega_z = 0$, the flow is irrotational and $\vec{V} = -\nabla\phi$

Then

$$u = -\frac{\partial\phi}{\partial x} \quad\text{and}\quad \phi = \int -u\,dx + f(y) = \int(-Ax^2 + 3By^2)\,dx + f(y)$$

$$\phi = -\frac{A}{3}x^3 + 3Bxy^2 + f(y)$$

$$v = -\frac{\partial\phi}{\partial y} \quad\text{and}\quad \phi = \int -v\,dy + g(x) = \int 2Axy\,dy + g(x)$$

$$\phi = Axy^2 + g(x)$$

Comparing the two expressions for ϕ we

- note that $Axy^2 = 3Bxy^2 \qquad (A = 1,\ B = \frac{1}{3})$
- conclude that $g(x) = -\frac{A}{3}x^3$, $f(y) = 0$

Hence $\phi = Axy^2 - \frac{A}{3}x^3$ or $\phi = 3Bxy^2 - \frac{A}{3}x^3$ ———— ϕ

Problem *6.100

***6.100** A flow field is represented by the stream function $\psi = x^5 - 10x^3y^2 + 5xy^4$. Find the corresponding velocity field. Show that this flow field is irrotational and obtain the potential function.

Given: Stream function

Find: Velocity field; Show flow is irrotational; Velocity potential

Solution:

Basic equations: Incompressibility because ψ exists $\quad u = \frac{\partial}{\partial y}\psi \quad v = -\frac{\partial}{\partial x}\psi \quad u = -\frac{\partial}{\partial x}\varphi \quad v = -\frac{\partial}{\partial y}\varphi$

Irrotationality $\quad \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

$$\psi(x,y) = x^5 - 10\cdot x^3\cdot y^2 + 5\cdot x\cdot y^4$$

$$u(x,y) = \frac{\partial}{\partial y}\psi(x,y) \qquad u(x,y) \rightarrow 20\cdot x\cdot y^3 - 20\cdot x^3\cdot y$$

$$v(x,y) = -\frac{\partial}{\partial x}\psi(x,y) \qquad v(x,y) \rightarrow 30\cdot x^2\cdot y^2 - 5\cdot x^4 - 5\cdot y^4$$

$$\frac{\partial}{\partial x}v(x,y) - \frac{\partial}{\partial y}u(x,y) \rightarrow 0$$

Hence flow is IRROTATIONAL

Hence $\quad u = -\frac{\partial}{\partial x}\varphi \quad$ so $\quad \varphi(x,y) = -\int u(x,y)\,dx + f(y) = 5\cdot x^4\cdot y - 10\cdot x^2\cdot y^3 + f(y)$

$\quad v = -\frac{\partial}{\partial y}\varphi \quad$ so $\quad \varphi(x,y) = -\int v(x,y)\,dy + g(x) = 5\cdot x^4\cdot y - 10\cdot x^2\cdot y^3 + y^5 + g(x)$

Comparing, the simplest velocity potential is then $\quad \varphi(x,y) = 5\cdot x^4\cdot y - 10\cdot x^2\cdot y^3 + y^5$

Given: Flow field represented by the potential function,

$$\phi = Ax^2 + Bxy - Ay^2$$

Find: (a) Verify that the flow is incompressible
(b) Determine the corresponding stream function, ψ

Solution:

The velocity field is given by $\vec{V} = -\nabla\phi$

$$\vec{V} = -\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(Ax^2 + Bxy - Ay^2) = -\hat{i}(2Ax + By) - \hat{j}(Bx - 2Ay)$$

If the flow is incompressible, then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}(-)(2Ax + By) + \frac{\partial}{\partial y}(-)(Bx - 2Ay) = -2A + 2A = 0$$

$\therefore$ flow is incompressible

From the definition of ψ, $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$

Thus,

$$u = -2Ax - By = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \psi = -\int (2Ax + By)\,dy + f(x)$$

$$\psi = -2Axy - B\frac{y^2}{2} + f(x)$$

Then,

$$v = -Bx + 2Ay = -\frac{\partial \psi}{\partial x} = 2Ay - \frac{df}{dx}$$

and $-\frac{df}{dx} = -Bx$ or $f = \frac{1}{2}Bx^2 + \text{constant}$

$$\therefore \psi = -2Axy - B\frac{y^2}{2} + B\frac{x^2}{2} + \text{constant}$$

Setting the constant equal to zero, we obtain

$$\psi = \frac{B}{2}(x^2 - y^2) - 2Axy$$

ψ

Problem *6.102

***6.102** Consider the flow field presented by the potential function $\phi = x^6 - 15x^4y^2 + 15x^2y^4 - y^6$. Verify that this is an incompressible flow and obtain the corresponding stream function.

Given: Velocity potential

Find: Show flow is incompressible; Stream function

Solution:

Basic equations: Irrotationality because φ exists $u = \frac{\partial}{\partial y}\psi$ $v = -\frac{\partial}{\partial x}\psi$ $u = -\frac{\partial}{\partial x}\varphi$ $v = -\frac{\partial}{\partial y}\varphi$

Incompressibility $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$

$$\varphi(x,y) = x^6 - 15\cdot x^4\cdot y^2 + 15\cdot x^2\cdot y^4 - y^6$$

$$u(x,y) = -\frac{\partial}{\partial x}\varphi(x,y) \qquad u(x,y) \to 60\cdot x^3\cdot y^2 - 6\cdot x^5 - 30\cdot x\cdot y^4$$

$$v(x,y) = -\frac{\partial}{\partial y}\varphi(x,y) \qquad v(x,y) \to 30\cdot x^4\cdot y - 60\cdot x^2\cdot y^3 + 6\cdot y^5$$

Hence $\frac{\partial}{\partial x}u(x,y) + \frac{\partial}{\partial y}v(x,y) \to 0$ Hence flow is INCOMPRESSIBLE

Hence $u = \frac{\partial}{\partial y}\psi$ so $\psi(x,y) = \int u(x,y)\,dy + f(x) = 20\cdot x^3\cdot y^3 - 6\cdot x^5\cdot y - 6\cdot x\cdot y^5 + f(x)$

$v = -\frac{\partial}{\partial x}\psi$ so $\psi(x,y) = -\int v(x,y)\,dx + g(y) = 20\cdot x^3\cdot y^3 - 6\cdot x^5\cdot y - 6\cdot x\cdot y^5 + g(y)$

Comparing, the simplest stream function is then $\psi(x,y) = 20\cdot x^3\cdot y^3 - 6\cdot x^5\cdot y - 6\cdot x\cdot y^5$

Problem *6.103

***6.103** Show that $f(z) = z^6$ (where z is the complex number $z = x + iy$) leads to a valid velocity potential (the real part of f) and a corresponding stream function (the imaginary part of f) of an irrotational and incompressible flow. Then show that the real and imaginary parts of df/dz yield u and $-v$, respectively.

Given: Complex function

Find: Show it leads to velocity potential and stream function of irrotational incompressible flow;
Show that df/dz leads to u and v

Solution:

Basic equations: Irrotationality because φ exists $\quad u = \frac{\partial}{\partial y}\psi \quad v = -\frac{\partial}{\partial x}\psi \quad u = -\frac{\partial}{\partial x}\varphi \quad v = -\frac{\partial}{\partial y}\varphi$

Incompressibility $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ Irrotationality $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

$$f(z) = z^6 = (x + i\cdot y)^6$$

Expanding $f(z) = x^6 - 15\cdot x^4\cdot y^2 + 15\cdot x^2\cdot y^4 - y^6 + i\cdot\left(6\cdot x\cdot y^5 + 6\cdot x^5\cdot y - 20\cdot x^3\cdot y^3\right)$

We are thus to check the following

$$\varphi(x,y) = x^6 - 15\cdot x^4\cdot y^2 + 15\cdot x^2\cdot y^4 - y^6 \qquad \psi(x,y) = 6\cdot x\cdot y^5 + 6\cdot x^5\cdot y - 20\cdot x^3\cdot y^3$$

$$u(x,y) = -\frac{\partial}{\partial x}\varphi(x,y) \qquad u(x,y) \rightarrow 60\cdot x^3\cdot y^2 - 6\cdot x^5 - 30\cdot x\cdot y^4$$

$$v(x,y) = -\frac{\partial}{\partial y}\varphi(x,y) \qquad v(x,y) \rightarrow 30\cdot x^4\cdot y - 60\cdot x^2\cdot y^3 + 6\cdot y^5$$

An alternative derivation of u and v is

$$u(x,y) = \frac{\partial}{\partial y}\psi(x,y) \qquad u(x,y) \rightarrow 6\cdot x^5 - 60\cdot x^3\cdot y^2 + 30\cdot x\cdot y^4$$

$$v(x,y) = -\frac{\partial}{\partial x}\psi(x,y) \qquad v(x,y) \rightarrow 60\cdot x^2\cdot y^3 - 30\cdot x^4\cdot y - 6\cdot y^5$$

Note that the values of u and v are of opposite sign using ψ and φ!different which is the same result using φ! To resolve this we could either let f = -φ+iψ; altenatively we could use a different definition of φ that many authors use:

$$u = \frac{\partial}{\partial x}\varphi \qquad v = \frac{\partial}{\partial y}\varphi$$

Hence $\frac{\partial}{\partial x}v(x,y) - \frac{\partial}{\partial y}u(x,y) \rightarrow 0$ Hence flow is IRROTATIONAL

Hence $\frac{\partial}{\partial x}u(x,y) + \frac{\partial}{\partial y}v(x,y) \rightarrow 0$ Hence flow is INCOMPRESSIBLE

Next we find $\frac{df}{dz} = \frac{d\left(z^6\right)}{dz} = 6\cdot z^5 = 6\cdot(x + i\cdot y)^5 = \left(6\cdot x^5 - 60\cdot x^3\cdot y^2 + 30\cdot x\cdot y^4\right) + i\cdot\left(30\cdot x^4\cdot y + 6\cdot y^5 - 60\cdot x^2\cdot y^3\right)$

Hence we see $\frac{df}{dz} = u - i\cdot v$ Hence the results are verified; $u = \mathrm{Re}\left(\frac{df}{dz}\right)$ and $v = -\mathrm{Im}\left(\frac{df}{dz}\right)$

These interesting results are explained in Problem 6.104!

Problem *6.104

*6.104 Show that *any* differentiable function $f(z)$ of the complex number $z = x + iy$ leads to a valid potential (the real part of f) and a corresponding stream function (the imaginary part of f) of an incompressible, irrotational flow. To do so, prove using the chain rule that $f(z)$ automatically satisfies the Laplace equation. Then show that $df/dz = u - iv$.

Given: Complex function

Find: Show it leads to velocity potential and stream function of irrotational incompressible flow; Show that df/dz leads to u and v

Solution:

Basic equations: $u = \frac{\partial}{\partial y}\psi \qquad v = -\frac{\partial}{\partial x}\psi \qquad u = -\frac{\partial}{\partial x}\varphi \qquad v = -\frac{\partial}{\partial y}\varphi$

First consider $\frac{\partial}{\partial x}f = \frac{\partial}{\partial x}z\cdot\frac{d}{dz}f = 1\cdot\frac{d}{dz}f = \frac{d}{dz}f \qquad (1)$ and also $\frac{\partial}{\partial y}f = \frac{\partial}{\partial y}z\cdot\frac{d}{dz}f = i\cdot\frac{d}{dz}f = i\cdot\frac{d}{dz}f \qquad (2)$

Hence $\frac{\partial^2}{\partial x^2}f = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}f\right) = \frac{d}{dz}\left(\frac{d}{dz}f\right) = \frac{d^2}{dz^2}f$ and $\frac{\partial^2}{\partial y^2}f = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}f\right) = i\cdot\frac{d}{dz}\left(i\cdot\frac{d}{dz}f\right) = -\frac{d^2}{dz^2}f$

Combining $\frac{\partial^2}{\partial x^2}f + \frac{\partial^2}{\partial y^2}f = \frac{d^2}{dz^2}f - \frac{d^2}{dz^2}f = 0$

Any differentiable function f(z) automatically satisfies the Laplace Equation; so do its real and imaginary parts!

We demonstrate derivation of velocities u and v

From Eq 1 $\frac{d}{dz}f = \frac{d}{dz}(\varphi + i\cdot\psi) = \frac{\partial}{\partial x}(\varphi + i\cdot\psi) = \frac{\partial}{\partial x}\varphi + i\cdot\frac{\partial}{\partial x}\psi = -u - i\cdot v$

From Eq 2 $\frac{d}{dz}f = \frac{d}{dz}(\varphi + i\cdot\psi) = \frac{1}{i}\cdot\frac{\partial}{\partial y}(\varphi + i\cdot\psi) = -i\cdot\frac{\partial}{\partial y}\varphi + \frac{\partial}{\partial y}\psi = i\cdot v + u$

There appears to be an incompatibilty here, but many authors define φ as $u = \frac{\partial}{\partial x}\varphi \qquad v = \frac{\partial}{\partial y}\varphi$ or in other words, as the negative of our definition

Alternatively, we can use out φ but set $f = -\varphi + i\cdot\psi$

Then

From Eq 1 $\frac{d}{dz}f = \frac{d}{dz}(\varphi + i\cdot\psi) = \frac{\partial}{\partial x}(\varphi + i\cdot\psi) = \frac{\partial}{\partial x}\varphi + i\cdot\frac{\partial}{\partial x}\psi = u - i\cdot v$

From Eq 2 $\frac{d}{dz}f = \frac{d}{dz}(\varphi + i\cdot\psi) = \frac{1}{i}\cdot\frac{\partial}{\partial y}(\varphi + i\cdot\psi) = -i\cdot\frac{\partial}{\partial y}\varphi + \frac{\partial}{\partial y}\psi = -i\cdot v + u$

Hence we have demonstrated that $\frac{df}{dz} = u - i\cdot v$ if we set $u = \frac{\partial}{\partial x}\varphi \qquad v = \frac{\partial}{\partial y}\varphi$

Problem *6.105

Given: Flow field represented by the velocity potential $\phi = Ax + Bx^2 - By^2$, where $A = 1\ m \cdot s^{-1}$, $B = 1\ s^{-1}$, and coordinates are measured in meters.

Find: (a) expression for the velocity field
(b) stream function
(c) pressure difference between points $(x_1, y_1) = (0,0)$ and $(x_2, y_2) = (1,2)$

Solution

The velocity field is determined from the velocity potential

$$\left.\begin{aligned} u &= -\partial\phi/\partial x = -A - 2Bx \\ v &= -\partial\phi/\partial y = 2By \end{aligned}\right\} \quad \vec{V} = -(A + 2Bx)\hat{i} + 2By\,\hat{j} \qquad \vec{V}$$

From the definition of the stream function, $u = \frac{\partial\psi}{\partial y}$, $v = -\frac{\partial\psi}{\partial x}$

Then

$$\psi = \int u\,dy + f(x) = \int -(A + 2Bx)\,dy + f(x)$$

$$\psi = -Ay - 2Bxy + f(x)$$

Also,

$$\psi = \int -v\,dx + g(y) = \int -2By\,dx + g(y)$$

$$\psi = -2Bxy + g(y)$$

Comparing the two expressions for ψ we conclude

$$f(x) = 0 \quad , \quad g(y) = -Ay$$

$$\therefore \psi = -(Ay + 2Bxy) \qquad \psi$$

Since $\nabla^2\phi = 2B - 2B = 0$, the flow is irrotational and the Bernoulli equation can be applied between any two points in the flow field

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 \qquad \left\{ \begin{aligned} &\text{Assume: } \rho = \text{constant} \\ &z_1 = z_2 \end{aligned} \right\}$$

$\vec{V}(0,0) = -A\hat{i} = -\hat{i}\ m/s \qquad V_{0,0} = 1\ m/s$

$\vec{V}(1,2) = -(A + 2B)\hat{i} + 4B\hat{j} = -3\hat{i} + 4\hat{j}\ m/s \qquad \therefore V_{1,2} = 5\ m/s$

$$\therefore p_1 - p_2 = \rho\left(\frac{V_2^2}{2} - \frac{V_1^2}{2}\right) = \frac{\rho}{2}(V_2^2 - V_1^2)$$

Assume fluid is water

$$p_1 - p_2 = \frac{1}{2} \times 999\frac{kg}{m^3}(25 - 1)\frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 12\ kN/m^2$$

Problem *6.106

Given: Incompressible flow field represented by $\psi = 3Ax^2y - Ay^3$ where $A = 1\ m^{-1}\cdot s^{-1}$

Show: that this flow field is irrotational

Find: the velocity potential ϕ

Plot: streamlines and potential lines, and visually verify that they are orthogonal

Solution:

For a 2-D incompressible, irrotational flow $\nabla^2\psi = 0$ (6.30)

For this flow field

$$\nabla^2\psi = \frac{\partial^2}{\partial x^2}(3Ax^2y - Ay^3) + \frac{\partial^2}{\partial y^2}(3Ax^2y - Ay^3) = 6Ay - 6Ay = 0 \quad \text{irrotational}$$

The velocity field is given by $\vec{V} = u\hat{i} + v\hat{j}$

$$u = \partial\psi/\partial y = 3Ax^2 - 3Ay^2 = 3A(x^2 - y^2)$$

$$v = -\partial\psi/\partial x = -6Axy$$

$$\vec{V} = 3A(x^2 - y^2)\hat{i} - 6Axy\hat{j}$$

The velocity potential is defined such that $u = -\frac{\partial\phi}{\partial x}$ & $v = -\frac{\partial\phi}{\partial y}$

Then,
$$\phi = -\int u\,dx + f(y) = -\int 3A(x^2 - y^2)\,dx + f(y) = -Ax^3 + 3Axy^2 + f(y) \quad \text{......(1)}$$

Also,
$$\phi = -\int v\,dy + g(x) = \int 6Axy\,dy + g(x) = 3Axy^2 + g(x) \quad \text{......(2)}$$

Equating expressions for ϕ (Eqs 1 and 2) we see that

$g(x) = -Ax^3$ and $f(y) = 0$ $\quad\therefore\ \phi = 3Axy^2 - Ax^3$ $\quad \phi$

Problem *6.107

Given: Flow field represented by the velocity potential $\phi = Ay^3 - Bx^2y$, where $A = \frac{1}{3}\,m^{-1}\cdot s^{-1}$, $B = 1\,m^{-1}\cdot s^{-1}$, and the coordinates are measured in meters.

Find: (a) expression for the magnitude of the velocity vector
(b) the stream function.

Plot: streamlines and potential lines, and visually verify that they are orthogonal.

Solution:

The velocity field is determined from the velocity potential

$$u = -\partial\phi/\partial x = 2Bxy = 2xy$$
$$v = -\partial\phi/\partial y = -3Ay^2 + Bx^2 = x^2 - y^2$$
$$\vec{V} = 2xy\,\hat{i} + (x^2 - y^2)\,\hat{j}$$

$$V = \left[u^2 + v^2\right]^{1/2} = \left[4x^2y^2 + (x^2-y^2)^2\right]^{1/2} = \left[4x^2y^2 + x^4 - 2x^2y^2 + y^4\right]^{1/2}$$

$$V = \left[x^4 + 2x^2y^2 + y^4\right]^{1/2} = \left[(x^2+y^2)^2\right]^{1/2} = x^2 + y^2 \quad \longleftarrow V$$

The stream function is defined such that $u = \frac{\partial\psi}{\partial y}$ and $v = -\frac{\partial\psi}{\partial x}$

Then, $\psi = \int u\,dy + f(x) = \int 2Bxy\,dy + f(x) = Bxy^2 + f(x)$ ---- (1)

Also, $\psi = \int -v\,dx + g(y) = \int (3Ay^2 - Bx^2)\,dx + g(y) = 3Axy^2 - \frac{B}{3}x^3 + g(y)$ -- (2)

Comparing the two expressions for ψ, we

- note that $Bxy^2 = 3Ax^2y$ $(B = 1,\ A = \frac{1}{3})$, and
- conclude that $f(x) = -\frac{B}{3}x^3$ and $g(y) = 0$

$\therefore\ \psi = Bxy^2 - \frac{B}{3}x^3$ or $\psi = 3Ax^2y - \frac{B}{3}x^3 \quad \longleftarrow \psi$

with $A = \frac{1}{3}$, $B = 1$

$$\psi = x^2y - \frac{x^3}{3} = x\left(y^2 - \frac{x^2}{3}\right)$$

For $\psi = 0$, $x = 0$ or $y = 0.577x$

For $\psi = -4$, $y^2 = \frac{x^2}{3} - \frac{4}{x}$

For $\psi = 4$, $y^2 = \frac{x^2}{3} + \frac{4}{x}$

See the next page for plots

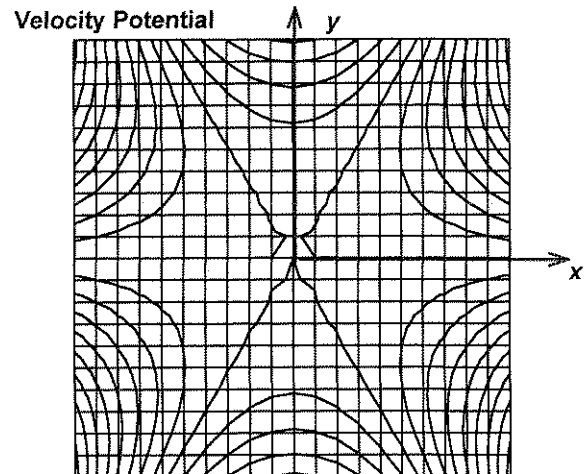
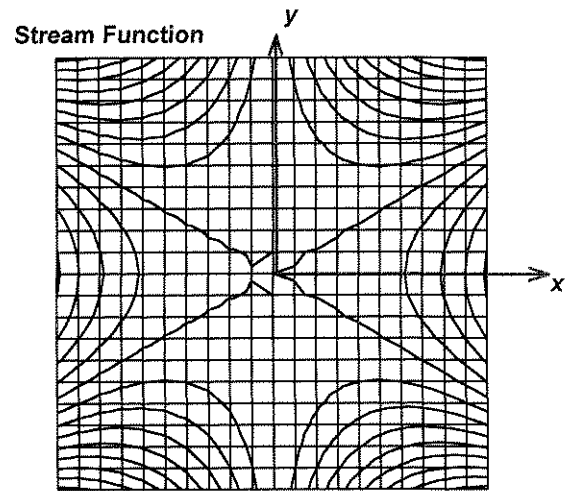
Problem *6.107

Using *Excel*, the stream function and velocity potential can be plotted.
The data below was obtained using the workbook for Example Problem 6.10.
Note the orthogonality of ψ and ϕ!

#NAME?	**Stream Function**

#NAME?	**Velocity Potential**

Note that the plot is
from x = -5 to 5 and y = -5 to 5

Problem *6.108

Given: Irrotational flow represented by $\psi = Bxy$, where $B = 0.25\ s^{-1}$ and the coordinates are measured in meters.

Find: (a) the rate of flow between points $(x_1, y_1) = (2,2)$ and $(x_2, y_2) = (3,3)$

(b) the velocity potential for this flow

Plot: streamlines and potential lines, and visually verify that they are orthogonal.

Solution:

The volume flow rate (per unit depth) between points ① and ② is given by

$$Q_{12} = \psi_2 - \psi_1 = B[x_2 y_2 - x_1 y_1] = 0.25\ s^{-1}[3m \times 3m - 2m \times 2m]$$

$$Q_{12} = 1.25\ m^3/s/m \quad \leftarrow Q_{12}$$

The velocity field is determined from the stream function

$$u = \partial\psi/\partial y = Bx \qquad v = -\partial\psi/\partial x = -By \qquad \therefore \vec{V} = Bx\hat{i} - By\hat{j}$$

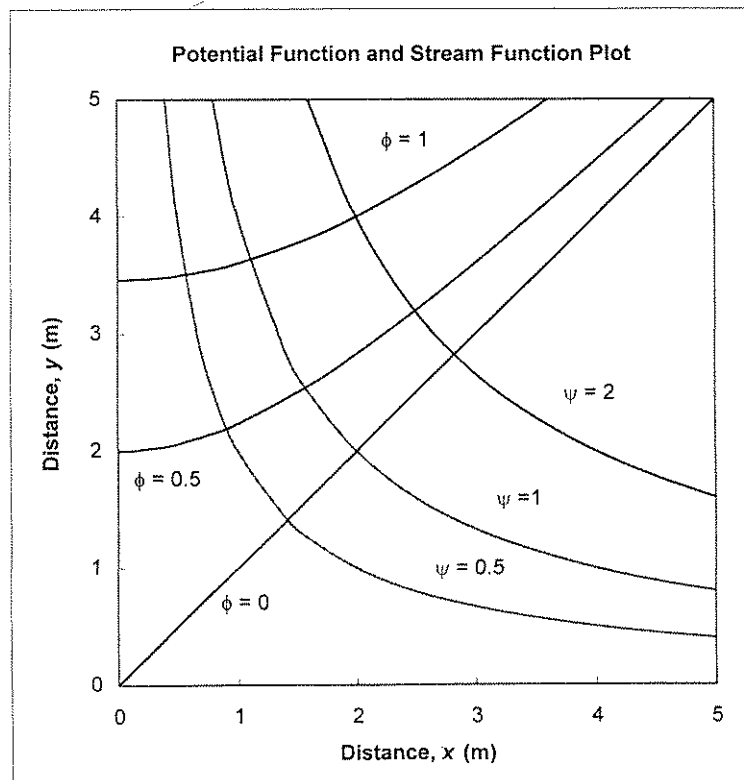
For irrotational flow $\vec{V} = -\nabla\phi$ and $u = -\partial\phi/\partial x$, $v = -\partial\phi/\partial y$ and

$$\phi = -\int u\,dx + f(y) = -\int Bx\,dx + f(y) = -\frac{B}{2}x^2 + f(y) \quad \ldots (1)$$

Also

$$\phi = -\int v\,dy + g(x) = \int By\,dy + g(x) = \frac{B}{2}y^2 + g(x) \quad \ldots (2)$$

Equating expressions for ϕ (Eqs 1 and 2) we conclude that

$$f(y) = \frac{B}{2}y^2, \quad g(x) = -\frac{B}{2}x^2 \quad \text{and} \quad \phi = \frac{B}{2}(y^2 - x^2) \quad \leftarrow \phi$$

Given: Two-dimensional, inviscid flow with velocity field $\vec{V} = (Ax+B)\hat{i} + (C-Ay)\hat{j}$, where $A = 3s^{-1}$, $B = 6m/s$, $C = 4m/s$ and the coordinates are measured in meters. The body force distribution is $\vec{B} = -g\hat{k}$; $\rho = 825 kg/m^3$.

Find: (a) if this is a possible incompressible flow
(b) stagnation point(s) of the flow field
(c) if the flow is irrotational
(d) the velocity potential (if one exists)
(e) pressure difference between origin and point $(x,y,z) = (2,2,2)$

Plot: a few streamlines in the upper half plane

Solution:

For incompressible flow $\nabla \cdot \vec{V} = 0$. For this flow

$$\nabla \cdot \vec{V} = \frac{\partial}{\partial x}(Ax+B) + \frac{\partial}{\partial y}(C-Ay) = A - A = 0$$

$\therefore$ velocity field represents possible incompressible flow.

At the stagnation point $u = v = 0$. $(\vec{V} = 0)$

$$u = 0 = (Ax+B) \quad \therefore x = -B/A = \frac{-6 m.s^{-1}}{3s^{-1}} = -2m$$

$$v = 0 = (C-Ay) \quad \therefore y = C/A = \frac{4 m.s^{-1}}{3s^{-1}} = 4/3 \, m$$

Stagnation point is at $(x,y) = (-2, 4/3)m$.

The fluid rotation (for a 2-D flow) is given by $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

For this flow $\omega_z = \frac{1}{2}\left[\frac{\partial(C-Ay)}{\partial x} - \frac{\partial(Ax+B)}{\partial y}\right] = 0$

$\therefore$ flow is irrotational

Then, $\vec{V} = -\nabla\phi$ and $u = -\partial\phi/\partial x$ and $v = -\partial\phi/\partial y$.

and $\phi = \int -u\,dx + f(y) = -\int(Ax+B)\,dx + f(y) = -A\frac{x^2}{2} - Bx + f(y)$ ...(1)

Also $\phi = -\int v\,dy + g(x) = -\int(C-Ay)\,dy + g(x) = A\frac{y^2}{2} - Cy + g(x)$ ...(2)

Equating the two expressions for ϕ (Eqs. 1 and 2) we note that $g(x) = -\left(A\frac{x^2}{2} + Bx\right)$ and $f(y) = A\frac{y^2}{2} - Cy$.

$$\therefore \phi = \frac{A}{2}(y^2 - x^2) - Bx - Cy \qquad \phi$$

Since the flow is irrotational we can apply the Bernoulli equation between any two points in the flow field.

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

At point 1 $(0,0,0)$, $\vec{V} = B\hat{i} + C\hat{j} = 6\hat{i} + 4\hat{j}$ m/s, $V_1^2 = 52 m^2/s^2$

At point 2 (2,2,2) $\vec{V}_2 = [3s^{-1}\times 2m + 6\,m/s]\hat{i} + [4\,m/s - 3s^{-1}\times 2m]\hat{j}$

$\vec{V}_2 = 12\hat{i} - 2\hat{j}\ m/s \quad , \quad V_2^2 = 148\ m^2/s^2$

$p_1 - p_2 = \frac{\rho}{2}(V_2^2 - V_1^2) + \rho g(z_2 - z_1) = \rho\left[\frac{(V_2^2 - V_1^2)}{2} + g(z_2 - z_1)\right]$

$= 825\ \frac{kg}{m^3} \times \left[\frac{1}{2}\times(148-52)\frac{m^2}{s^2} + 9.81\frac{m}{s^2}\times(2m)\right]\times\frac{N\cdot s^2}{kg\cdot m}$

$p_1 - p_2 = 55.8\ kPa$ ← Δp

The stream function is defined such that $u = \frac{\partial\psi}{\partial y}$, $v = -\frac{\partial\psi}{\partial x}$

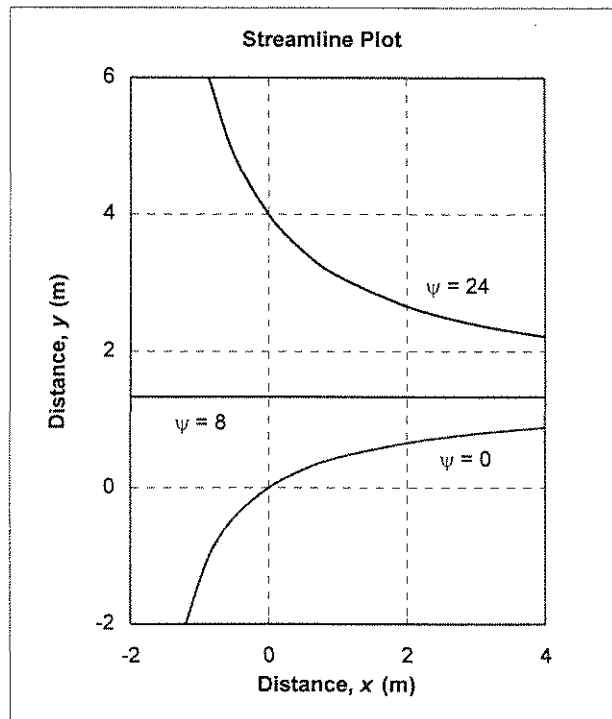
Then, $\psi = \int u\,dy + f(x) = \int (Ax+B)\,dy + f(x) = Axy + By + f(x)$ ----(1)

Also, $\psi = -\int v\,dx + g(y) = \int(-C + Ay)\,dx + g(y) = -Cx + Axy + g(y)$ ----(2)

Equating the two expressions for ψ (Eqs 1 and 2) we note that

$f(x) = -Cx$, $g(y) = By$ and $\therefore\ \psi = Axy + By - Cx$ ← ψ

The stagnation streamline goes through the stagnation point $(-2, \frac{4}{3})$

$\psi_{stag} = 3s^{-1}\times(-2m)\times\frac{4}{3}m + 6m\cdot s^{-1}\times\frac{4}{3}m - 4m\cdot s^{-1}\times(-2m) = 8\ m^2/s$ ← ψ_{stag}

Given: Flow past a circular cylinder of Example Problem 6.11.

Find: (a) Show that $V_r = 0$ along the lines $(r,\theta) = (r, \pm\pi/2)$
(b) Plot V_θ/U versus r for $r \geq a$ along line $(r, \pi/2)$
(c) Find distance beyond which the influence of the cylinder on the velocity is less that 1% of U

<u>Solution</u>.

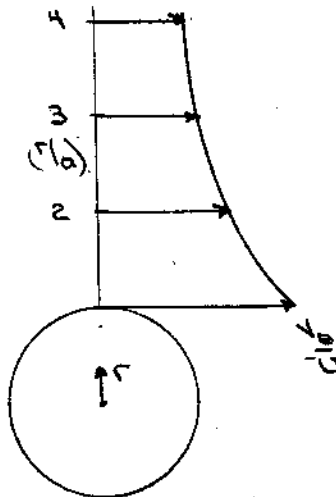
From Example Problem 6.11

$$\vec{V} = \left(-\frac{\Lambda\cos\theta}{r^2} + U\cos\theta\right)\hat{i}_r + \left(-\frac{\Lambda\sin\theta}{r^2} - U\sin\theta\right)\hat{i}_\theta \quad \ldots\ldots (1)$$

Then $V_r = \left(-\frac{\Lambda}{r^2} + U\right)\cos\theta$. For $\theta = \pm\frac{\pi}{2}$, $\cos\theta = 0$ and $V_r = 0$

$V_\theta = -\left(\frac{\Lambda}{r^2} + U\right)\sin\theta$, but $\frac{\Lambda}{U} = a^2$

$\therefore V_\theta = -\left(\frac{a^2}{r^2} + 1\right)U\sin\theta$ For $\theta = \pi/2$.

$$\frac{V_\theta}{U} = -\left(1 + \frac{a^2}{r^2}\right)$$

$$\vec{V} = U\cos\theta\left(1 - \frac{a^2}{r^2}\right)\hat{i} - U\sin\theta\left(1 + \frac{a^2}{r^2}\right)\hat{j}$$

For $\theta = \pi/2$

$\frac{V}{U} = 1 + \frac{a^2}{r^2}$ If $\frac{V}{U} = 1.01$ then $\frac{a^2}{r^2} = 0.01$ or $\frac{a}{r} = 0.1$

$\therefore \frac{V}{U} < 1\%$ for $r > 10a$

Problem *6.111

[3]

Consider flow around a circular cylinder with freestream velocity from right to left and a counterclockwise free vortex. Show that the lift force on the cylinder can be expressed as $F_L = -\rho U\Gamma$, as illustrated in Example 6.12.

Open-Ended Problem Statement: Consider flow around a circular cylinder with freestream velocity from right to left and a counterclockwise free vortex. Show that the lift force on the cylinder can be expressed as $F_L = -\rho U\Gamma$, as illustrated in Example 6.12.

Discussion: The only change in this flow from the flow of Example 6.12 is that the directions of the freestream velocity and the vortex are changed. This changes the sign of the freestream velocity from U to $-U$ and the sign of the vortex strength from K to $-K$. Consequently the signs of both terms in the equation for lift are changed. Therefore the direction of the lift force remains unchanged.

The analysis of Example 6.12 shows that only the term involving the vortex strength contributes to the lift force. Therefore the expression for lift obtained with the changed freestream velocity and vortex strength is identical to that derived in Example 6.12. Thus the general solution of Example 6.12 holds for any orientation of the freestream and vortex velocities. For the present case, $F_L = -\rho U\Gamma$, as shown for the general case in Example 6.12.

Problem *6.112

Given: A tornado is modelled by the superposition of a sink (strength, $q = 2800\ m^2/sec$) and a free vortex (strength, $K = 5600\ m^2/sec$)

Find: (a) Expressions for ψ and ϕ
(b) Estimate the radius beyond which the flow may be treated as incompressible.
(c) Find the gage pressure at that radius.

Solution:

$$\psi = \psi_{si} + \psi_{vo} = -\frac{q}{2\pi}\theta - \frac{K}{2\pi}\ln r \qquad \psi$$

$$\phi = \phi_{si} + \phi_{vo} = \frac{q}{2\pi}\ln r - \frac{K}{2\pi}\theta \qquad \phi$$

$\vec{V} = V_r\hat{i}_r + V_\theta\hat{i}_\theta$ $\qquad V_{r\,si} = -\frac{q}{2\pi r}$, $V_{r\,vo} = 0$; $V_{\theta\,si} = 0$, $V_{\theta\,vo} = \frac{K}{2\pi r}$

$$\therefore \vec{V} = -\frac{q}{2\pi r}\hat{i}_r + \frac{K}{2\pi r}\hat{i}_\theta$$

Then

$$V = (V_r^2 + V_\theta^2)^{1/2} = \left[\left(-\frac{q}{2\pi}\right)^2 + \left(\frac{K}{2\pi}\right)^2\right]^{1/2}\frac{1}{r}$$

For incompressible flow $M \le 0.3$. For standard air this corresponds to $V < 102\ m/sec$. Then, for incompressible flow

$$V = 102\ m/sec < \left[q^2 + K^2\right]^{1/2}\frac{1}{2\pi r}$$

or

$$r > \left[q^2 + K^2\right]^{1/2}\frac{1}{2\pi}\times\frac{s}{102\ m} = \left[(2800)^2 + (5600)^2\right]^{1/2}\frac{m^2}{s^2}\times\frac{1}{2\pi}\times\frac{s}{102\ m}$$

$$r > 9.77\ m \qquad r$$

To determine the gage pressure at this radius, apply the Bernoulli equation for irrotational flow

$$\frac{p_\infty}{\rho} + \frac{V_\infty^2}{2} + gz_\infty = \frac{p}{\rho} + \frac{V^2}{2} + gz \qquad \text{assume } \Delta z = 0 \quad (V_\infty^2 \approx 0)$$

Then

$$p_{gage} = p - p_\infty = -\frac{1}{2}\rho V^2 = -\frac{1}{2}\times 1.225\frac{kg}{m^3}\times(102)^2\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}$$

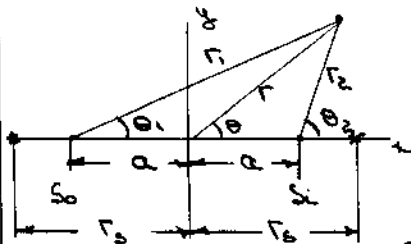
$$p_{gage} = -6.37\ kPa \quad (\text{for standard air}) \qquad p_{gage}$$

Problem *6.113

Given: Flow past a Rankine body is formed from the superposition of a uniform flow ($\bar{U} = 20\,m/s$) in the $+x$ direction and a source and a sink of equal strengths ($q = 3\pi\ m^2/s$) located on the x axis at $x = -a$ and $x = a$, respectively.

Find: (a) expressions for ψ, ϕ, and $\vec{V}$
(b) the value of ψ = constant on the stagnation streamline.
(c) the stagnation points if $a = 0.3\,m$.

Solution:

$$\psi = \psi_{so} + \psi_{si} + \psi_{uf} = \frac{q}{2\pi}\theta_1 - \frac{q}{2\pi}\theta_2 + \bar{U}y$$

$$\psi = \frac{q}{2\pi}(\theta_1 - \theta_2) + \bar{U} r \sin\theta \quad \text{— } \psi$$

$$\phi = \phi_{so} + \phi_{si} + \phi_{uf} = -\frac{q}{2\pi}\ln r_1 + \frac{q}{2\pi}\ln r_2 - \bar{U}x$$

$$\phi = \frac{q}{2\pi}\ln\frac{r_2}{r_1} - \bar{U} r\cos\theta \quad \text{— } \phi$$

$$u = u_{so} + u_{si} + u_{uf} = \frac{q}{2\pi r_1}\cos\theta_1 - \frac{q}{2\pi r_2}\cos\theta_2 + \bar{U}$$

$$v = v_{so} + v_{si} + v_{uf} = \frac{q}{2\pi r_1}\sin\theta_1 - \frac{q}{2\pi r_2}\sin\theta_2$$

$$\vec{V} = u\hat{i} + v\hat{j} = \left\{\frac{q}{2\pi}\left(\frac{\cos\theta_1}{r_1} - \frac{\cos\theta_2}{r_2}\right) + \bar{U}\right\}\hat{i} + \frac{q}{2\pi}\left(\frac{\sin\theta_1}{r_1} - \frac{\sin\theta_2}{r_2}\right)\hat{j} \quad \text{— } \vec{V}$$

At stagnation point $\vec{V} = 0$ $\quad y = 0 \quad \theta_1 = \theta_2 = 0$

$r_2 = r_s - a \quad , r_1 = r_s + a$

$$\therefore u = 0 = \frac{q}{2\pi}\left(\frac{1}{r_s + a} - \frac{1}{r_s - a}\right) + \bar{U} = \frac{q}{2\pi}\left[\frac{(r_s - a) - (r_s + a)}{(r_s^2 - a^2)}\right] + \bar{U}$$

$$0 = -\frac{qa}{\pi(r_s^2 - a^2)} + \bar{U} \quad \text{or} \quad (r_s^2 - a^2) = \frac{qa}{\pi\bar{U}}$$

$$r_s = \left(a^2 + \frac{qa}{\pi\bar{U}}\right)^{1/2} = a\left(1 + \frac{q}{\pi\bar{U}a}\right)^{1/2}$$

For $a = 0.3\,m$

$$r = 0.3\,m\left[1 + \frac{3\pi}{\pi}\frac{m^2}{s}\cdot\frac{s}{20\,m}\cdot\frac{1}{0.3\,m}\right]^{1/2} = 0.367\,m$$

Stagnation points located at $\theta = 0, \pi \quad r = 0.367\,m$ —

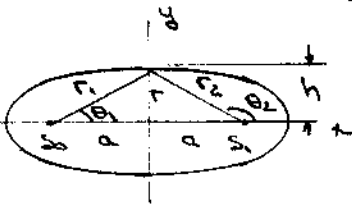
Since $\psi = \frac{q}{2\pi}(\theta_1 - \theta_2) + \bar{U}y$ and $\theta_1 = \theta_2$ & $y = 0$ at stagnation

$$\psi_{stag} = 0 \quad \text{—}$$

Problem *6.114

Given: Flow past a Rankine body is formed from the superposition of a uniform flow ($U = 20$ m/s) in the +x direction, and a source and a sink of equal strengths ($q = 3\pi$ m²/s) located on the x axis at $x = -a$ and $x = a$, respectively

Find: (a) the half width of the body
(b) $\vec{V}$ and p at the points $(0, \pm h)$

Solution:

$$\psi = \psi_{so} + \psi_{si} + \psi_{uf} = \frac{q}{2\pi}(\theta_1 - \theta_2) + U r \sin\theta$$

At stagnation point $\theta_1 = \theta_2$ and $\theta = 0, \pi$

$\therefore \psi_{stag} = 0$ and equation of stag. streamline is

$$0 = \frac{q}{2\pi}(\theta_1 - \theta_2) + U r \sin\theta$$

or

$$r = \frac{q}{2\pi}\frac{(\theta_2 - \theta_1)}{U \sin\theta}$$

At half width, $\theta = \frac{\pi}{2}$, $\theta_2 = \pi - \theta_1$, and $r = h = \frac{q}{2\pi}\frac{[(\pi - \theta_1) - \theta_1]}{U}$

$$\therefore hU = \frac{q}{2\pi}[\pi - 2\theta_1] = \frac{q}{2} - \frac{q\theta_1}{\pi} \quad \text{or} \quad \theta_1 = \frac{\pi}{2} - \frac{Uh\pi}{q}$$

Since $h = a\tan\theta_1$

$$\frac{h}{a} = \tan\left(\frac{\pi}{2} - \frac{Uh\pi}{q}\right) = \cot\left(\frac{Uh\pi}{q}\right)$$

Substituting values, $\frac{h}{0.3} = \cot\left(\frac{20}{3}h\right)$. Trial and error solution gives

$$h = 0.1615 \text{ m} \qquad h$$

The velocity field is given by $\vec{V} = \hat{i}u + \hat{j}v$

$$\vec{V} = \left\{\frac{q}{2\pi}\left(\frac{\cos\theta_1}{r_1} - \frac{\cos\theta_2}{r_2}\right) + U\right\}\hat{i} + \frac{q}{2\pi}\left(\frac{\sin\theta_1}{r_1} - \frac{\sin\theta_2}{r_2}\right)\hat{j}$$

At $(0, h)$, $r_1 = r_2$, $\theta_2 = \pi - \theta_1$ $\therefore \sin\theta_2 = \sin\theta_1$, $\cos\theta_2 = -\cos\theta_1$

and $\vec{V} = \left(\frac{q\cos\theta_1}{r_1} + U\right)\hat{i}$

$$\theta_1 = \tan^{-1}\frac{h}{a} = \tan^{-1}\frac{0.1615}{0.3} = 28.3^\circ \qquad r_1 = [a^2 + h^2]^{1/2} = [0.3^2 + 0.1615^2]^{1/2} = 0.341 \text{ m}$$

$$\vec{V} = \left(\frac{q\cos\theta_1}{r_1} + U\right)\hat{i} = \left(3\pi\,\frac{m^2}{sec} \times \frac{\cos 28.3^\circ}{0.341\,m} + 20\,\frac{m}{s}\right)\hat{i} = 44.3\,\hat{i}\ \text{m/s} \qquad V$$

To find the gage pressure apply the Bernoulli equation between the point at conditions at ∞

$$\frac{p_\infty}{\rho} + \frac{U^2}{2} = \frac{p}{\rho} + \frac{V^2}{2}$$

$$p_{gage} = p - p_\infty = \frac{1}{2}\rho(U^2 - V^2) = \frac{1}{2} \times 1.225\,\frac{kg}{m^3}\left[(20)^2 - (44.3)^2\right]\frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$p_{gag} = -957 \text{ N/m}^2 \qquad p_{gag}$$

Problem *6.115

Given: Flow field formed by superposition of a uniform flow in the + x direction ($U = 10$ m/s) and a counterclockwise vortex, with strength $K = 16\pi$ m²/s, located at the origin

Find: (a) ψ, ϕ, and $\vec{V}$ for the flow field
(b) stagnation point(s)

Plot: streamlines and lines of constant potential

Solution:

$$\psi = \psi_{uf} + \psi_v = Uy - \frac{K}{2\pi}\ln r = Ur\sin\theta - \frac{K}{2\pi}\ln r \quad \leftarrow \psi$$

$$\phi = \phi_{uf} + \phi_v = -Ux - \frac{K}{2\pi}\theta = -Ur\cos\theta - \frac{K}{2\pi}\theta \quad \leftarrow \phi$$

$$V_r = -\frac{\partial\phi}{\partial r} = U\cos\theta \;, \quad V_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta + \frac{K}{2\pi r}$$

$$\vec{V} = U\cos\theta\,\hat{e}_r + \left(\frac{K}{2\pi r} - U\sin\theta\right)\hat{e}_\theta \quad \leftarrow \vec{V}$$

At stagnation point, $\vec{V} = 0$

$V_r = 0$ at $\theta = \pm\frac{\pi}{2}$; $V_\theta = 0$ on $r = \frac{K}{2\pi U\sin\theta}$

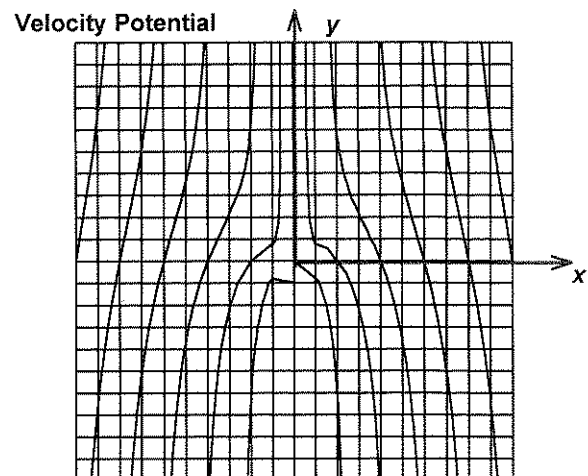
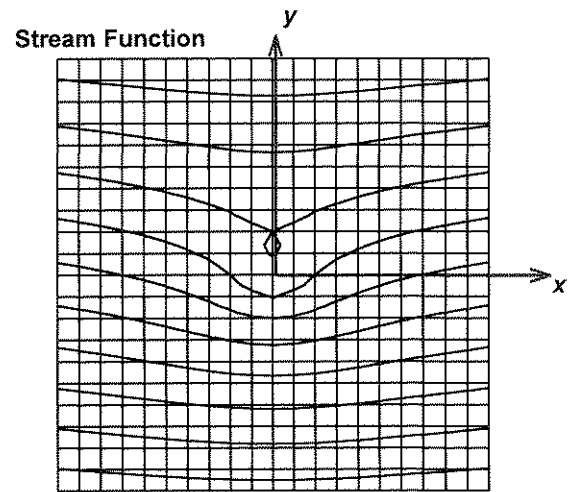
$\therefore \vec{V} = 0$ at $(r, \theta) = \frac{K}{2\pi U}, \pi/2$ $\leftarrow$ Stagnation

See the next page for plots

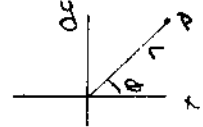
Problem *6.115

Using *Excel*, the stream function and velocity potential can be plotted.
The data below was obtained using the workbook for Example Problem 6.10.
Note the orthogonality of ψ and ϕ!

Note that the plot is
from $x = -5$ to 5 and $y = -5$ to 5

Problem *6.116

Given: Flow field formed by combining a uniform flow in the +x direction ($U = 50$ m/s) and a sink (of strength, $q = 90$ m²/s) at the origin.

Find: the net force per unit depth needed to hold in place (in standard air) the surface shape formed by the stagnation streamline

Solution:

$$\psi = \psi_{uf} + \psi_{si} = Uy - \frac{q}{2\pi}\theta = Ur\sin\theta - \frac{q}{2\pi}\theta \quad \text{----(1)}$$

$u = u_{uf} + u_{si}$; $u_{uf} = U$, $u_{si} = -V_r\cos\theta = -\frac{q}{2\pi r}\frac{x}{r}$ $\quad \therefore u = U - \frac{q}{2\pi}\frac{x}{r^2}$

$v = v_{uf} + v_{si}$; $v_{uf} = 0$, $v_{si} = -V_r\sin\theta = -\frac{q}{2\pi r}\frac{y}{r}$ $\quad \therefore v = -\frac{q}{2\pi}\frac{y}{r^2}$

$$\therefore \vec{V} = u\hat{i} + v\hat{j} = \left(U - \frac{q}{2\pi}\frac{x}{r^2}\right)\hat{i} - \frac{q}{2\pi}\frac{y}{r^2}\hat{j}$$

At the stagnation point, $\vec{V} = 0$

$\therefore -\frac{q}{2\pi}\frac{y}{r^2} = 0$, i.e. $y = 0$. Also $U - \frac{q}{2\pi}\frac{x}{r^2} = 0 \quad \therefore U = \frac{q}{2\pi}\frac{x}{(x^2+y^2)}$

and $x_{stag} = \frac{q}{2\pi U} = 90\frac{m^2}{s} \times \frac{1}{2\pi} \times \frac{s}{50\,m} = 0.286\,m$

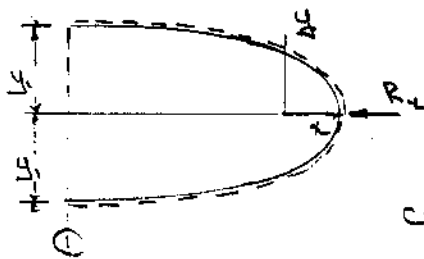
At stagnation point, $y = 0$ and $\theta = 0$. From eq (1), then $\psi_{stag} = 0$

The equation of the stagnation streamline is then,

$$\psi = 0 = Ur\sin\theta - \frac{q}{2\pi}\theta \quad \text{or} \quad r_{stag} = \frac{q\theta}{2\pi U\sin\theta}$$

Since $y = r\sin\theta$, then along the stagnation streamline $y = \frac{q\theta}{2\pi U}$.

Far upstream, $\theta \to \pi$ and $y = y_1 \to \frac{q}{2U}$.

The surface shape formed by the stagnation streamline is then as follows:

There is no flow across this streamline. The flow in through the left face must be equal to the flow (q) which leaves through the sink at the origin.

Applying the x momentum equation to the CV shown. R_x is force required to hold shape in place

$$-R_x = \int u\,\rho\vec{V}\cdot d\vec{A} = -U\dot{m}_1 = -U\rho q b$$

$$\therefore \frac{R_x}{b} = \rho q U$$

For standard air $\rho = 1.225$ kg/m³ and

$$\frac{R_x}{b} = 1.225\frac{kg}{m^3} \times 90\frac{m^2}{s} \times 50\frac{m}{s} \times \frac{N\cdot s^2}{kg\cdot m} = 5.51\ kN/m$$

$$\vec{R}_x/b = -5.51\,\hat{i}\ kN/m \quad \longleftarrow \quad R_x/b$$

Problem *6.117

Given: Flow field obtained by combining a uniform flow in the +x direction ($U = 30$ m/s) and a source (of strength $q = 150$ m²/s) located at the origin.

Plot: the ratio of the local velocity V to the free stream velocity U as a function of θ along the stagnation streamline.

Find: (a) points on the stagnation streamline where the velocity reaches its maximum value

(b) gage pressure at this location if $\rho = 1.2$ kg/m³

Solution:

Superposition of a uniform flow and source gives flow around a half body.

$$\psi = \psi_{uf} + \psi_{so} = Uy + \frac{q}{2\pi}\theta = Ur\sin\theta + \frac{q}{2\pi}\theta \quad (1)$$

$$u = u_{uf} + u_{so}\,;\; u_{uf} = U\,;\; u_{so} = V_r\cos\theta = \frac{q}{2\pi r}\frac{x}{r} \qquad \therefore u = U + \frac{q}{2\pi}\frac{x}{r^2}$$

$$v = v_{uf} + v_{so}\,;\; v_{uf} = 0\,;\; v_{so} = V_r\sin\theta = \frac{q}{2\pi r}\frac{y}{r} \qquad \therefore v = \frac{q}{2\pi}\frac{y}{r^2}$$

$$\therefore \vec{V} = u\hat{i} + v\hat{j} = \left(U + \frac{q}{2\pi}\frac{x}{r^2}\right)\hat{i} + \frac{q}{2\pi}\frac{y}{r^2}\hat{j} \quad (2)$$

Then,

$$V^2 = u^2 + v^2 = \left(U + \frac{q}{2\pi r}\cos\theta\right)^2 + \left(\frac{q}{2\pi r}\sin\theta\right)^2$$

$$= U^2 + \left(\frac{q}{2\pi r}\right)^2\cos^2\theta + \frac{Uq}{\pi r}\cos\theta + \left(\frac{q}{2\pi r}\right)^2\sin^2\theta$$

$$V^2 = U^2 + \left(\frac{q}{2\pi r}\right)^2 + \frac{Uq}{\pi r}\cos\theta \quad (3)$$

To determine the equation of the stagnation streamline, we locate the stagnation point ($\vec{V} = 0$). From Eq. 2 $y = 0$ and

$$U + \frac{q}{2\pi}\frac{x}{r^2} = 0 = U + \frac{q}{2\pi}\frac{x}{(x^2+y^2)} = U + \frac{q}{2\pi x} \quad \text{and} \quad x_{stag} = -\frac{q}{2\pi U}$$

$$x_{stag} = -\frac{q}{2\pi U} = -\frac{1}{2\pi}\times 150\,\frac{m^2}{s}\times\frac{s}{30\,m} = -0.796\,m$$

At the stagnation point $y = 0$ and $\theta = \pi$. From Eq. 1 $\psi_{stag} = \frac{q}{2}$

the equation of the stagnation streamline is then:

$$\psi_{stag} = \frac{q}{2} = Ur\sin\theta + \frac{q}{2\pi}\theta \quad . \text{ Solving for } r\text{, we obtain}$$

$$r = \frac{1}{U\sin\theta}\left(\frac{q}{2} - \frac{q}{2\pi}\theta\right) = \frac{q(\pi-\theta)}{2\pi U\sin\theta} \quad (4)$$

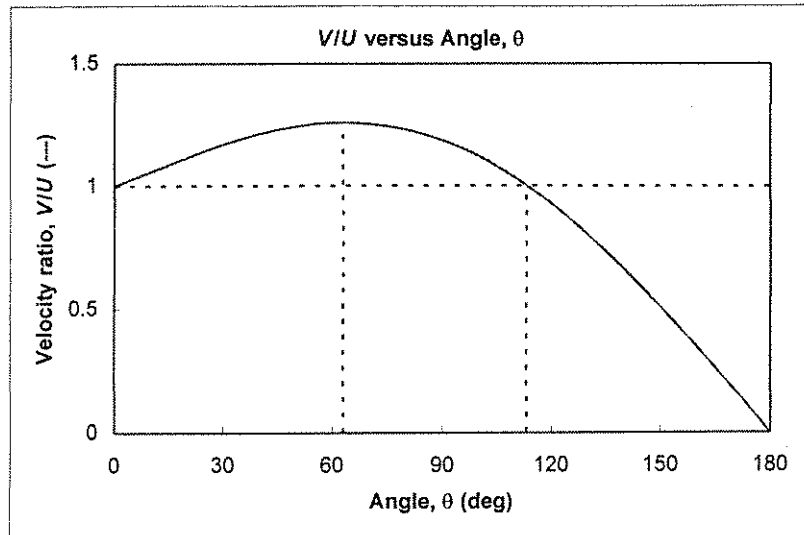
Substituting this value of r into the expression for V^2 [Eq. 3] we obtain

$$V^2 = U^2 + \left[\frac{q}{2\pi}\times\frac{2\pi U\sin\theta}{q(\pi-\theta)}\right]^2 + \frac{Uq\cos\theta}{\pi}\times\frac{2\pi U\sin\theta}{q(\pi-\theta)}$$

$$V^2 = U^2 + \frac{U^2\sin^2\theta}{(\pi-\theta)^2} + \frac{2U^2\sin\theta\cos\theta}{(\pi-\theta)} = U^2\left[1 + \frac{\sin^2\theta}{(\pi-\theta)^2} + \frac{2\sin\theta\cos\theta}{(\pi-\theta)}\right]$$

Along the stagnation streamline

$$\frac{V}{U} = \left[1 + \frac{\sin^2\theta}{(\pi-\theta)^2} + \frac{2\sin\theta\cos\theta}{(\pi-\theta)}\right]^{1/2} \quad (5)$$

V/U is plotted as a function of θ

From the plot we see that V/U is a maximum at $\theta = 63°$ (also at $\theta = 297°$ from symmetry with respect to the x axis.

At $\theta = 63°$, Eq 5 gives $V/U_{max} = 1.26$

, Eq. 4 gives $r = 150\frac{m^2}{s} \times \frac{(\pi - 0.35\pi)}{2\pi \sin 63°} \times \frac{s}{30 m} = 1.82 m$.

Thus $V = V_{max}$ at $r = 1.82 m$ and $\theta = 63°, 297°$ ← $(r,\theta)_{V_{max}}$

To determine the gage pressure at this point, write the Bernoulli equation between a point upstream and the point of maximum velocity.

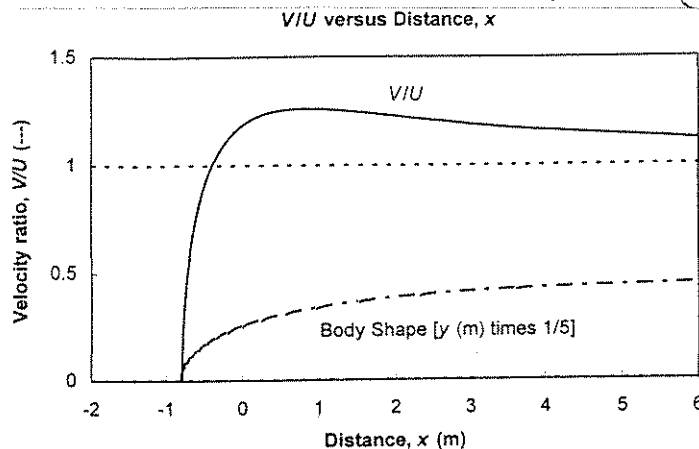
$$\frac{p_\infty}{\rho} + \frac{U^2}{2} = \frac{p}{\rho} + \frac{V_{max}^2}{2}$$

$$\therefore p - p_\infty = \frac{\rho}{2}\left[U^2 - V_{max}^2\right] = \frac{1}{2}\rho U^2\left[1 - \left(\frac{V_{max}}{U}\right)^2\right]$$

$$= \frac{1}{2} \times 1.2\frac{kg}{m^3} \times (30)^2\frac{m^2}{s^2}\left[1 - \left(1.26\right)^2\right] \times \frac{N \cdot s^2}{kg \cdot m}.$$

$p - p_\infty = 317 N/m^2$ ← p_{gage}

Note: From the plot we see that $V/U = 1.0$, and hence $p = p_\infty$, at $\theta = 113°$. The corresponding r is 1.01 m.

Problem *6.118

Given: Flow field obtained by superposing a uniform flow in the $+x$ direction ($U = 25$ m/s) and a source (of strength q) at the origin. Stagnation point is at $x = -1.0$ m.

Find: (a) expressions for ψ, ϕ, $\vec{V}$
(b) source strength, q.

Plot: streamlines and potential lines.

Solution:

$\psi = \psi_{u.f} + \psi_{so} = Uy + \frac{q}{2\pi}\theta = Ur\sin\theta + \frac{q}{2\pi}\theta$ — ψ

$\phi = \phi_{u.f} + \phi_{so} = -Ux - \frac{q}{2\pi}\ln r = -Ur\cos\theta - \frac{q}{2\pi}\ln r$ — ϕ

$u = u_{u.f} + u_{so}$; $u_{u.f} = U$; $u_{so} = V_r\cos\theta = \frac{q}{2\pi r}\frac{x}{r}$ $\quad\therefore u = U + \frac{q}{2\pi}\frac{x}{r^2}$

$v = v_{u.f} + v_{so}$; $v_{u.f} = 0$; $v_{so} = V_r\sin\theta = \frac{q}{2\pi r}\frac{y}{r}$ $\quad\therefore v = \frac{q}{2\pi}\frac{y}{r^2}$

$\vec{V} = u\hat{i} + v\hat{j} = \left\{U + \frac{q}{2\pi}\frac{x}{(x^2+y^2)}\right\}\hat{i} + \frac{q}{2\pi}\frac{y}{(x^2+y^2)}\hat{j}$ — $\vec{V}$

At the stagnation point $\vec{V} = 0$ $\quad x = -1.0$ m $\quad y = 0$ ($v = 0$).

For $u = 0 = U + \frac{q}{2\pi}\frac{x}{(x^2+y^2)}$ $\quad\therefore q = -2\pi U x_{stag}$

$q = -2\pi \times 25\frac{m}{s} \times (-1.0\,m) = 50\pi$ m²/s — q

At the stagnation point, $\theta = \pi$ $\quad\therefore \psi_{stag} = \frac{q}{2\pi}\theta = \frac{q}{2}$

The equation of the stagnation streamline is then

$q/2 = Ur\sin\theta + \frac{q}{2\pi}\theta$ $\quad$ and $\quad r = \frac{q(\pi-\theta)}{2\pi U\sin\theta}$

At $\theta = \pi/2$, $\quad r = \frac{q}{4U} = 50\pi\frac{m^2}{s} \times \frac{1}{4} \times \frac{s}{25\,m} = \pi/2$

Far downstream $\theta \to 0$ and the y coordinate of the body

$y = r\sin\theta = \frac{q(\pi-\theta)}{2\pi U}$ approaches $\frac{q}{2U} = \frac{50\pi}{2\times 25} = \pm\pi$ m — $y_{\theta\to 0}$

See the next page for plots

Problem *6.118

Using *Excel*, the stream function and velocity potential can be plotted.
The data below was obtained using the workbook for Example Problem 6.10.
Note the orthogonality of ψ and ϕ!

#NAME?	Stream Function
#NAME?	Velocity Potential

Note that the plot is
from x = -5 to 5 and y = -5 to 5